PROJECTIONS ONTO L^p-BERGMAN SPACES OF REINHARDT DOMAINS

DEBRAJ CHAKRABARTI & LUKE D. EDHOLM

Abstract. For $1 < p < \infty$, a new projection operator is constructed from the L^p space of a Reinhardt domain to the L^p-Bergman space, by emulating the representation of the usual L^2-Bergman projection as an orthogonal series, but by using a monomial Schauder basis of the L^p-Bergman space instead of the L^2 orthonormal basis of the standard L^2-Bergman space. Such a projection operator is expected to have better L^p-mapping behavior than the classical Bergman projection. The existence and superior mapping properties of this new operator are verified on a class of domains on which the classical Bergman projection has poor behavior. On this class of domains, the dual of the L^p-Bergman space is identified with an L^q-Bergman space with weight.

1. Introduction

1.1. **The Bergman projection on L^p.** Given a domain $\Omega \subset \mathbb{C}^n$, the *Bergman projection* $\boldsymbol{B}^\Omega$ is the orthogonal projection from $L^2(\Omega)$ onto the *Bergman space* $A^2(\Omega) = L^2(\Omega) \cap \mathcal{O}(\Omega)$, the subspace of holomorphic square-integrable functions. The Bergman projection can be represented by integration against the *Bergman kernel*:

$$\boldsymbol{B}^\Omega f(z) = \int_\Omega B^\Omega(z,w) f(w) dV(w), \qquad f \in L^2(\Omega), \tag{1.1}$$

where dV stands for the Lebesgue measure. The Bergman kernel enjoys remarkable reproducing, invariance and extremal properties and is closely related to the $\bar\partial$-Neumann problem (see e.g. [Ber70, FK72, Kra13], etc.). These properties make the Bergman kernel an important tool in the study of boundary behavior of holomorphic functions and maps. Bergman spaces can be naturally defined on all complex manifolds, in contrast with Hardy spaces, whose construction is tied to distinguished measures on the boundary of a domain, e.g., the Haar measure on the unit circle in the case of the classical Hardy space $H^p(\mathbb{D})$.

Inspired by Hardy spaces, it is natural to consider the space of p-th power integrable holomorphic functions $A^p(\Omega)$ of a domain $\Omega \subset \mathbb{C}^n$. These have also been known as *(L^p-) Bergman spaces* since the 1970s, though S. Bergman only studied the square integrable setting. In view of M. Riesz's classical result on the L^p-boundedness of the Szegő projection for $1 < p < \infty$, it is also natural to ask whether the Bergman projection (restricted to $L^2(\Omega) \cap L^p(\Omega)$) can be continuously extended as a bounded linear projection operator $\boldsymbol{B}_p^\Omega$ from $L^p(\Omega)$ onto $A^p(\Omega)$. When Ω is a ball in $\mathbb{C}^n$, this turns out to be the case (see [ZJ64, FR74]), and the same remains true in many classes of smoothly bounded pseudoconvex domains ([PS77, NRSW89, MS94, McN94] etc.) In these cases, the extended operator turns out to be even *absolutely bounded*, in the sense that the associated operator

$$(\boldsymbol{B}_p^\Omega)^+ f(z) = \int_\Omega \left|B^\Omega(z,w)\right| f(w) dV(w), \qquad f \in L^p(\Omega),$$

2020 *Mathematics Subject Classification.* 32A36, 46B15, 32A70, 32A25.

The first author was supported in part by US National Science Foundation grant number DMS-2153907, and by a gift from the Simons Foundation (number 706445).

The second author was supported in part by Austrian Science Fund (FWF): AI0455721.

is a bounded linear operator on $L^p(\Omega)$. The proofs typically use some version of *Schur's test* for the L^p-boundedness of an operator defined by a positive integral kernel (see Proposition 5.11 below).

On the other hand, there are many examples of domains for which the extended Bergman projection $\boldsymbol{B}_p^\Omega$ fails to define a bounded projection from $L^p(\Omega)$ onto $A^p(\Omega)$ for some (and sometimes for all!) $p \neq 2$ (see [Bar84, FKP99, FKP01, Zey13, EM17] and the survey [Zey20]). Recent studies of the Bergman projection in certain classes of Reinhardt domains ([CZ16, Edh16, EM16, Che17, CEM19, EM20, HW20, Zha21a, Zha21b, Mon21, BCEM22] etc.) shed more light on this phenomenon, revealing that the L^p-behavior of the Bergman projection that one sees on, e.g., strongly pseudoconvex domains (see [PS77]) breaks down on bounded Reinhardt domains whose boundary passes through the center of rotational symmetry, a simple example being the well-known *Hartogs triangle* $\{|z_1| < |z_2| < 1\} \subset \mathbb{C}^2$. In such a bounded domain it is possible that there are indices $1 < p_1 < p_2 < \infty$ such that the linear subspace $A^{p_2}(\Omega)$ is not dense in the Bergman space $A^{p_1}(\Omega)$. This phenomenon is impossible in smoothly bounded pseudoconvex domains (see [Cat80]), and may perhaps constitute a glimpse of an L^p-function theory where the geometry of the Banach space L^p replaces the Hilbert space idea of orthogonality. In the Reinhardt domains studied in this paper, Laurent series representations can be used to clarify some of these phenomena. For example, the fact that $A^{p_2}(\Omega)$ is not necessarily dense in $A^{p_1}(\Omega)$ can be thought to be a manifestation of the fact that there may be Laurent monomials whose p_1-th power is integrable but not the p_2-th power.

One of the first tasks of such an L^p-function theory would be to study possible analogs of the Bergman projection and kernel. In this paper, we take the point of view that the extended Bergman projection $\boldsymbol{B}_p^\Omega$ *does not* necessarily constitute the best solution to the problem of constructing a projection operator from $L^p(\Omega)$ onto $A^p(\Omega)$. We propose that a different operator, which we call the *Monomial Basis Projection (MBP)* may be expected to have better behavior. This projection can be represented as an integral operator against the *Monomial Basis Kernel (MBK)*. We now explain the construction of these objects.

1.2. **Projection operators associated to bases.** Let L be a Hilbert space, let A be a closed subspace of L and let $\{e_j\}$ be a complete orthogonal set in A. (All Hilbert or Banach spaces in this paper are assumed to be separable.) Letting $\phi_j = \frac{e_j}{\|e_j\|}$, we see that $\{\phi_j\}$ is an orthonormal basis of A, and the orthogonal projection $\boldsymbol{P}$ from L to A may be represented by the series convergent in the norm of L:

$$\boldsymbol{P}x = \sum_j \langle x, \phi_j \rangle \phi_j = \sum_j \frac{\langle x, e_j \rangle e_j}{\|e_j\|^2}, \qquad x \in L. \tag{1.2}$$

Since $\boldsymbol{P}x$ is defined geometrically as the point in A nearest to x, this representation is independent of the particular complete orthogonal set $\{e_j\}$. When $L = L^2(\Omega)$, $A = A^2(\Omega)$, (1.2) leads to the well-known formulas for the Bergman kernel

$$B^\Omega(z, w) = \sum_j \phi_j(z)\overline{\phi_j(w)} = \sum_j \frac{e_j(z)\overline{e_j(w)}}{\|e_j\|^2}.$$

The analog of a complete orthogonal set of a Hilbert space in the setting of a general Banach space is a *Schauder basis* (see [LT77]). A sequence $\{e_j\}_{j=1}^\infty$ in a complex Banach space A is called a Schauder basis if for each $x \in A$, there is a unique sequence $\{c_j\}_{j=1}^\infty$ of complex numbers such that $x = \sum_{j=1}^\infty c_j e_j$, where the series converges in the norm-topology of A. In fact, there are bounded linear functionals $a_j : A \to \mathbb{C}$ such that $c_j = a_j(x)$, generalizing the Fourier coefficients $a_j(x) = \frac{\langle x, e_j \rangle}{\|e_j\|^2}$ in the special case when A is Hilbert.

When L is a Banach space, A a closed subspace, and $\{e_j\}_{j=1}^\infty$ a Schauder basis of A, one might attempt to define a projection operator from L onto A by emulating (1.2):

$$\boldsymbol{P}x = \sum_j \widetilde{a}_j(x)e_j, \qquad x \in L, \tag{1.3}$$

where $\widetilde{a}_j : L \to \mathbb{C}$ is a Hahn-Banach (norm-preserving) extension of $a_j : A \to \mathbb{C}$, generalizing the functional $x \mapsto \frac{\langle x, e_j\rangle}{\|e_j\|^2}$ on L that occurs in (1.2). When it exists, we call a projection operator of the type in (1.3) a *basis projection* determined by the Schauder basis $\{e_j\}$.

If $\mathbb{T}$ is the unit circle with the Haar measure, the classical Szegő projection from $L^p(\mathbb{T})$ onto the Hardy space $H^p(\mathbb{D})$ for $1 < p < \infty$ is in fact a basis projection (see Proposition 2.9 below) and our goal in this paper is to construct basis projections from $L^p(\Omega)$ to $A^p(\Omega)$, where Ω is a Reinhardt domain.

Note that if $p \neq 2$, the classical attempt to extend the Bergman projection to L^p by continuity, even when successful, leads to a projection operator which is not *a basis projection*; see Proposition 4.3. This is an underlying reason for the deficiencies of the Bergman projection in L^p-spaces.

1.3. **The Monomial Basis Projection.** Formula (1.3) is purely formal, as there is no guarantee that, given a Schauder basis of subspace, a basis projection onto the subspace determined by the given basis exists. Several technical obstacles need to be overcome to actually construct a basis projection onto the L^p-Bergman space of a Reinhardt domain:

(1) Unlike an orthogonal projection, the basis projection of (1.3) depends not only on the range subspace A of the projection, but also (in principle) on the choice of the Schauder basis $\{e_j\}$ determining the projection. A Banach space need not have a Schauder basis, but in the Bergman space $A^p(\Omega)$ of a Reinhardt domain $\Omega \subset \mathbb{C}^n$, there is a distinguished Schauder basis closely tied to geometry and function theory. This is the collection of *Laurent monomials* in $A^p(\Omega)$, the functions $z \mapsto z_1^{\alpha_1} z_2^{\alpha_2} \dots z_n^{\alpha_n}$ where $\alpha_j \in \mathbb{Z}, 1 \leq j \leq n$. The fact that these monomials form a Schauder basis was proved in [CEM19], and is recalled in Theorem 2.16 below in a slightly more general form. The projection operator from $L^p(\Omega)$ to $A^p(\Omega)$ defined in terms of this "monomial basis" by formula (1.3) is the main topic of this paper, and will be called the *Monomial Basis Projection (MBP)*.

(2) A Hahn-Banach extension of a linear functional in general is far from unique, but in our application, where we extend functionals defined on $A^p(\Omega)$ to $L^p(\Omega)$, *we do have uniqueness*; see Propositions 2.3 and 2.4 below. This means $\boldsymbol{P}$ is unambiguously defined by (1.3) once the order of summation of the multiple series is clarified.

(3) Such a projection $\boldsymbol{P}$ would not only be tied to the Schauder basis elements $\{e_j\}$, but also to their *ordering*, as the convergence of the series in (1.3) is in general *conditional*. This is analogous, and related to, the conditional convergence of multiple Fourier series in $L^p(\mathbb{T}^n)$ (see, e.g., [Kra99]). An exception to the conditional convergence of (1.3) is when L is a Hilbert space, when the convergence is *unconditional*, i.e., the series converges no matter how it is rearranged. Further, the Laurent monomials $z \mapsto z_1^{\alpha_1} z_2^{\alpha_2} \dots z_n^{\alpha_n}$ with respect to which the MBP is defined, are indexed by the associated multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$ and do not admit a natural linear ordering. In order to accommodate these complexities, we begin by extending the notions of Schauder bases and basis projections, using multi-indices and specifying the kinds of partial sums that are allowed in the series representation with respect to the basis (see Section 2 and in particular Definitions 2.1 and 2.6). It turns out that in spite of conditional convergence, we still have a great deal of freedom in rearranging the summation order in the series representation of the MBP, which comes from the integral representation of the MBP (see Section 4.2 below). This is useful in proving facts about the MBP, such as the transformation law in Theorem 6.27.

(4) None of this guarantees the formal series (1.3) converges for each $x \in L$. Showing that (1.3) defines a bounded operator on L requires us to show by direct estimation that the partial summation operators are uniformly bounded in the operator norm on L. In our application to Bergman spaces, the problem is simplified because of the availability of an integral representation of the MBP, and in an interesting class of domains we will prove not only the existence of the MBP, but also its *absolute boundedness*, i.e., the boundedness of the operator defined by the absolute value of the integral kernel representing the MBP. As we remarked earlier, this is a property of the Bergman projection on many domains.

1.4. **General notation, definitions and conventions.** We pause to introduce notation and definitions to be used throughout the paper.

(1) Unless otherwise indicated, Ω will denote a bounded Reinhardt domain in $\mathbb{C}^n$ with center of symmetry at 0. Let $|\Omega| \subset \mathbb{R}^n$ denote its *Reinhardt Shadow*, i.e., the set of points

$$|\Omega| = \{(|z_1|, \dots, |z_n|) \in \mathbb{R}^n : z \in \Omega\}.$$

(2) We always assume that the index p satisfies $1 < p < \infty$, and let q be the index Hölder-conjugate to p, i.e. $\frac{1}{p} + \frac{1}{q} = 1$.

(3) For a domain $U \subset \mathbb{C}^n$ and a measurable function $\lambda : U \to [0, \infty]$ which is positive a.e. (the *weight*), we set for a measurable function f:

$$\|f\|^p_{L^p(U,\lambda)} = \|f\|^p_{p,\lambda} = \int_U |f|^p \, \lambda \, dV, \tag{1.4}$$

where, dV denotes the Lebesgue measure, and functions equal a.e. are identified. We let $L^p(U, \lambda)$ be the space of functions f for which $\|f\|_{p,\lambda} < \infty$, which is well-known to be a Banach space in its natural norm.

Let $A^p(U, \lambda)$ be the subspace of $L^p(U, \lambda)$ consisting of holomorphic functions:

$$A^p(U, \lambda) = L^p(U, \lambda) \cap \mathcal{O}(U).$$

We will only consider weights $\lambda : U \to [0, \infty]$ which are *admissible* in the sense that *Bergman's inequality* holds in $A^p(U, \lambda)$, i.e., for each compact set $K \subset U$, there is a constant $C_K > 0$ such that for each $f \in A^p(U, \lambda)$ we have

$$\sup_K |f| \le C_K \|f\|_{L^p(U,\lambda)} . \tag{1.5}$$

It is easy to see that if λ is a positive continuous function on U then it is an admissible weight on U; we also consider more general admissible weights.

If λ is an admissible weight on U, a standard argument shows that $A^p(U, \lambda)$ is a closed subspace of $L^p(U, \lambda)$, and therefore a Banach space. It is called a *weighted Bergman space* on U with weight λ, and generalizes the unweighted A^p-spaces from Section 1.1.

(4) We are interested in Reinhardt domains Ω and phenomena which are invariant under the Reinhardt (rotational) symmetry of Ω. Therefore, we consider only those weights λ on Ω which are admissible, and *multi-radial* in the sense that there is a function ℓ on the Reinhardt shadow $|\Omega|$ such that

$$\lambda(z_1, \dots, z_n) = \ell(|z_1|, \dots, |z_n|),$$

so that λ depends only on the absolute values of the coordinates.

(5) For $\alpha \in \mathbb{Z}^n$, we denote by e_α the Laurent monomial of exponent α:

$$e_\alpha(z) = z_1^{\alpha_1} \dots z_n^{\alpha_n}. \tag{1.6}$$

(6) We define the set of *p-allowable indices with respect to λ* to be the collection

$$\mathcal{S}_p(\Omega, \lambda) = \{\alpha \in \mathbb{Z}^n : e_\alpha \in L^p(\Omega, \lambda)\} . \tag{1.7}$$

(7) If $\lambda \equiv 1$, we abbreviate $L^p(\Omega, 1)$, $A^p(\Omega, 1)$ and $\mathcal{S}_p(\Omega, 1)$ by $L^p(\Omega)$, $A^p(\Omega)$ and $\mathcal{S}_p(\Omega)$, respectively.

(8) The map $\chi_p : \mathbb{C}^n \to \mathbb{C}^n$ defined as

$$\chi_p(\zeta) = \left(\zeta_1 \left|\zeta_1\right|^{p-2}, \cdots, \zeta_n \left|\zeta_n\right|^{p-2}\right) \tag{1.8}$$

arises naturally in our investigations, and will be referred to as the *twisting map*. Notice that for $p = 2$, the twisting map reduces to the identity. For a function f we denote by $\chi_p^* f$ its pullback under χ_p:

$$\chi_p^* f = f \circ \chi_p. \tag{1.9}$$

1.5. **The Monomial Basis Kernel.** When it exists, the MBP of $A^p(\Omega, \lambda)$ is a bounded surjective projection and will be written $\boldsymbol{P}^{\Omega}_{p,\lambda} : L^p(\Omega, \lambda) \to A^p(\Omega, \lambda)$. With the aim of a representation of the MBP by an integral formula analogous to (1.1), we define the *Monomial Basis Kernel* of $A^p(\Omega, \lambda)$ (abbreviated *MBK*) by the formal series on $\Omega \times \Omega$ given as:

$$K^{\Omega}_{p,\lambda}(z, w) = \sum_{\alpha \in \mathcal{S}_p(\Omega,\lambda)} \frac{e_\alpha(z) \cdot \overline{\chi_p^* e_\alpha(w)}}{\|e_\alpha\|^p_{p,\lambda}}. \tag{1.10}$$

For $p = 2$, the MBK $K^{\Omega}_{2,\lambda}$ is the classical weighted Bergman kernel B^{Ω}_{λ} of $A^2(\Omega, \lambda)$, and the series (1.10) converges uniformly on compact subsets of $\Omega \times \Omega$. For a general $1 < p < \infty$, we show in Theorem 3.6 that when Ω is pseudoconvex, the series (1.10) converges locally normally on $\Omega \times \Omega$, so the unordered sum makes sense. The similarity of the definition of the MBK with the series representation of the Bergman kernel means that many properties of the latter have "twisted" analogs for the MBK; see Section 3.4. The MBK affords an integral representation of the MBP when it exists, by the formula

$$\boldsymbol{P}^{\Omega}_{p,\lambda}(f)(z) = \int_{\Omega} f(w) K^{\Omega}_{p,\lambda}(z, w) \lambda(w) dV(w), \qquad f \in L^p(\Omega, \lambda).$$

See Theorem 4.1 and Proposition 4.10 below.

When $\boldsymbol{P}^{\Omega}_{p,\lambda}$ is *absolutely bounded* on $L^p(\Omega, \lambda)$ in the sense of Definition 4.23 below, we can give a natural description of the dual of $A^p(\Omega, \lambda)$ *as a space of holomorphic functions*, as a weighted Bergman space on a Reinhardt domain $\Omega^{(p-1)}$ associated to the domain Ω; see Proposition 4.36 below. The duality is realized in terms of the "twisted pairing"

$$\{f, g\}_{p,\lambda} = \int_{\Omega} f \cdot \overline{\chi_p^*(g)} \, \lambda \, dV,$$

rather than the usual L^2-style paring of spaces, where the notation is as in (1.8) and (1.9).

1.6. **Transformation laws.** Bell's transformation law ([Bel81, Bel82]) for the Bergman kernel under proper holomorphic maps is a key ingredient in the study of the boundary behavior of proper holomorphic mappings. In the setting of Reinhardt domains, the holomorphic maps we need to consider should preserve the Reinhardt structure, and therefore are the *monomial maps*, each of whose components is a monomial function (see Section 6.3 below, and [NP21, BCEM22]). Monomial maps define proper holomorphic maps which are of "quotient type", i.e. there is an abelian group Γ of biholomorphic automorphisms such that the monomial mapping $\Phi : \Omega_1 \to \Omega_2$ may be identified with the natural projection onto the quotient Ω_1/Γ, which is biholomorphic to Ω_2. In Section 6 below we study the transformation of the MBP by monomial maps, leading to the transformation law of Theorem 6.27. A technical issue in these results arises from the conditional convergence of series that need to be rearranged, but this issue is resolved by Corollary 4.22.

We go on to prove in Section 7.2 a transformation law for the *Absolute Monomial Basis Operator* (abbreviated *AMBO*), the integral operator with kernel given by the absolute value of the MBK. Notice that if $B^U : U \times U \to \mathbb{C}$ is the Bergman kernel of a (not necessarily Reinhardt) domain U in $\mathbb{C}^n$, then we can reconstruct B^U from the nonnegative function $A^U = \left|B^U\right|$ on $U \times U$. Further, if $\phi : U \to V$ is a biholomorphic mapping, we have, by taking absolute values in the formula for biholomorphic transformation of Bergman kernels, and using the relation $\det D\phi = |\det \phi'|^2$ between the real and the complex Jacobian determinants, the transformation law

$$A^U(z,w) = (\det D\phi(z))^{\frac{1}{2}} A^V(\phi(z), \phi(w))(\det D\phi(w))^{\frac{1}{2}}, \qquad z, w \in \Omega_1.$$

This relation (which can be invariantly interpreted in terms of external tensor products of $\frac{1}{2}$-density bundles) immediately leads to a transformation law for the absolute Bergman operators of U and V (i.e. the integral operators whose kernels are A^U and A^V, respectively.) We prove an analogous transformation law for the AMBO under monomial maps in Theorem 7.11. This result, apart from its intrinsic interest, plays a crucial role in the proof of absolute boundedness on monomial polyhedra in Section 8.

1.7. **Explicit examples in one and several dimensions.** In Section 5 and Section 8 we give actual examples of Reinhardt domains on which the MBP both exists and exhibits better behavior than the L^p-extended Bergman projection. In Section 5 we show that for $1 < p < \infty$, the MBP on the disc and the punctured disc exist, are absolutely bounded and are surjective onto the Bergman space. For the disc, this shows that the L^p-behavior of the MBP is at least as good as that of the classical Bergman projection. For the punctured disc, the behavior is better, since the classical Bergman projection is not surjective onto $A^p(\mathbb{D}^*)$ if $1 < p < 2$, despite it being a bounded operator on $L^p(\mathbb{D}^*)$.

In Section 8 we consider nonsmooth pseudoconvex Reinhardt domains called *monomial polyhedra* (see [NP09, BCEM22]). A bounded domain $\mathscr{U} \subset \mathbb{C}^n$ is a monomial polyhedron in our sense, if there are exactly n monomials $e_{\alpha^1}, \dots, e_{\alpha^n}$ (see (1.6) for notation) such that

$$\mathscr{U} = \{z \in \mathbb{C}^n : |e_{\alpha^1}(z)| < 1, \dots, |e_{\alpha^n}(z)| < 1\}.$$

The ur-example of a monomial polyhedron is the Hartogs triangle $\mathbb{H} = \{|z_1| < |z_2| < 1\} \subset \mathbb{C}^2$. In [BCEM22] it was shown that there is an integer $\kappa(\mathscr{U})$ associated to each monomial polyhedron $\mathscr{U}$ such that the Bergman projection is bounded in the L^p-norm if and only if

$$\frac{2\kappa(\mathscr{U})}{\kappa(\mathscr{U})+1} < p < \frac{2\kappa(\mathscr{U})}{\kappa(\mathscr{U})-1}. \tag{1.11}$$

A key ingredient of the proof is a "resolution of singularities" of a monomial polyhedron by a proper holomorphic map of quotient type (see Proposition 8.5). In contrast, this same resolution of singularities is used to prove Theorem 8.1 below, to show that the MBP is even *absolutely* bounded on $L^p(\mathscr{U})$ for each $1 < p < \infty$. This leads to the representation of the dual space $A^p(\mathscr{U})'$ as a *weighted* Bergman space on $\mathscr{U}$ (see Theorem 8.24).

In Section 9 there is a detailed discussion of the deficiencies of the L^p-extended Bergman projection in certain model settings. This gives an opportunity for a side-by-side comparison of the Bergman projection and the Monomial Basis Projection, where the failures of the former properly frame the successes of the latter.

1.8. **Conclusions and future work.** Since Barrett's work on a nonpseudoconvex "worm" domain with Hartogs symmetry ([Bar84]), and especially after the recent results on singular Reinhardt domains mentioned above, the failure of the Bergman projection on certain domains to satisfy L^p-estimates has seemed to indicate fundamental problems with the definition of the projection operator on L^p-Bergman spaces. Here, we suggest a possible way of resolving this puzzle by *redefining* the projection operator from L^p to A^p, and showing that this idea gives good results for at least one class of interesting domains. The next step in this investigation should be to extend this to further classes of interesting domains, such as the "irrational Hartogs triangles" (see [EM17]), smoothly bounded pseudoconvex Reinhardt domains, etc. It will also be interesting to extend this idea to Hartogs domains in the natural way, and see what this means for the original Barrett example.

2. The Monomial Basis Projection

2.1. **Schauder bases.** Since our application will use bases indexed by multi-indices, we introduce a slightly more general notion of a Schauder basis than the classical one described above in Section 1.2. For a multi-index $\alpha \in \mathbb{Z}^n$, let $|\alpha|_\infty = \max_{1\leq j\leq n} |\alpha_j|$.

Definition 2.1. Let A be a Banach space, n a positive integer and $\mathfrak{A} \subset \mathbb{Z}^n$ a set of multi-indices. A collection $\{e_\alpha : \alpha \in \mathfrak{A}\}$ of elements of A is said to form a *Schauder basis* of A if for each $f \in A$, there are complex numbers $\{c_\alpha : \alpha \in \mathfrak{A}\}$ such that

$$f = \lim_{N\to\infty} \sum_{\substack{|\alpha|_\infty \leq N \\ \alpha\in\mathfrak{A}}} c_\alpha e_\alpha, \tag{2.2}$$

where the sequence of partial sums converges to f in the norm-topology of A.

The sums on the right hand side of (2.2) whose limit is taken are called *square partial sums*. More general partial sums can be considered in this definition, and will be needed in Section 4.2 below.

Adapting the classical proof ([LT77, Proposition 1.a.2]), is not difficult to see that for each $\alpha \in \mathfrak{A}$, the map $a_\alpha : A \to \mathbb{C}$ assigning to an element $x \in A$ the coefficient c_α of the series (2.2) is a bounded linear functional on A. The collection of functionals $\{a_\alpha : \alpha \in \mathfrak{A}\}$ is called the set of *coefficient functionals dual to the basis* $\{e_\alpha : \alpha \in \mathfrak{A}\}$. In the literature, they are also called *biorthogonal functionals* or *coordinate functionals*.

2.2. **Unique Hahn-Banach extension.** Recall that a normed linear space is said to be *strictly convex*, if for distinct vectors f, g of unit norm, we have $\|f + g\| < 2$, i.e. the midpoint of a chord of the unit sphere of A lies in the open unit ball. We will need the following simple observation:

Proposition 2.3 ([Tay39]). *If L is a Banach space such that its normed dual L' is strictly convex, and $f : A \to \mathbb{C}$ is a bounded linear functional on a subspace $A \subset L$, then f admits a* unique *norm-preserving extension as a linear functional on L.*

Proof. That at least one functional extending f and having the same norm exists is the content of the Hahn-Banach theorem. Without loss of generality, the norm of f as an element of A' is 1. Suppose that f admits two distinct extensions $f_1, f_2 \in L'$ such that $\|f_1\|_{L'} = \|f_2\|_{L'} = 1$. Then $g = \frac{1}{2}(f_1 + f_2)$ is yet another extension of f to an element of L', so $\|g\|_{L'} \geq \|f\|_{A'} = 1$. On the other hand, thanks to the strict convexity of L', we have $\|g\|_{L'} < \frac{1}{2} \cdot 2 = 1$. This contradiction shows that $f_1 = f_2$. □

The examples of unique Hahn-Banach extensions in this paper arise from the following:

Proposition 2.4. *Let $(X, \mathcal{F}, \mu)$ be a measure space, and $1 < p < \infty$. The dual of $L^p(\mu)$ is strictly convex.*

Proof. Since the dual of $L^p(\mu)$ can be isometrically identified with $L^q(\mu)$ where q is the exponent conjugate to p, it suffices to check that $L^q(\mu)$ is strictly convex. Let f, g be distinct elements of $L^q(\mu)$ such that $\|f\|_q = \|g\|_q = 1$. Suppose we have $\|f + g\|_q = 2 = \|f\|_q + \|g\|_q$, so that we have equality in the Minkowski triangle inequality for $L^q(\mu)$. It is well-known that equality occurs in the Minkowski triangle inequality only if $f = cg$ for some $c > 0$. But since $\|f\|_q = \|g\|_q = 1$ this gives that $c = 1$, which is a contradiction since $f \neq g$. Therefore $\|f + g\|_q < 2$ showing that $L^q(\mu)$ is strictly convex. □

Corollary 2.5. *Let A and E be closed linear subspaces of $L^p(\mu)$ where $A \subset E$. Then each bounded linear functional on A admits a unique Hahn-Banach extension to E.*

Proof. Let $f : A \to \mathbb{C}$ be a bounded linear functional. Suppose for a contradiction, that there are two distinct functionals f_1, f_2 on E extending f such that $\|f_1\| = \|f_2\| = \|f\|$. By the Hahn-Banach theorem, there are extensions $\widetilde{f}_1, \widetilde{f}_2$ of f_1 and f_2 respectively to $L^p(\mu)$ such that $\left\|\widetilde{f}_1\right\| = \left\|\widetilde{f}_2\right\| = \|f_1\| = \|f_2\| = \|f\|$. Therefore the functional f on A admits two distinct norm-preserving extensions to all of $L^p(\mu)$ which contradicts Proposition 2.3. □

2.3. **Basis projections.** Let L be Banach space such that its dual is strictly convex, A be a closed subspace, the collection $\{e_\alpha : \alpha \in \mathfrak{A}\}$ a Schauder basis of A in the sense of Definition 2.1, and let $\{a_\alpha : \alpha \in \mathfrak{A}\}$ be the coefficient functionals dual to this Schauder basis. Let $\widetilde{a}_\alpha : L \to \mathbb{C}$ be the unique Hahn-Banach extension of the functional $a_\alpha : A \to \mathbb{C}$, where uniqueness follows by Propositon 2.3.

Definition 2.6. A bounded linear projection operator $\boldsymbol{P}$ from L onto A is called the *basis projection* determined by $\{e_\alpha : \alpha \in \mathfrak{A}\}$, if for each $f \in L$, we have a series representation convergent in the norm of L given by

$$\boldsymbol{P}f = \lim_{N\to\infty} \sum_{\substack{|\alpha|_\infty \le N \\ \alpha\in\mathfrak{A}}} \widetilde{a}_\alpha(f)e_\alpha. \tag{2.7}$$

The following quasi-trivial observation is often useful:

Proposition 2.8. *Let $\{e_\alpha : \alpha \in \mathfrak{A}\}$ be a Schauder basis of A and $\{c_\alpha : \alpha \in \mathfrak{A}\}$ a family of nonzero complex numbers. The "scaled" collection $\{c_\alpha e_\alpha : \alpha \in \mathfrak{A}\}$ is also a Schauder basis of A, and the basis projection $\boldsymbol{P}' : L \to A$ determined by the scaled Schauder basis exists if and only if the basis projection $\boldsymbol{P} : L \to A$ determined by the original Schauder basis exists. When this happens the operators coincide, i.e., $\boldsymbol{P}f = \boldsymbol{P}'f$ for every $f \in L$.*

Proof. The fact that the scaled collection is a Schauder basis follows on rewriting the series expansion of an element $f \in A$ with respect to the Schauder basis:

$$f = \lim_{N\to\infty} \sum_{\substack{|\alpha|_\infty \le N \\ \alpha\in\mathfrak{A}}} a_\alpha(f)e_\alpha = \lim_{N\to\infty} \sum_{\substack{|\alpha|_\infty \le N \\ \alpha\in\mathfrak{A}}} \frac{a_\alpha(f)}{c_\alpha} \cdot (c_\alpha e_\alpha).$$

The statements about the basis projections with respect to the two bases follow on noting that, thanks to the uniqueness of norm-preserving extensions, the Hahn-Banach extension of the functional $\frac{1}{c_\alpha} a_\alpha : A \to \mathbb{C}$ to L is given by $\frac{1}{c_\alpha}\widetilde{a}_\alpha$, where $\widetilde{a}_\alpha$ is the Hahn-Banach extension of a_α as above. □

In view of Proposition 2.8, we could always assume that a Schauder basis is *normalized*, i.e., each basis element has norm 1. In this case the dual coefficient functionals are also easily seen to be normalized. However, in view of the intended application to Bergman spaces, we choose *not* to make this simplification.

2.4. **The Szegő projection on $L^p(\mathbb{T})$ as a basis projection.** Let $1 < p < \infty$, let $L = L^p(\mathbb{T})$, the L^p-space of the circle with the normalized Haar measure $\frac{1}{2\pi}d\theta$, and let $A = H^p(\mathbb{D})$, the Hardy space of the unit disc, the subspace of $L^p(\mathbb{T})$ consisting of those elements of $L^p(\mathbb{T})$ which are boundary values of holomorphic functions in the disc. Let $\tau_\alpha(e^{i\theta}) = e^{i\alpha\theta}$, $\alpha \in \mathbb{Z}$, denote the α-th trigonometric monomial on $\mathbb{T}$. It is well-known that $\{\tau_\alpha : \alpha \geq 0\}$ is a (normalized) Schauder basis of $H^p(\mathbb{D})$, i.e., the partial sums of the Fourier series of a function in $H^p(\mathbb{D})$ converge in the norm $L^p(\mathbb{T})$.

Proposition 2.9. *For $1 < p < \infty$, the basis projection from $L^p(\mathbb{T})$ onto $H^p(\mathbb{D})$ determined by the Schauder basis $\{\tau_\alpha\}_{\alpha=0}^\infty$ exists, and is the Szegő projection.*

Proof. The coefficient functionals on $H^p(\mathbb{D})$ dual to the Schauder basis $\{\tau_\alpha : \alpha \geq 0\}$ are precisely the Fourier coefficient functionals $\{a_\alpha\}_{\alpha=0}^\infty$:

$$a_\alpha(f) = \int_0^{2\pi} f(e^{i\theta})e^{-i\alpha\theta}\frac{d\theta}{2\pi}, \qquad f \in H^p(\mathbb{D}). \tag{2.10}$$

Notice that for $f \in H^p(\mathbb{D})$, we have

$$|a_\alpha(f)| \leq \int_0^{2\pi} \left|f(e^{i\theta})\right| \frac{d\theta}{2\pi} \leq \|f\|_{L^p(\mathbb{T})} \|1\|_{L^q(\mathbb{T})} = \|f\|_{L^p(\mathbb{T})}, \tag{2.11}$$

where q is the Hölder conjugate of p, and we use Hölder's inequality along with the fact that the measure is a probability measure. Therefore $\|a_\alpha\| \leq 1$. But since $\|\tau_\alpha\|_{L^p(\mathbb{T})} = 1$, and $a_\alpha(\tau_\alpha) = 1$, it follows that $\|a_\alpha\| = 1$. We now claim that the Hahn-Banach extension $\widetilde{a}_\alpha : L^p(\mathbb{T}) \to \mathbb{C}$ of the coefficient functional $a_\alpha : H^p(\mathbb{D}) \to \mathbb{C}$ is still the Fourier coefficient functional:

$$\widetilde{a}_\alpha(f) = \int_0^{2\pi} f(e^{i\theta})e^{-i\alpha\theta}\frac{d\theta}{2\pi}, \qquad f \in L^p(\mathbb{T}).$$

Indeed, $\widetilde{a}_\alpha$ is an extension of a_α, and repeating the argument of (2.11) shows $\|\widetilde{a}_\alpha\| = 1$, and thus it is a Hahn-Banach extension. Uniqueness follows from Propositions 2.3 and 2.4.

Let $\boldsymbol{S}$ denote the basis projection from $L^p(\mathbb{T})$ onto $H^p(\mathbb{D})$ and let $f \in L^p(\mathbb{T})$ be a trigonometric polynomial. Then formula (2.7) in this case becomes:

$$\boldsymbol{S}f(e^{i\phi}) = \sum_{\alpha=0}^\infty \left(\int_0^{2\pi} f(e^{i\theta})e^{-i\alpha\theta}\frac{d\theta}{2\pi}\right)e^{i\alpha\phi} = \int_0^{2\pi} \frac{f(e^{i\theta})}{1 - e^{i(\phi-\theta)}} \cdot \frac{d\theta}{2\pi}.$$

This shows that on the trigonometric polynomials, the basis projection coincides with the Szegő projection, which is well-known to be represented by the singular integral at the end of the above chain of equalities. But as the Szegő projection is bounded from $L^p(\mathbb{T})$ onto $H^p(\mathbb{D})$, it follows that the basis projection exists and is equal to the Szegő projection. □

In contrast with the above, it is shown in Corollary 4.3 that the Bergman projection on a Reinhardt domain, if it admits an extension to L^p, $p \neq 2$, is necessarily different from the corresponding basis projection determined by the monomial basis.

2.5. **Invariance properties of the basis projection.** Let L be a Banach space and let $\{e_\alpha\}$ be a Schauder basis of the closed subspace A. If the basis projection determined by $\{e_\alpha\}$ exists, it is determined the metric geometry of L, A and $\{e_\alpha\}$, and is consequently preserved by isometric isomorphism. We note here that basis projections are also preserved by a slightly more general class of maps.

If L, M are Banach spaces, by a *homothetic isomorphism* $\boldsymbol{T} : L \to M$ we mean a bijection such that there is a $C > 0$ satisfying

$$\|\boldsymbol{T}f\|_M = C\|f\|_L, \qquad \text{for every } f \in L. \tag{2.12}$$

Such a map clearly amounts to nothing more than a change of unit of measurement, and thus preserves all metric relations. We record here for future use the following quasi-trivial observation, which is obvious from the previous remark.

Proposition 2.13. *Let $\boldsymbol{T} : L \to M$ be a homothetic isomorphism of Banach spaces, and suppose that the dual of L is strictly convex. Suppose that A is a closed subspace of L and that $\{e_\alpha\}$ is a Schauder basis of A. Then we have the following:*

(1) The dual of M is strictly convex.
(2) $\boldsymbol{T}(A)$ is a closed subspace of M, and $\{\boldsymbol{T}(e_\alpha)\}$ is a Schauder basis of A.
(3) The basis projection $\boldsymbol{P}$ from L to A determined by the basis $\{e_\alpha\}$ exists if and only the basis projection $\boldsymbol{Q}$ from M to $\boldsymbol{T}(A)$ determined by the basis $\{\boldsymbol{T}(e_\alpha)\}$ exists.
(4) If either of the basis projections $\boldsymbol{P}, \boldsymbol{Q}$ exists (and therefore by the previous part, both exist), then the following diagram of Banach spaces and linear operators commutes:

$$\begin{array}{ccc} L & \xrightarrow[\cong]{\boldsymbol{T}} & M \\ \Big\downarrow{\scriptstyle \boldsymbol{P}} & & \Big\downarrow{\scriptstyle \boldsymbol{Q}} \\ A & \xrightarrow[\cong]{\boldsymbol{T}} & \boldsymbol{T}(A). \end{array}$$

2.6. **The Monomial Basis Projection.** On a Reinhardt domain Ω each holomorphic function $f \in \mathcal{O}(\Omega)$ has a unique *Laurent expansion*

$$f = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha e_\alpha, \tag{2.14}$$

where $c_\alpha \in \mathbb{C}$ and the series converges locally normally, i.e., for each compact $K \subset \Omega$, the sum $\sum_\alpha \|c_\alpha e_\alpha\|_K < \infty$, where $\|\cdot\|_K = \sup_K |\cdot|$ is the sup norm (see e.g. [Ran86]). It follows that (2.14) converges uniformly on compact subsets of Ω. Define

$$a_\alpha : \mathcal{O}(\Omega) \to \mathbb{C}, \qquad a_\alpha(f) = c_\alpha \tag{2.15}$$

where c_α is as above in (2.14). The functional a_α is called the α-th *Laurent coefficient functional* of the domain Ω.

The following result shows that the Laurent monomials (under an appropriate ordering) form a Schauder basis of the Bergman space $A^p(\Omega, \lambda)$, where Ω is a Reinhardt domain in $\mathbb{C}^n$ and λ is an admissible multi-radial weight. The unweighted version of this result (the case $\lambda \equiv 1$) was proved in [CEM19], inspired by the case of the disc considered in [Zhu91]. This more general result is proved in exactly the same way, by replacing the implicit weight $\lambda \equiv 1$ in [CEM19, Theorem 3.11] with a general multi-radial weight λ. A key ingredient of the proof is the fact that the Laurent polynomials are dense in $A^p(\Omega, \lambda)$. In [CEM19] this is proved using a duality argument, but there is also an alternative approach based on Cesàro summability of power series (see [CD22, Theorem 2.5].) Recall that the notation and conventions established in Section 1.4 are in force throughout the paper.

Theorem 2.16. *The collection of Laurent monomials $\{e_\alpha : \alpha \in \mathcal{S}_p(\Omega, \lambda)\}$ forms a Schauder basis of the Bergman space $A^p(\Omega, \lambda)$. The functionals dual to this basis are the coefficient functionals $\{a_\alpha : \alpha \in \mathcal{S}_p(\Omega, \lambda)\}$, and the norm of $a_\alpha : A^p(\Omega, \lambda) \to \mathbb{C}$ is given by*

$$\|a_\alpha\|_{A^p(\Omega,\lambda)'} = \frac{1}{\|e_\alpha\|_{p,\lambda}}. \tag{2.17}$$

Thus, if $f \in A^p(\Omega, \lambda)$, the Laurent series of f written as $\sum_{\alpha \in \mathbb{Z}^n} a_\alpha(f) e_\alpha$ consists only of terms corresponding to monomials $e_\alpha \in A^p(\Omega, \lambda)$, i.e., if $\alpha \notin \mathcal{S}_p(\Omega, \lambda)$, then $a_\alpha(f) = 0$.

In Theorem 2.16 (see also the proof in [CEM19, Theorem 3.11]), square partial sums arise in the following way: the uniform boundedness of the square partial sums in the operator norm of L^p is reduced to the boundedness of partial sums of Fourier series on functions in $L^p(\mathbb{T}^n)$ of the n-dimensional torus. Square partial sums are the simplest choice of partial sums under which such uniform boundedness in the L^p-operator norm can be proved, but it must be emphasized that uniform boundedness in L^p still holds for other partial orderings of monomials. This crucial point is expounded upon in Section 4.2.

We are ready to formally define the main object of this paper:

Definition 2.18. A bounded linear projection $\boldsymbol{P}^{\Omega}_{p,\lambda}$ from $L^p(\Omega,\lambda)$ onto $A^p(\Omega,\lambda)$ is called the *Monomial Basis Projection* of $A^p(\Omega,\lambda)$, if for $f \in L^p(\Omega,\lambda)$ it admits the series representation convergent in the norm of $L^p(\Omega,\lambda)$ given by

$$\boldsymbol{P}^{\Omega}_{p,\lambda}(f) = \lim_{N\to\infty} \sum_{\substack{|\alpha|_\infty \le N \\ \alpha\in\mathcal{S}_p(\Omega,\lambda)}} \widetilde{a}_\alpha(f) e_\alpha, \tag{2.19}$$

where $\widetilde{a}_\alpha : L^p(\Omega,\lambda) \to \mathbb{C}$ is the unique Hahn-Banach extension of the coefficient functional $a_\alpha : A^p(\Omega,\lambda) \to \mathbb{C}$.

Therefore the Monomial Basis Projection is the basis projection of Definition 2.6 determined by the monomial Schauder basis $\{e_\alpha : \alpha \in \mathcal{S}_p(\Omega,\lambda)\}$.

3. The monomial basis kernel

In this section, we introduce the Monomial Basis Kernel, prove its existence on pseudoconvex Reinhardt domains and establish some of its properties. It will be shown in Section 4 that this kernel is the integral kernel representing the Monomial Basis Projection.

3.1. Properties of the twisting map.

We begin with a list of properties of the twisting map χ_p introduced in (1.8). Recall that $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

(1) *The map χ_p is a homeomorphism of $\mathbb{C}^n$ with itself, and its inverse is the map χ_q.*

Proof. Notice that if $w = \chi_p(z)$, then for each j we have

$$w_j\,|w_j|^{q-2} = \left(z_j\,|z_j|^{p-2}\right)\cdot \left|z_j|z_j|^{p-2}\right|^{q-2} = z_j\,|z_j|^{p-2+(p-1)(q-2)} = z_j,$$

since $p-2+(p-1)(q-2) = pq-p-q = 0$. So $\chi_q \circ \chi_p$ is the identity, and similarly $\chi_p \circ \chi_q$ is also the identity. □

(2) *The map χ_p is a diffeomorphism away from the union of the coordinate hyperplanes $\{z_j = 0\}$, $1 \le j \le n$. The Jacobian determinant of χ_p (as a mapping of the real vector space $\mathbb{C}^n$) is given by*

$$\eta_p(\zeta) = \det(D\chi_p) = (p-1)^n\,|\zeta_1\cdot\dots\cdot\zeta_n|^{2p-4}\,. \tag{3.1}$$

Proof. This follows from direct computation. □

(3) *Let $\Omega \subset \mathbb{C}^n$ be a Reinhardt domain. Then χ_p restricts to a homeomorphism*

$$\chi_p : \Omega \to \Omega^{(p-1)}, \tag{3.2}$$

where $\Omega^{(p-1)}$ is a Reinhardt power of Ω as in (3.9) below, which in this case becomes

$$\Omega^{(p-1)} = \left\{ z \in \mathbb{C}^n : \left(|z_1|^{\frac{1}{p-1}},\dots,|z_n|^{\frac{1}{p-1}}\right) \in \Omega \right\}.$$

Proof. Notice that in each coordinate, the map $z \mapsto z\,|z|^{p-2}$ is represented in polar coordinates as $re^{i\theta} \mapsto r^{p-1}e^{i\theta}$. The claim now follows from the definition of $\Omega^{(p-1)}$ above. Notice also that from item (1) above, the map

$$\chi_q : \Omega^{(p-1)} \to \Omega. \tag{3.3}$$

gives the inverse homeomorphism. □

3.2. **Existence of the Monomial Basis Kernel.** Recall the definition of the MBK as a formally defined series given by (1.10). Since

$$\begin{aligned}\chi_p^* e_\alpha(w) = e_\alpha\left(w_1 |w_1|^{p-2}, \dots, w_n |w_n|^{p-2}\right) &= (w_1 |w_1|^{p-2})^{\alpha_1} \dots (w_n |w_n|^{p-2})^{\alpha_n} \\ &= (w_1^{\alpha_1} \cdots w_n^{\alpha_n}) |w_1^{\alpha_1} \cdots w_n^{\alpha_n}|^{p-2} \\ &= e_\alpha(w) |e_\alpha(w)|^{p-2}, \end{aligned} \tag{3.4}$$

we can alternatively express the MBK as the series

$$K^\Omega_{p,\lambda}(z,w) = \sum_{\alpha \in \mathcal{S}_p(\Omega,\lambda)} \frac{e_\alpha(z)\overline{e_\alpha(w)}\, |e_\alpha(w)|^{p-2}}{\|e_\alpha\|^p_{p,\lambda}}. \tag{3.5}$$

We now give a sufficient condition for the convergence of this series.

Theorem 3.6. *Let Ω be a* pseudoconvex *Reinhardt domain in $\mathbb{C}^n$ and λ be an admissible multi-radial weight function on Ω. The series* (3.5) *defining $K^\Omega_{p,\lambda}(z,w)$ converges locally normally on $\Omega \times \Omega$.*

We need two lemmas for the proof of the above result. The first may itself be considered an analog for Laurent series of the well-known Abel's lemma on the domain of convergence of a Taylor series ([Ran86, p. 14]):

Lemma 3.7. *Let $\Omega \subset \mathbb{C}^n$ be a Reinhardt domain, define $\mathcal{S}(\Omega) = \{\alpha \in \mathbb{Z}^n : e_\alpha \in \mathcal{O}(\Omega)\}$, and for coefficients $a_\alpha \in \mathbb{C}$, $\alpha \in \mathcal{S}(\Omega)$, let*

$$\sum_{\alpha \in \mathcal{S}(\Omega)} a_\alpha e_\alpha \tag{3.8}$$

be a formal Laurent series on Ω. Suppose that for each $z \in \Omega$ there is a $C > 0$ such that for each $\alpha \in \mathcal{S}(\Omega)$ we have $|a_\alpha e_\alpha(z)| \le C$. Then the series (3.8) *converges locally normally on Ω.*

Proof. See Lemma 1.6.3 and Proposition 1.6.5 of [JP08, Section 1.6]. $\square$

Given a Reinhardt domain $\Omega \subset \mathbb{C}^n$ and a number $m > 0$, define the *m-th Reinhardt power* of Ω to be the Reinhardt domain given by

$$\Omega^{(m)} = \left\{ z \in \mathbb{C}^n : \left(|z_1|^{\frac{1}{m}}, \dots, |z_n|^{\frac{1}{m}}\right) \in \Omega \right\}, \tag{3.9}$$

a particular case of which we saw in (3.2) and (3.3) above. If Ω is also assumed to be pseudoconvex, then for each $m > 0$ the domain $\Omega^{(m)}$ is also pseudoconvex. Indeed, recall first that the logarithmic shadow of Ω is the subset $\log(\Omega)$ of $\mathbb{R}^n$ given by

$$\log(\Omega) = \{(\log|z_1|, \dots, \log|z_n|) : z \in \Omega\}. \tag{3.10}$$

Recall also that Ω is pseudoconvex if and only if the set $\log(\Omega)$ is convex, and Ω is "weakly relatively complete" ([JP08, Theorem 1.11.13 and Proposition 1.11.6]). But we easily see that the condition of weak relative completeness is preserved by the construction of Reinhardt powers, and

$$\log\left(\Omega^{(m)}\right) = \{(m\log|z_1|, \dots, m\log|z_n|) : z \in \Omega\} = m\log(\Omega)$$

is itself convex, if $\log(\Omega)$ is convex. So $\Omega^{(m)}$ is pseudoconvex if and only if Ω is pseudoconvex.

The second result needed in the proof of Theorem 3.6 is the following:

Lemma 3.11. *Let A be a Banach space of holomorphic functions on Ω and suppose that for each $z \in \Omega$ the evaluation functional $\phi_z : A \to \mathbb{C}$ given by $\phi_z(f) = f(z)$ for $f \in A$ is continuous. Then for $m > 0$, the following series converges locally normally on $\Omega^{(m)}$:*

$$\sum_{\substack{\alpha\in\mathbb{Z}^n \\ e_\alpha \in A}} \frac{e_\alpha}{\|e_\alpha\|_A^m}.$$

Proof. Let $z \in \Omega^{(m)}$ so that there is $\zeta \in \Omega$ such that $|z_j| = |\zeta_j|^m$ for each j. If $\phi_\zeta : A \to \mathbb{C}$ is the evaluation functional, there is a constant $C > 0$ such that $|\phi_\zeta(f)| \le C \|f\|_A$ for each $f \in A$. Then for each $\alpha \in \mathbb{Z}^n$ such that $e_\alpha \in A$ we have

$$\frac{|e_\alpha(z)|}{\|e_\alpha\|_A^m} = \left(\frac{|e_\alpha(\zeta)|}{\|e_\alpha\|_A}\right)^m = \left(\frac{\phi_\zeta(e_\alpha)}{\|e_\alpha\|_A}\right)^m \le C^m.$$

The result now follows by Lemma 3.7. □

Proof of Theorem 3.6. Let $t_j = z_j \overline{w_j} |w_j|^{p-2}$, $1 \le j \le n$, and $t = (t_1, \dots, t_n)$. Then the series for the MBK given in (3.5) assumes the form

$$K^\Omega_{p,\lambda}(z,w) = \sum_{\alpha\in\mathcal{S}_p(\Omega,\lambda)} \frac{t^\alpha}{\|e_\alpha\|^p_{p,\lambda}}. \tag{3.12}$$

Since Bergman's inequality (1.5) holds by definition for admissible weights, the point evaluations are bounded on $A^p(\Omega,\lambda)$. Lemma 3.11 therefore guarantees the series in (3.12) above converges locally normally on $\Omega^{(p)}$ defined in (3.9). It thus suffices to show that the image of the map $\Omega \times \Omega \to \mathbb{C}^n$ given by

$$(z,w) \longmapsto (t_1, \dots, t_n)$$

coincides with $\Omega^{(p)}$, since then the image of a compact set $K \subset \Omega \times \Omega$ is a compact subset of $\Omega^{(p)}$, on which the series (3.12) is known to converge normally.

Now consider the logarithmic shadow $\log(\Omega \times \Omega) = \log(\Omega) \times \log(\Omega)$ defined in (3.10). Thanks to the log-convexity of pseudoconvex Reinhardt domains, what we want to prove is equivalent to saying that the map from $\log(\Omega) \times \log(\Omega) \to \mathbb{R}^n$ given by

$$(\xi, \eta) \longmapsto \xi + (p-1)\eta \tag{3.13}$$

has image exactly $p\log(\Omega) = \{p\theta : \theta \in \log(\Omega)\} = \log\left(\Omega^{(p)}\right)$. But since $\log(\Omega)$ is convex, the map on $\log(\Omega) \times \log(\Omega)$ given by

$$(\xi, \eta) \longmapsto \tfrac{1}{p}\xi + \left(1 - \tfrac{1}{p}\right)\eta$$

has image contained in $\log(\Omega)$. Taking $\xi = \eta$ we see that the image is exactly $\log(\Omega)$. Therefore the image of (3.13) is precisely $p\log(\Omega)$ and we have proved that the series (3.5) converges locally normally on $\Omega \times \Omega$. □

3.3. **Admissible weights with singularities along the axes.** Recall the notion of an admissible weight from Section 1.4, item (3). While positive continuous functions λ are always admissible, we also encounter multi-radial weights which may vanish or blow up along the axes, which we show are also admissible. Let $Z \subset \mathbb{C}^n$ denote the union of the coodinate hyperplanes

$$Z = \{z \in \mathbb{C}^n : z_j = 0 \text{ for some } 1 \le j \le n\}.$$

Proposition 3.14. *Let U be a domain in $\mathbb{C}^n$ and let $U^* = U \setminus Z$. Suppose that $\lambda : U \to [0,\infty]$ is a measurable function on U such that the restriction $\lambda|_{U^*}$ is an admissible weight on U^*. Then λ is an admissible weight on U.*

Proof. Assume that $U\cap Z \neq \varnothing$, since otherwise there is nothing to show, and set $\lambda^* = \lambda|_{U^*}$. If $f \in A^p(U,\lambda)$, then since λ^* is admissible on U^*, if a compact K is contained in U^*, there exists a $C_K > 0$ such that

$$\sup_K |f| \leq C_K \|f\|_{A^p(U^*,\lambda^*)} = C_K \|f\|_{A^p(U,\lambda)}.$$

To complete the proof, we need to show that for each $\zeta \in U \cap Z$, there is a compact neighborhood K of ζ in U such that (1.5) holds for each $f \in A^p(U,\lambda)$. Now, there is a polydisc centered at ζ

$$P = \{z \in \mathbb{C}^n : |z_j - \zeta_j| < r,\, 1 \leq j \leq n\}$$

such that the closure $\overline{P}$ is contained in U. We can assume further that the radius $r > 0$ is chosen so that it is distinct from each of the nonnegative numbers $|\zeta_j|$, $1 \leq j \leq n$. Then the "distinguished boundary"

$$T = \{z \in \mathbb{C}^n : |z_j - \zeta_j| = r,\, 1 \leq j \leq n\}$$

of this polydisc satisfies the condition that $T \subset U^*$. Therefore for each $f \in \mathcal{O}(U)$ and each $w \in P$, we have the Cauchy representation:

$$f(w) = \frac{1}{(2\pi i)^n} \int_T \frac{f(z_1,\dots,z_n)}{(z_1 - w_1)\dots(z_n - w_n)}\, dz_1 \dots dz_n \tag{3.15}$$

where the integral is an n-times repeated contour integral on T (which is the product of n circles). Now suppose that K is a compact subset of P containing the center ζ, and let $\rho > 0$ be such that $|z_j - w_j| \geq \rho$ for each $z \in T$ and $w \in K$. Then for $w \in K$, a sup-norm estimate on (3.15) gives

$$|f(w)| \leq \frac{1}{(2\pi)^n} \cdot \frac{\sup_T |f|}{\rho^n} (2\pi r)^n \leq \left(\frac{r}{\rho}\right)^n \cdot \|f\|_{A^p(U^*,\lambda^*)} = \left(\frac{r}{\rho}\right)^n \cdot \|f\|_{A^p(U,\lambda)}$$

where we used the fact that λ^* is admissible on U^*. The result follows. ☐

3.4. **Properties of the Monomial Basis Kernel.** The MBK of the space $A^p(\Omega,\lambda)$ coincides with the Bergman kernel B^Ω_λ with the same weight when $p = 2$, and for other p it enjoys many properties analogous to that of the Bergman kernel, up to an appearance of the twisting function χ_p of (1.8). This is not surprising given the defining series (1.10).

(1) The Bergman kernel is holomorphic in the first slot, i.e., for $w \in \Omega$, $z \mapsto B^\Omega_\lambda(z,w)$ is holomorphic on Ω. Similarly, for $w \in \Omega$, the function $z \mapsto K^\Omega_{p,\lambda}(z,w)$ is holomorphic on Ω, since by Theorem 3.6, for each compact $L \subset \Omega$ the series (1.10) converges uniformly on the set $L \times \{w\}$, and the partial sums are clearly holomorphic on Ω.

(2) The Bergman kernel is anti-holomorphic in the second slot, i.e., for $z \in \Omega$, the function $w \mapsto \overline{B^\Omega_\lambda(z,w)}$ is holomorphic. There is a corresponding "twisted" property of the MBK: There is a function $\widetilde{K}^\Omega_{p,\lambda} : \Omega \times \Omega^{(p-1)} \to \mathbb{C}$, holomorphic in the first slot and anti-holomorphic in the second slot, such that

$$K^\Omega_{p,\lambda}(z,w) = \widetilde{K}^\Omega_{p,\lambda}(z, \chi_q(w)), \quad z,w \in \Omega.$$

Here the Reinhardt domain $\Omega^{(p-1)}$ is the Reinhardt power as in (3.9). In fact,

$$\widetilde{K}^\Omega_{p,\lambda}(z,\zeta) = \sum_{\alpha \in \mathcal{S}_p(\Omega,\lambda)} \frac{e_\alpha(z)\cdot \overline{e_\alpha(\zeta)}}{\|e_\alpha\|^p_{p,\lambda}}, \qquad z \in \Omega, \quad \zeta \in \Omega^{(p-1)},$$

and the assertion follows from Theorem 3.6 and the fact that (3.2) is a homeomorphism.

For $z \in \Omega$, the function $B_\lambda^\Omega(z,\cdot) \in L^2(\Omega,\lambda)$. When the MBP of $A^p(\Omega,\lambda)$ exists, Theorem 4.1 shows that $K_{p,\lambda}^\Omega(z,\cdot) \in L^q(\Omega,\lambda)$. For $w \in \Omega$, the function $B_\lambda^\Omega(\cdot,w) \in A^2(\Omega,\lambda)$. One can show that $K_{p,\lambda}^\Omega(\cdot,w)$ belongs to $A^p(\Omega,\lambda)$ whenever the MBP is absolutely bounded.

(3) The following generalizes the conjugate symmetry property $B_\lambda^\Omega(z,w) = \overline{B_\lambda^\Omega(w,z)}$ of the Bergman kernel to the MBK: Suppose the weight λ on Ω is continuous and positive on Ω. Then the weight $\eta_q \cdot \chi_q^*\lambda$, where $\eta_q = \det(D\chi_q)$ is as in (3.1), is admissible on $\Omega^{(p-1)}$ and

$$K_{p,\lambda}^\Omega(\chi_q(z),w) = \overline{K_{q,\eta_q\cdot\chi_q^*\lambda}^{\Omega^{(p-1)}}(\chi_p(w),z)}, \qquad z \in \Omega^{(p-1)},\ w \in \Omega. \tag{3.16}$$

When $\lambda \equiv 1$ this becomes

$$K_{p,1}^\Omega(\chi_q(z),w) = \overline{K_{q,\eta_q}^{\Omega^{(p-1)}}(\chi_p(w),z)}, \qquad z \in \Omega^{(p-1)},\ w \in \Omega.$$

In other words, combining the switching of variables with the twisting map χ_p, we still have a form of conjugate symmetry of $K_{p,\lambda}^\Omega$, though on the right hand side we do have the MBK of a different domain $\Omega^{(p-1)}$ with a different weight $\eta_q \cdot \chi_q^*\lambda$.

Proof. Since λ is positive and continuous and η_q and χ_q are given by the formulas in (3.1) and (1.8), respectively, the weight $\eta_q \cdot \chi_q^*\lambda$ on $\Omega^{(p-1)}$ is easily seen to be continuous and strictly positive away from the axial hyperplanes. Thus, by Proposition 3.14 above, it is admissible. To see (3.16), we first note that (see (3.4) above):

$$\left|\chi_q^* e_\alpha(\zeta)\right|^p = |e_\alpha(\chi_q(\zeta))|^p = |e_\alpha(\zeta)|^{(q-1)p} = |e_\alpha(\zeta)|^q,$$

and therefore using χ_q to change of variables, we have

$$\begin{aligned}\|e_\alpha\|_{L^p(\Omega,\lambda)}^p = \int_\Omega |e_\alpha|^p\,\lambda\,dV &= \int_{\Omega^{(p-1)}} |e_\alpha(\chi_q(\zeta))|^p\,\eta_q(\zeta)\lambda(\chi_q(\zeta))\,dV(\zeta)\\ &= \|e_\alpha\|_{L^q(\Omega^{(p-1)},\eta_q\cdot\chi_q^*\lambda)}^q,\end{aligned}$$

which in particular shows the equality of the sets $\mathcal{S}_p(\Omega,\lambda) = \mathcal{S}_q(\Omega^{(p-1)},\eta_q\cdot\chi_q^*\lambda)$ of allowable indices. Therefore, from (1.10), we have for $z \in \Omega^{(p-1)}, w \in \Omega$:

$$\begin{aligned}K_{p,\lambda}^\Omega(\chi_q(z),w) &= \sum_{\alpha\in\mathcal{S}_p(\Omega,\lambda)} \frac{e_\alpha(\chi_q(z))\overline{\chi_p^* e_\alpha(w)}}{\|e_\alpha\|_{L^p(\Omega,\lambda)}^p}\\ &= \sum_{\alpha\in\mathcal{S}_q(\Omega^{(p-1)},\eta_q\cdot\chi_q^*\lambda)} \overline{\frac{e_\alpha(\chi_p(w))\overline{\chi_q^* e_\alpha(z)}}{\|e_\alpha\|_{L^q(\Omega^{(p-1)},\eta_q\cdot\chi_q^*\lambda)}^q}} = \overline{K_{q,\eta_q\cdot\chi_q^*\lambda}^{\Omega^{(p-1)}}(\chi_p(w),z)}.\end{aligned}$$

□

(4) The function $z \mapsto \log K_{p,\lambda}^\Omega(z,z)$ is plurisubharmonic on Ω, generalizing the well-known property of the Bergman kernel.

Proof. Let ℓ^p denote the Banach space of p-th power summable functions on the set of indices $\mathcal{S}_p(\Omega,\lambda)$, so that

$$\ell^p = \left\{c : \mathcal{S}_p(\Omega,\lambda) \to \mathbb{C} : \|c\|_{\ell^p}^p = \sum |c(\alpha)|^p < \infty\right\},$$

where the sum is over $\alpha \in \mathcal{S}_p(\Omega,\lambda)$. For $z \in \Omega$, consider the map $\nu(z) : \mathcal{S}_p(\Omega,\lambda) \to \mathbb{C}$ defined by

$$\nu(z)(\alpha) = \frac{e_\alpha(z)}{\|e_\alpha\|_{p,\lambda}}, \qquad \text{for } \alpha \in \mathcal{S}_p(\Omega,\lambda).$$

We claim that $\nu(z) \in \ell^p$ for each $z \in \Omega$. Indeed, the definition of the MBK in (1.10) and Theorem 3.6 together show for $z \in \Omega$ that

$$\sum_\alpha \frac{|e_\alpha(z)|^p}{\|e_\alpha\|^p_{p,\lambda}} = K^\Omega_{p,\lambda}(z,z) < \infty. \tag{3.17}$$

We therefore have a map $\nu : \Omega \to \ell^p$, which we claim is a Banach-valued holomorphic function. Let $\varphi : \ell^p \to \mathbb{C}$ be a bounded linear functional, meaning there is a $\mu \in \ell^q$ such that for $z \in \Omega$ we have $\varphi(\nu(z)) = \sum_\alpha \frac{e_\alpha(z)\overline{\mu(\alpha)}}{\|e_\alpha\|^p_{p,\lambda}}$. Summing over $\alpha \in \mathcal{S}_p(\Omega,\lambda)$, we have

$$\left| \sum_\alpha \frac{e_\alpha(z)\overline{\mu(\alpha)}}{\|e_\alpha\|^p_{p,\lambda}} \right| \le \sum_\alpha \frac{|e_\alpha(z)|\,|\mu(\alpha)|}{\|e_\alpha\|^p_{p,\lambda}} \le \|\varphi\|_{\mathrm{op}} \|\nu(z)\|_{\ell^p},$$

so by Lemma 3.7 the function $z \mapsto \varphi(\nu(z))$ is represented by convergent Laurent series and is therefore holomorphic. Therefore ν is itself holomorphic as an ℓ^p-valued map (see, e.g., [Muj86, Theorem 8.12]). But if ν is a Banach-valued holomorphic function, the function $\log\|\nu\|$ is known to be plurisubharmonic (see, e.g., [Muj86, Corollary 34.10]). By the definition of ν and (3.17), we see that

$$\log \|\nu(z)\|_{\ell^p} = \log \left(\sum_\alpha \frac{|e_\alpha(z)|^p}{\|e_\alpha\|^p_{p,\lambda}} \right)^{\frac{1}{p}} = \frac{1}{p} \log K^\Omega_{p,\lambda}(z,z),$$

and the result follows. □

(5) The MBK of a product Reinhardt domain can be obtained from the MBKs of its factors in the same way as in the case of the Bergman kernel. Suppose that $n = n' + n''$, and the domain Ω can be represented as a product $\Omega = \Omega' \times \Omega''$, where Ω' and Ω'' are Reinhardt domains in $\mathbb{C}^{n'}$ and $\mathbb{C}^{n''}$, respectively. Suppose further that the weight λ has a tensor product representation as $\lambda(z) = \lambda'(z')\lambda''(z'')$, where λ, λ'' are admissible multi-radial weight functions on Ω' and Ω'', respectively. Then λ is easily seen to be admissible, and the MBK of $A^p(\Omega,\lambda)$ factors into the product of the MBKs of $A^p(\Omega',\lambda')$ and $A^p(\Omega'',\lambda'')$:

$$K^\Omega_{p,\lambda}(z,w) = K^{\Omega'}_{p,\lambda'}(z',w')K^{\Omega''}_{p,\lambda''}(z'',w''). \tag{3.18}$$

Formula (3.18) follows from (1.10) by decomposing $\alpha = (\alpha',\alpha'')$, where $\alpha \in \mathbb{Z}^n$, $\alpha' \in \mathbb{Z}^{n'}$, $\alpha'' \in \mathbb{Z}^{n''}$, and $z = (z',z'')$, where $z \in \mathbb{C}^n$, $z' \in \mathbb{C}^{n'}$, $z'' \in \mathbb{C}^{n''}$, and noticing that

$$e_\alpha(z) = e_{\alpha'}(z')e_{\alpha''}(z''), \tag{3.19a}$$

$$e_\alpha(\chi_p(w)) = e_{\alpha'}(\chi_p(w'))e_{\alpha''}(\chi_p(w'')), \tag{3.19b}$$

$$\|e_\alpha\|_{L^p(\Omega,\lambda)} = \|e_{\alpha'}\|_{L^p(\Omega',\lambda')} \|e_{\alpha''}\|_{L^p(\Omega'',\lambda'')}. \tag{3.19c}$$

4. Integral representation of the Monomial Basis Projection

In this section we prove that the MBP admits an integral representation by the MBK via (4.2). Recall that the MBK of $A^p(\Omega,\lambda)$ is guaranteed to exist by Theorem 3.6 when Ω is a pseudoconvex Reinhardt domain and λ is an admissible multi-radial weight. In Proposition 4.10 it is shown that a necessary and sufficient condition for the existence of the MBP is that the integral operator on the right hand side of (4.2) admits estimates in $L^p(\Omega,\lambda)$. This representation gives a more tractable way of dealing with the MBP, which was originally defined as the limit of partial sums in (2.19).

General Hypothesis for Section 4. *We continue to follow the conventions and the notation of Section 1.4. In particular, $\Omega \subset \mathbb{C}^n$ is a pseudoconvex Reinhardt domain, λ an admissible multi-radial weight on Ω, $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, etc.*

Theorem 4.1. *If the Monomial Basis Projection $\boldsymbol{P}^{\Omega}_{p,\lambda} : L^p(\Omega, \lambda) \to A^p(\Omega, \lambda)$ exists, then*

$$\boldsymbol{P}^{\Omega}_{p,\lambda}(f)(z) = \int_{\Omega} K^{\Omega}_{p,\lambda}(z, w) f(w)\lambda(w) dV(w), \qquad f \in L^p(\Omega, \lambda), \tag{4.2}$$

and for each $z \in \Omega$, we have $K^{\Omega}_{p,\lambda}(z, \cdot) \in L^q(\Omega, \lambda)$.

When $p = 2$, this is simply the representation of the Bergman projection $\boldsymbol{B}^{\Omega}_{\lambda}$ of $A^2(\Omega, \lambda)$ by its Bergman kernel. But unlike this special case, the existence of the MBP of $A^p(\Omega, \lambda)$ for $p \neq 2$ is not guaranteed by abstract Hilbert-space theory. We note a related consequence of Theorem 4.1, which should be contrasted with Proposition 2.9:

Corollary 4.3. *Suppose the Bergman projection $\boldsymbol{B}^{\Omega}_{\lambda} : L^2(\Omega, \lambda) \to A^2(\Omega, \lambda)$ extends by continuity to a bounded operator $\boldsymbol{B}^{\Omega}_{p,\lambda} : L^p(\Omega, \lambda) \to A^p(\Omega, \lambda)$, $p \neq 2$. The extension $\boldsymbol{B}^{\Omega}_{p,\lambda}$ is not the basis projection determined by the monomial Schauder basis $\{e_\alpha : \alpha \in \mathcal{S}_p(\Omega, \lambda)\}$.*

Proof. This is immediate, since the operator $\boldsymbol{B}^{\Omega}_{p,\lambda}$ is still given by the formula

$$\boldsymbol{B}^{\Omega}_{p,\lambda}(f)(z) = \int_{\Omega} B^{\Omega}_{\lambda}(z, w) f(w)\lambda(w) dV(w),$$

where B^{Ω}_{λ} is the weighted Bergman kernel. This kernel is distinct from the MBK given by formula (1.10) except when $p = 2$. □

4.1. **Hahn-Banach extensions and integral formulas.** By Proposition 2.4, the dual of $L^p(\Omega, \lambda)$ is strictly convex. Proposition 2.3 thus guarantees that each coefficient functional in the set $\{a_\alpha : \alpha \in \mathcal{S}_p(\Omega, \lambda)\}$ dual to the monomial Schauder basis $\{e_\alpha : \alpha \in \mathcal{S}_p(\Omega, \lambda)\}$ has a unique Hahn-Banach extension to a functional $\widetilde{a}_\alpha : L^p(\Omega, \lambda) \to \mathbb{C}$. We begin by identifying this extension:

Proposition 4.4. *For $\alpha \in \mathcal{S}_p(\Omega, \lambda)$, let g_α be the function defined on Ω by*

$$g_\alpha = \frac{\chi_p^* e_\alpha}{\|e_\alpha\|^p_{p,\lambda}} = \frac{e_\alpha \, |e_\alpha|^{p-2}}{\|e_\alpha\|^p_{p,\lambda}}. \tag{4.5}$$

Then the unique Hahn-Banach extension $\widetilde{a}_\alpha : L^p(\Omega, \lambda) \to \mathbb{C}$ of the coefficient functional $a_\alpha : A^p(\Omega, \lambda) \to \mathbb{C}$ is given by

$$\widetilde{a}_\alpha(f) = \int_{\Omega} f \cdot \overline{g_\alpha} \, \lambda dV, \qquad f \in L^p(\Omega, \lambda). \tag{4.6}$$

Proof. First we compute the norm of g_α in $L^q(\Omega, \lambda)$:

$$\|g_\alpha\|^q_{q,\lambda} = \frac{1}{\|e_\alpha\|^{pq}_{p,\lambda}} \int_{\Omega} |e_\alpha|^{(p-1)q} \, \lambda \, dV = \frac{1}{\|e_\alpha\|^{pq}_{p,\lambda}} \|e_\alpha\|^p_{p,\lambda} = \frac{1}{\|e_\alpha\|^{pq-p}_{p,\lambda}} = \frac{1}{\|e_\alpha\|^q_{p,\lambda}}.$$

It follows that $g_\alpha \in L^q(\Omega, \lambda)$ and the linear functional in (4.6) satisfies $\widetilde{a}_\alpha \in L^p(\Omega, \lambda)'$, and its norm is given by

$$\|\widetilde{a}_\alpha\|_{L^p(\Omega,\lambda)'} = \|g_\alpha\|_{q,\lambda} = \frac{1}{\|e_\alpha\|_{p,\lambda}}. \tag{4.7}$$

By (2.17), we have $\|a_\alpha\|_{A^p(\Omega,\lambda)'} = \|\widetilde{a}_\alpha\|_{L^p(\Omega,\lambda)'}$. To complete the proof it remains to show that $\widetilde{a}_\alpha$ is an extension of a_α.

By Theorem 2.16, the linear span of $\{e_\beta : \beta \in \mathcal{S}_p(\Omega, \lambda)\}$ is dense in $A^p(\Omega, \lambda)$. Therefore we only need to show that for each $\beta \in \mathcal{S}_p(\Omega, \lambda)$, we have $\widetilde{a}_\alpha(e_\beta) = a_\alpha(e_\beta)$. Since λ is multi-radial, there is a function ℓ on the Reinhardt shadow $|\Omega|$ such that $\lambda(z) = \ell(|z_1|, \dots, |z_n|)$.

And since $g_\alpha \in L^q(\Omega, \lambda)$ and $e_\beta \in L^p(\Omega, \lambda)$, the product $e_\beta \overline{g_\alpha} \in L^1(\Omega, \lambda)$ and Fubini's theorem therefore implies

$$\int_\Omega e_\beta \overline{g_\alpha} \lambda dV = \frac{1}{\|e_\alpha\|_{p,\lambda}^p} \int_{|\Omega|} r^\beta (r^\alpha)^{p-1} \left(\int_{\mathbb{T}^n} e^{i\langle \beta - \alpha, \theta \rangle} d\theta \right) r_1 r_2 \dots r_n \ell dr_1 \dots dr_n, \tag{4.8}$$

where $d\theta = d\theta_1 \dots d\theta_n$ is the natural volume element of the unit torus $\mathbb{T}^n$. First suppose that $\beta \neq \alpha$, so that the integral over $\mathbb{T}^n$ on the right hand side of (4.8) vanishes. Then we have $\int_\Omega e_\beta \overline{g_\alpha} \lambda dV = 0 = a_\alpha(e_\beta)$. If $\beta = \alpha$, (4.8) gives

$$\begin{aligned}
\int_\Omega e_\alpha \overline{g_\alpha} \lambda dV &= \frac{1}{\|e_\alpha\|_{p,\lambda}^p} \int_{|\Omega|} r^\beta (r^\alpha)^{p-1} \left(\int_{\mathbb{T}^n} d\theta \right) r_1 r_2 \dots r_n \ell dr_1 \dots dr_n \\
&= \frac{1}{\|e_\alpha\|_{p,\lambda}^p} \cdot (2\pi)^n \int_{|\Omega|} (r^\alpha)^p r_1 r_2 \dots r_n \ell dr_1 \dots dr_n \\
&= \frac{1}{\|e_\alpha\|_{p,\lambda}^p} \cdot \|e_\alpha\|_{p,\lambda}^p = 1 = a_\alpha(e_\alpha).
\end{aligned}$$

It follows that $\widetilde{a}_\alpha$ is a norm preserving extension of a_α. Since this extension is unique, the result follows. □

Observe that by combining (3.4) and (4.5), the MBK of $A^p(\Omega, \lambda)$ can be written as

$$K_{p,\lambda}^\Omega(z, w) = \sum_{\alpha \in \mathcal{S}_p(\Omega, \lambda)} e_\alpha(z) \overline{g_\alpha(w)}. \tag{4.9}$$

We now establish our necessary and sufficient condition for the existence of the MBP:

Proposition 4.10. *Define an integral operator on $C_c(\Omega)$ by*

$$\boldsymbol{Q} f(z) = \int_\Omega K_{p,\lambda}^\Omega(z, w) f(w) \lambda(w) dV(w), \qquad f \in C_c(\Omega). \tag{4.11}$$

The MBP of $A^p(\Omega, \lambda)$ exists if and only if $\boldsymbol{Q}$ satisfies L^p-estimates, i.e., there is a constant $C > 0$ such that for each $f \in C_c(\Omega)$ we have the inequality

$$\|\boldsymbol{Q} f\|_{p,\lambda} \leq C \|f\|_{p,\lambda}. \tag{4.12}$$

Proof. Recall that $\Omega \subset \mathbb{C}^n$ is a pseudoconvex Reinhardt domain and λ is an admissible multi-radial weight. The function $K_{p,\lambda}^\Omega$ is continuous on $\Omega \times \Omega$ by Theorem 3.6, so the integral in (4.11) exists for each $z \in \Omega$. Item (1) in Section 3.4 shows $z \mapsto K_{p,\lambda}^\Omega(z, w)$ is holomorphic for each $w \in \Omega$, implying $\boldsymbol{Q} f$ is holomorphic for $f \in C_c(\Omega)$, for instance, by applying Morera's theorem in each variable, or equivalently, by applying $\bar{\partial}$ to both sides.

Let $f \in C_c(\Omega)$. Since the series for $K_{p,\lambda}^\Omega$ converges absolutely and uniformly on the compact subset $\{z\} \times \operatorname{supp}(f) \subset \Omega \times \Omega$, (4.9) gives

$$\begin{aligned}
\boldsymbol{Q} f(z) &= \int_\Omega \Bigg(\sum_{\alpha \in \mathcal{S}_p(\Omega, \lambda)} e_\alpha(z) \overline{g_\alpha(w)} \Bigg) f(w) \lambda(w)\, dV(w) \\
&= \sum_{\alpha \in \mathcal{S}_p(\Omega, \lambda)} \left(\int_\Omega f(w) \overline{g_\alpha(w)} \lambda(w)\, dV(w) \right) e_\alpha(z) = \sum_{\alpha \in \mathcal{S}_p(\Omega, \lambda)} \widetilde{a}_\alpha(f) e_\alpha(z).
\end{aligned} \tag{4.13}$$

The series (4.13) converges unconditionally and is the Laurent series of the holomorphic function $\boldsymbol{Q} f$. It is therefore uniformly convergent for z in compact subsets of Ω.

Suppose now that the MBP $\boldsymbol{P}^{\Omega}_{p,\lambda} : L^p(\Omega, \lambda) \to A^p(\Omega, \lambda)$ exists, which by Definition 2.18 is a bounded, surjective, linear projection given by the following limit of partial sums, convergent in $A^p(\Omega, \lambda)$:

$$\boldsymbol{P}^{\Omega}_{p,\lambda} f = \lim_{N\to\infty} \sum_{\substack{|\alpha|_\infty \le N\\ \alpha\in\mathcal{S}_p(\Omega,\lambda)}} \widetilde{a}_\alpha(f) e_\alpha, \qquad f \in L^p(\Omega, \lambda). \tag{4.14}$$

Since convergence in $A^p(\Omega, \lambda)$ implies uniform convergence on compact subsets, it follows that for $f \in C_c(\Omega)$, $\boldsymbol{Q}f = \boldsymbol{P}^{\Omega}_{p,\lambda} f$. Therefore $\boldsymbol{Q}$ satisfies L^p-estimates, i.e. (4.12) holds.

Conversely, suppose that (4.12) holds. Then $\boldsymbol{Q}$ can then be extended by continuity to an operator $\widetilde{\boldsymbol{Q}}$ on $L^p(\Omega, \lambda)$ with the same norm. We claim that $\widetilde{\boldsymbol{Q}}$ is the MBP.

If $f \in L^p(\Omega, \lambda)$, we can find a sequence $\{f_j\} \subset C_c(\Omega)$ such that $f_j \to f$ in $L^p(\Omega, \lambda)$. Each $\boldsymbol{Q}f_j \in A^p(\Omega, \lambda)$ and (by definition) $\boldsymbol{Q}f_j \to \widetilde{\boldsymbol{Q}}f$ in $L^p(\Omega, \lambda)$. But this implies $\boldsymbol{Q}f_j \to \widetilde{\boldsymbol{Q}}f$ uniformly on compact subsets, so the limit $\widetilde{\boldsymbol{Q}}f$ is holomorphic, and thus the range of $\widetilde{\boldsymbol{Q}}$ is contained in $A^p(\Omega, \lambda)$. A direct computation now shows $\widetilde{\boldsymbol{Q}}e_\alpha = e_\alpha$ for $\alpha \in \mathcal{S}_p(\Omega, \lambda)$, and it follows that $\widetilde{\boldsymbol{Q}}$ is a surjective projection from $L^p(\Omega, \lambda)$ to $A^p(\Omega, \lambda)$.

If $f \in C_c(\Omega)$, then $\boldsymbol{Q}f = \widetilde{\boldsymbol{Q}}f \in A^p(\Omega, \lambda)$ and by Theorem 2.16 the Laurent series expansion of $\widetilde{\boldsymbol{Q}}f$ given by (4.13) converges (as a sequence of square partial sums) in $A^p(\Omega, \lambda)$:

$$\widetilde{\boldsymbol{Q}}f = \lim_{N\to\infty} \sum_{\substack{|\alpha|_\infty \le N\\ \alpha\in\mathcal{S}_p(\Omega,\lambda)}} \widetilde{a}_\alpha(f) e_\alpha. \tag{4.15}$$

For a general $g \in L^p(\Omega, \lambda)$, $\widetilde{\boldsymbol{Q}}g \in A^p(\Omega, \lambda)$ and so again by Theorem 2.16,

$$\widetilde{\boldsymbol{Q}}g = \lim_{N\to\infty} \sum_{\substack{|\alpha|_\infty \le N\\ \alpha\in\mathcal{S}_p(\Omega,\lambda)}} a_\alpha(\widetilde{\boldsymbol{Q}}g) e_\alpha. \tag{4.16}$$

It follows that on $C_c(\Omega)$ we have the identity $a_\alpha \circ \boldsymbol{Q} = \widetilde{a}_\alpha$. This relation extends by continuity to give $a_\alpha \circ \widetilde{\boldsymbol{Q}} = \widetilde{a}_\alpha$ as functionals on $L^p(\Omega, \lambda)$. Then (4.16) becomes

$$\widetilde{\boldsymbol{Q}}g = \lim_{N\to\infty} \sum_{\substack{|\alpha|_\infty \le N\\ \alpha\in\mathcal{S}_p(\Omega,\lambda)}} \widetilde{a}_\alpha(g) e_\alpha,$$

i.e., $\widetilde{\boldsymbol{Q}}$ is the MBP, as we wanted to show. □

Proof of Theorem 4.1. Since the MBP exists, by Proposition 4.10 the operator $\boldsymbol{Q}$ of (4.11) satisfies L^p-estimates. Then, by the continuity of point-evaluation in $A^p(\Omega, \lambda)$, for each $z \in \Omega$ the map $g \mapsto \boldsymbol{Q}g(z)$ is a bounded linear functional on $L^p(\Omega, \lambda)$. Formula (4.11) representing this functional now shows that $K^{\Omega}_{p,\lambda}(z, \cdot) \in L^q(\Omega, \lambda)$. Standard techniques of real analysis (cutting off and mollification) gives us a sequence $\{f_j\} \subset C_c(\Omega)$ such that $f_j \to f$ in $L^p(\Omega, \lambda)$. Therefore for each $z \in \Omega$, the sequence $\{K^{\Omega}_{p,\lambda}(z, \cdot) f_j(\cdot)\} \subset C_c(\Omega)$ converges in $L^1(\Omega, \lambda)$ to the limit $K^{\Omega}_{p,\lambda}(z, \cdot) f(\cdot)$. Since integration against the weight λ is a bounded linear functional on $L^1(\Omega, \lambda)$, we obtain (4.2) in the limit. □

4.2. **Subkernels and subprojections.** It is possible to introduce a slightly more general notion of Schauder basis and basis projection than was given in Definition 2.1. In Section 6, for example, we need to allow for partial sums other than the square partial sums of (2.2), which were used for simplicity of notation and exposition. We give the following definitions in the context of the Bergman space $A^p(\Omega, \lambda)$, keeping in mind that the General Hypothesis of Section 4 is still in force.

Let $\mathfrak{B}$ be a subset of $\mathcal{S}_p(\Omega,\lambda)$, the set of p-allowable multi-indices. Let $A^p_{\mathfrak{B}}(\Omega,\lambda)$ denote the closed subspace of $A^p(\Omega,\lambda)$ generated by the monomials $\{e_\alpha : \alpha \in \mathfrak{B}\}$, i.e., $A^p_{\mathfrak{B}}(\Omega,\lambda)$ is the closure in $A^p(\Omega,\lambda)$ of the span (finite linear combinations) of $\{e_\alpha : \alpha\in\mathfrak{B}\}$. It is easy to see that any function $f \in A^p_{\mathfrak{B}}(\Omega,\lambda)$ must have a Laurent series of the form

$$f = \sum_{\alpha\in\mathfrak{B}} a_\alpha(f) e_\alpha,$$

i.e., only monomials indexed by $\mathfrak{B}$ occur in the sum. Suppose now we are given an increasing family of subsets $\mathfrak{B}_1 \subset \mathfrak{B}_2 \subset \cdots \subset \mathfrak{B}$ such that $\bigcup_{N=1}^\infty \mathfrak{B}_N = \mathfrak{B}$. Let us call such a family $\{\mathfrak{B}_N\}_{N=1}^\infty$ an *exhaustion* of $\mathfrak{B}$. We say that the set $\{e_\alpha : \alpha \in \mathfrak{B}\}$ together with the exhaustion $\{\mathfrak{B}_N\}_{N=1}^\infty$ forms a *Schauder basis in the wider sense* if for each $f \in A^p_{\mathfrak{B}}(\Omega,\lambda)$, we have the following limit of partial sums

$$\lim_{N\to\infty} \left\| \sum_{\alpha\in\mathfrak{B}_N} \widetilde{a}_\alpha(f)e_\alpha - f \right\|_{L^p(\Omega,\lambda)} = 0. \tag{4.17}$$

When this limit holds for all $f \in A^p_{\mathfrak{B}}(\Omega,\lambda)$, call $\{\mathfrak{B}_N\}_{N=1}^\infty$ a *Schauder exhaustion* of $\mathfrak{B}$.

Independent of the choice of exhaustion, the set $\mathfrak{B}$ gives rise to an integral kernel. Let

$$K_{\mathfrak{B}}(z,w) = \sum_{\alpha\in\mathfrak{B}} \frac{e_\alpha(z)\cdot \overline{\chi_p^* e_\alpha(w)}}{\|e_\alpha\|^p_{p,\lambda}}, \tag{4.18}$$

be called the *subkernel* associated to $\mathfrak{B}$. Since the series (1.10) is locally normally convergent by Theorem 3.6, it follows that the same is true for the subseries defining $K_{\mathfrak{B}}$. It is not difficult to see that versions of the first four properties of the MBK listed in Section 3.4 continue to hold for the kernel $K_{\mathfrak{B}}$.

We say that a bounded projection operator $\boldsymbol{P}_{\mathfrak{B}}$ from $L^p(\Omega,\lambda)$ to $A^p_{\mathfrak{B}}(\Omega,\lambda)$ is the *basis projection determined by* $\mathfrak{B}$ *and Schauder exhaustion* $\{\mathfrak{B}_N\}_{N=1}^\infty$ if for $f\in L^p(\Omega,\lambda)$,

$$\boldsymbol{P}_{\mathfrak{B}} f = \lim_{N\to\infty} \sum_{\alpha\in\mathfrak{B}_N} \widetilde{a}_\alpha(f) e_\alpha, \tag{4.19}$$

where the limit is in the norm of $L^p(\Omega,\lambda)$, and as before $\widetilde{a}_\alpha$ is the Hahn-Banach extension of the coefficient functional $a_\alpha : A^p_{\mathfrak{B}}(\Omega,\lambda) \to \mathbb{C}$ to a linear functional on $L^p(\Omega,\lambda)$. We call $\boldsymbol{P}_{\mathfrak{B}}$ a *subprojection*. Notice that the exhaustion $\{\mathfrak{B}_N\}_{N=1}^\infty$ is used on the right hand side of (4.19), but is suppressed in the notation $\boldsymbol{P}_{\mathfrak{B}}$ (with good reason, see Corollary 4.22 below).

With these definitions we state an analog of Theorem 4.1 and Proposition 4.10:

Proposition 4.20. *Let $\mathfrak{B} \subset \mathcal{S}_p(\Omega,\lambda)$ and suppose $\{e_\alpha : \alpha\in\mathfrak{B}\}$ and its exhaustion $\{\mathfrak{B}_N\}_{N=1}^\infty$ form a monomial Schauder basis in the wider sense, as described above. If the subprojection $\boldsymbol{P}_{\mathfrak{B}} : L^p(\Omega,\lambda)\to A^p_{\mathfrak{B}}(\Omega,\lambda)$ exists, then for each $z\in\Omega$ we have $K_{\mathfrak{B}}(z,\cdot) \in L^q(\Omega,\lambda)$ and*

$$\boldsymbol{P}_{\mathfrak{B}}(f)(z) = \int_\Omega K_{\mathfrak{B}}(z,w) f(w)\lambda(w)dV(w), \qquad f\in L^p(\Omega,\lambda). \tag{4.21}$$

Further, the subprojection $\boldsymbol{P}_{\mathfrak{B}}$ exists if and only if the operator $\boldsymbol{Q}_{\mathfrak{B}}$ given by

$$\boldsymbol{Q}_{\mathfrak{B}} g(z) = \int_\Omega K_{\mathfrak{B}}(z,w) g(w)\lambda(w)dV(w), \qquad g\in C_c(\Omega),$$

admits estimates in the $L^p(\Omega,\lambda)$-norm.

Proof. First notice that when $\mathfrak{B} = \mathcal{S}_p(\Omega,\lambda)$ and $\{\mathfrak{B}_N\}_{N=1}^\infty$ is the Schauder exhaustion of $\mathcal{S}_p(\Omega,\lambda)$ by square partial sums, this result is simply Theorem 4.1 and Proposition 4.10 combined. The general case is proved in exactly the same way, on noting that:

(1) Independently of the choice of exhaustion $\{\mathfrak{B}_N\}_{N=1}^{\infty}$, the coefficient functionals dual to the monomial Schauder basis $\{e_\alpha : \alpha \in \mathfrak{B}\}$ are the Laurent coefficient functionals $\{a_\alpha : \alpha \in \mathfrak{B}\}$. It is not difficult to see that these functionals are characterized by the condition $a_\alpha(e_\beta) = \delta_\alpha^\beta$, where δ_α^β is the Kronecker Delta, i.e. $\delta_\alpha^\alpha = 1$ for all α and $\delta_\alpha^\beta = 0$ if $\alpha \neq \beta$. The exhaustion $\{\mathfrak{B}_N\}_{N=1}^{\infty}$ used to define the sequence of partial sums is only important in that it must correspond to a Schauder basis of monomials for $A^p_{\mathfrak{B}}(\Omega, \lambda)$.

(2) The series (4.18) is absolutely convergent, and therefore can be rearranged in any fashion we want. In particular, the analog of (4.15) for $\boldsymbol{Q}_{\mathfrak{B}} f$ holds with partial sums determined by the exhaustion $\{\mathfrak{B}_N\}_{N=1}^{\infty}$.

(3) The only way in which the exhaustion by square partial sums in (4.14) was used in the proof of Proposition 4.10 was through the fact that square partial sums of Laurent series of a function in $A^p(\Omega, \lambda)$ converge in the $L^p(\Omega, \lambda)$ norm, i.e., they form a Schauder basis of $A^p(\Omega, \lambda)$ in the sense of Definition 2.1.

It follows that all the arguments in the proofs of Proposition 4.10 and Theorem 4.1 go over *mutatis mutandis* to this new situation. □

Proposition 4.20 has an immediate consequence:

Corollary 4.22. *Let $\mathfrak{B} \subset \mathcal{S}_p(\Omega, \lambda)$. If the subprojection $\boldsymbol{P}_{\mathfrak{B}} : L^p(\Omega, \lambda) \to A^p_{\mathfrak{B}}(\Omega, \lambda)$ exists with respect to some Schauder exhaustion $\{\mathfrak{B}_N\}_{N=1}^{\infty}$ of $\mathfrak{B}$, it exists for every Schauder exhaustion of $\mathfrak{B}$ and is independent of that choice.*

In other words, suppose $\{\mathfrak{B}_N\}_{N=1}^{\infty}$ and $\{\mathfrak{C}_N\}_{N=1}^{\infty}$ are two Schauder exhaustions of $\mathfrak{B}$. Then for each $f \in A^p_{\mathfrak{B}}(\Omega, \lambda)$,

$$\lim_{N\to\infty} \sum_{\alpha \in \mathfrak{B}_N} a_\alpha(f) e_\alpha = f = \lim_{N\to\infty} \sum_{\alpha \in \mathfrak{C}_N} a_\alpha(f) e_\alpha,$$

with convergence in $L^p(\Omega, \lambda)$. If the subprojection $\boldsymbol{P}_{\mathfrak{B}}$ corresponding to one of these exhaustions exists, they both exist and for $f \in L^p(\Omega, \lambda)$,

$$\lim_{N\to\infty} \sum_{\alpha \in \mathfrak{B}_N} \widetilde{a}_\alpha(f) e_\alpha = \boldsymbol{P}_{\mathfrak{B}} f = \lim_{N\to\infty} \sum_{\alpha \in \mathfrak{C}_N} \widetilde{a}_\alpha(f) e_\alpha,$$

since both are represented by the integral formula on the right hand side of (4.21). In particular, these considerations apply to the full MBP of $A^p(\Omega, \lambda)$, and therefore, the MBP can be defined with respect to any ordering of the monomials in which they form a Schauder basis of $A^p(\Omega, \lambda)$ in the wider sense.

4.3. **Absolute integral kernel operators.** Let $(X, \mathcal{F}, \mu)$ be a measure space, $1 < p < \infty$, and suppose we are given an operator $\boldsymbol{T}$ *formally* defined on $L^p(\mu)$ by an integral kernel A:

$$\boldsymbol{T} f(z) = \int_X A(z, w) f(w) d\mu(w).$$

We make no assumption about the existence of the integral, so $\boldsymbol{T}$ is in general defined on a linear subspace of $L^p(\mu)$ (which may degenerate to the zero subspace).

Definition 4.23. The *absolute operator* corresponding to $\boldsymbol{T}$ is the formally defined operator on $L^p(\mu)$ corresponding to the absolute value of the kernel $|A(z, w)|$, denoted by

$$\boldsymbol{T}^+ f(z) = \int_X |A(z, w)|\, f(w) d\mu(w). \tag{4.24}$$

The operator $\boldsymbol{T}$ is said to be *absolutely bounded* on $L^p(\mu)$, if $\boldsymbol{T}^+$ is everywhere defined on $L^p(\mu)$ and defines a bounded linear operator from $L^p(\mu)$ to itself.

Notice that if a formally defined integral operator $\boldsymbol{T}$ is absolutely bounded on $L^p(\mu)$, it is clearly bounded, though the converse statement is not true.

4.4. **Adjoints and Duality.** Recall the twisted pairing from Section 1.5:

$$\{f,g\}_{p,\lambda} = \int_\Omega f \cdot \overline{\chi_p^*(g)}\, \lambda\, dV, \tag{4.25}$$

with notation as in (1.8) and (1.9). Recalling from (3.1) that $\eta_q = \det D\chi_q$, we have:

Proposition 4.26. *The pairing given by* (4.25)

$$(f,g) \mapsto \{f,g\}_{p,\lambda}, \qquad f \in L^p(\Omega,\lambda), \quad g \in L^q(\Omega^{(p-1)}, \eta_q \cdot \chi_q^* \lambda)$$

is a sesquilinear duality paring of Banach spaces.

Proof. Recall that the pairing of $f \in L^p(\Omega,\lambda)$ with $h \in L^q(\Omega,\lambda)$ given by $(f,h) \mapsto \int_\Omega f\overline{h}\lambda dV$ is a sesquilinear duality pairing. Also recall from (3.2) and (3.3) that $\chi_q : \Omega^{(p-1)} \to \Omega$ is a diffeomorphism outside a set of measure zero, with inverse $\chi_p : \Omega \to \Omega^{(p-1)}$, which is itself a diffeomorphism outside a set of measure zero. It therefore suffices to show that

$$\chi_q^* : L^q(\Omega,\lambda) \to L^q(\Omega^{(p-1)}, \eta_q \cdot \chi_q^* \lambda) \tag{4.27}$$

is an (isometric) isomorphism of Banach spaces. Calculation shows

$$\begin{aligned} \|h\|^q_{L^q(\Omega,\lambda)} = \int_\Omega |h|^q\, \lambda dV &= \int_{\Omega^{(p-1)}} |h \circ \chi_q(w)|^q\, \eta_q(w)\lambda(\chi_q(w))dV(w) \\ &= \left\|\chi_q^*(h)\right\|^q_{L^q(\Omega^{(p-1)},\eta_q\cdot\chi_q^*\lambda)}. \end{aligned} \tag{4.28}$$

Since the inverse map χ_p^* of χ_q^* exists, it is surjective and the result follows by the closed-graph theorem. □

Following the general definition (4.24), define the *Absolute Monomial Basis Operator* (abbreviated AMBO), the integral operator obtained by integrating against the absolute value of the MBK:

$$(\boldsymbol{P}^\Omega_{p,\lambda})^+ f(z) = \int_\Omega \left|K^\Omega_{p,\lambda}(z,w)\right| f(w)\lambda(w)dV(w), \qquad f \in C_c(\Omega). \tag{4.29}$$

We say the MBP $\boldsymbol{P}^\Omega_{p,\lambda}$ is *absolutely bounded* if the AMBO $(\boldsymbol{P}^\Omega_{p,\lambda})^+$ is bounded in $L^p(\Omega,\lambda)$.

For the remainder of the section our main focus is on the case of $\lambda \equiv 1$, where the pairing (4.25) assumes the form

$$\{f,g\}_{p,1} = \int_\Omega f \cdot \overline{\chi_p^*(g)}\, dV, \qquad f \in L^p(\Omega), \quad g \in L^q(\Omega^{(p-1)}, \eta_q). \tag{4.30}$$

Proposition 4.31. *If the MBP $\boldsymbol{P}^\Omega_{p,1} : L^p(\Omega) \to A^p(\Omega)$ is absolutely bounded in $L^p(\Omega)$, then under the pairing* (4.25) *with $\lambda \equiv 1$, the adjoint of $\boldsymbol{P}^\Omega_{p,1}$ is the weighted MBP*

$$\boldsymbol{P}^{\Omega^{(p-1)}}_{q,\eta_q} : L^q(\Omega^{(p-1)}, \eta_q) \to A^q(\Omega^{(p-1)}, \eta_q).$$

Proof. We have for $f \in L^p(\Omega)$ and $g \in L^q(\Omega^{(p-1)}, \eta_q)$:

$$\begin{aligned} \left\{\boldsymbol{P}^\Omega_{p,1} f, g\right\}_{p,1} = \int_\Omega \boldsymbol{P}^\Omega_{p,1} f \cdot \overline{\chi_p^* g}\, dV &= \int_\Omega \left(\int_\Omega K^\Omega_{p,1}(z,w) f(w)\, dV(w)\right) \overline{g(\chi_p(z))}\, dV(z) &(4.32) \\ &= \int_\Omega \left(\int_\Omega K^\Omega_{p,1}(z,w) \overline{g(\chi_p(z))} dV(z)\right) f(w)\, dV(w), &(4.33) \end{aligned}$$

where the change in order of integration can be justified as follows. By the assumption that $\boldsymbol{P}_{p,1}^{\Omega}$ is absolutely bounded on $L^p(\Omega)$, we see that the function on Ω given by

$$z \longmapsto \int_{\Omega} \left|K_{p,1}^{\Omega}(z,w)\right| \cdot |f(w)|\, dV(w)$$

is in $L^p(\Omega)$. Since $g \in L^q(\Omega^{(p-1)}, \eta_q)$, using Tonelli's theorem we see that

$$\begin{aligned}
&\int_{\Omega\times\Omega} \left|K_{p,1}^{\Omega}(z,w) g(\chi_p(z)) f(w)\right| dV(z,w) \\
&= \int_{\Omega} \left(\int_{\Omega} \left|K_{p,1}^{\Omega}(z,w)\right| \cdot |f(w)|\, dV(z)\right) |g(\chi_p(z))|\, dV(w) < \infty,
\end{aligned}$$

by Proposition 4.26. Now Fubini's theorem gives that (4.32) = (4.33).

Now make a change of variables $z = \chi_q(\zeta)$ where $\zeta \in \Omega^{(p-1)}$ to obtain that

$$\begin{aligned}
(4.33) &= \int_{\Omega} \left(\int_{\Omega^{(p-1)}} K_{p,1}^{\Omega}(\chi_q(\zeta), w) \overline{g(\zeta)}\, \eta_q(\zeta)\, dV(\zeta)\right) f(w) dV(w) \\
&= \int_{\Omega} \left(\int_{\Omega^{(p-1)}} \overline{K_{q,\eta_q}^{\Omega^{(p-1)}}(\chi_p(w), \zeta)} \cdot \overline{g(\zeta)}\, \eta_q(\zeta) dV(\zeta)\right) f(w) dV(w) \\
&= \int_{\Omega} \overline{\left(\int_{\Omega^{(p-1)}} K_{q,\eta_q}^{\Omega^{(p-1)}}(\chi_p(w), \zeta) \cdot g(\zeta)\, \eta_q(\zeta) dV(\zeta)\right)} f(w) dV(w) &\quad (4.34)\\
&= \int_{\Omega} f(w) \overline{\boldsymbol{P}_{q,\eta_q}^{\Omega^{(p-1)}} g(\chi_p(w))}\, dV(w) &\quad (4.35)\\
&= \int_{\Omega} f \cdot \overline{\chi_p^* \left(\boldsymbol{P}_{q,\eta_q}^{\Omega^{(p-1)}} g\right)} dV = \left\{f, \boldsymbol{P}_{q,\eta_q}^{\Omega^{(p-1)}} g\right\}_{p,1}.
\end{aligned}$$

Note that the second line follows from (3.16). The fact that (4.35) = (4.34) can be justified as follows. For a $g \in L^q(\Omega^{(p-1)}, \eta_q)$, the quantity in (4.34) is finite for each $f \in L^p(\Omega)$, since by the above computations it is equal to the finite quantity $\left\{\boldsymbol{P}_{p,1}^{\Omega} f, g\right\}_{p,1}$. Therefore we see that for each $g \in L^q(\Omega^{(p-1)}, \eta_q)$, we have

$$\left(w \longmapsto \int_{\Omega^{(p-1)}} K_{q,\eta_q}^{\Omega^{(p-1)}}(\chi_p(w), \zeta) g(\zeta) \eta_q(\zeta) dV(\zeta)\right) \in L^q(\Omega),$$

so that the linear map

$$g \longmapsto \int_{\Omega^{(p-1)}} K_{q,\eta_q}^{\Omega^{(p-1)}}(\chi_p(\cdot), \zeta) g(\zeta) \eta_q(\zeta) dV(\zeta)$$

is bounded from $L^q(\Omega^{(p-1)}, \eta_q)$ to $L^q(\Omega)$ by the closed graph theorem (since the integral operator is easily seen to be closed). Composing with the (isometric) bounded linear map $\chi_q^* : L^q(\Omega) \to L^q(\Omega^{(p-1)}, \eta_q)$ we see that the operator on $L^q(\Omega^{(p-1)}, \eta_q)$ given by

$$g \longmapsto \int_{\Omega^{(p-1)}} g(\zeta) K_{q,\eta_q}^{\Omega^{(p-1)}}(\cdot, \zeta) \eta_q(\zeta) dV(\zeta)$$

is bounded on $L^q(\Omega^{(p-1)}, \eta_q)$. Now Proposition 4.10 shows (4.35) = (4.34). $\square$

Proposition 4.36. *If the Monomial Basis Projection $\boldsymbol{P}_{p,1}^{\Omega} : L^p(\Omega) \to A^p(\Omega)$ is absolutely bounded in $L^p(\Omega)$, the pairing $\{\cdot,\cdot\}_{p,1}$ of* (4.30) *restricts to a duality pairing*

$$A^p(\Omega) \times A^q(\Omega^{(p-1)}, \eta_q) \to \mathbb{C}.$$

Proof. We show that the conjugate-linear continuous map $A^q(\Omega^{(p-1)}, \eta_q) \to A^p(\Omega)'$ given by $h \mapsto \{\cdot, h\}_{p,1}$ is a homeomorphism of Banach spaces. To see surjectivity, let $\phi \in A^p(\Omega)'$, let $\widetilde{\phi} : L^p(\Omega) \to \mathbb{C}$ be its Hahn-Banach extension, and let $g \in L^q(\Omega^{(p-1)}, \eta_q)$ be such that $\widetilde{\phi}(f) = \{f, g\}_{p,1}$. The existence of g follows from Proposition 4.26, and we see that for each $f \in A^p(\Omega)$ we have

$$\phi(f) = \widetilde{\phi}(f) = \{f, g\}_{p,1} = \{\boldsymbol{P}^{\Omega}_{p,1} f, g\}_{p,1} = \{f, \boldsymbol{P}^{\Omega^{(p-1)}}_{q,\eta_q} g\}_{p,1},$$

so the surjectivity follows since $\boldsymbol{P}^{\Omega^{(p-1)}}_{q,\eta_q} g \in A^q(\Omega^{(p-1)}, \eta_q)$. Now if $h \in A^q(\Omega^{(p-1)}, \eta_q)$ is in the null-space of this map, i.e., for each $f \in A^p(\Omega)$ we have $\{f, h\}_{p,1} = 0$, then for $g \in L^p(\Omega)$:

$$\{g, h\}_{p,1} = \{g, \boldsymbol{P}^{\Omega^{(p-1)}}_{q,\eta_q} h\}_{p,1} = \{\boldsymbol{P}^{\Omega}_{p,1} g, h\}_{p,1} = 0.$$

This shows that $h = 0$, so the mapping is injective. □

Remark 4.37. More generally, a slight adaptation of Proposition 4.31 can show the following: let $\Omega \subset \mathbb{C}^n$ be a pseudoconvex Reinhardt domain and $\lambda > 0$ be a continuous multi-radial weight. If the MBP $\boldsymbol{P}^{\Omega}_{p,\lambda} : L^p(\Omega, \lambda) \to A^p(\Omega, \lambda)$ is absolutely bounded in $L^p(\Omega, \lambda)$, then under the pairing (4.25), the adjoint of $\boldsymbol{P}^{\Omega}_{p,\lambda}$ is the weighted MBP

$$\boldsymbol{P}^{\Omega^{(p-1)}}_{q,\eta_q \cdot \chi_q^* \lambda} : L^q(\Omega^{(p-1)}, \eta_q \cdot \chi_q^* \lambda) \to A^q(\Omega^{(p-1)}, \eta_q \cdot \chi_q^* \lambda).$$

This allows for the formulation of a duality statement generalizing Proposition 4.36. ◊

5. (Sub)-projections on the disc and punctured disc

We compute here explicitly for $1 < p < \infty$ certain subkernels of the Monomial Basis Kernels associated to $A^p(\mathbb{D}, \mu_\gamma)$ and $A^p(\mathbb{D}^*, \mu_\gamma)$, where $\mu_\gamma(z) = |z|^\gamma$, and using this show that the associated subprojections (including the full Monomial Basis Projections) of these spaces both exist and are absolutely bounded. The subkernels and subprojections considered have application to monomial polyhedra in Section 8. It is easily checked by Proposition 3.14 that μ_γ is an admissible weight on both $\mathbb{D}$ and $\mathbb{D}^*$.

5.1. **Arithmetic progression subkernels on $\mathbb{D}$ and $\mathbb{D}^*$.** Let $a, b \in \mathbb{Z}$ with b positive, $U = \mathbb{D}$ or $\mathbb{D}^*$, $1 < p < \infty$ and $\mu_\gamma(z) = |z|^\gamma$, $\gamma \in \mathbb{R}$. Consider the set of integers

$$\mathcal{A}(U, p, \gamma, a, b) = \{\alpha \in \mathbb{Z} : \alpha \equiv a \mod b\} \cap \mathcal{S}_p(U, \mu_\gamma), \tag{5.1}$$

where, as usual $\mathcal{S}_p(U, \mu_\gamma) \subset \mathbb{Z}$ is the set of α such that $e_\alpha \in A^p(U, \mu_\gamma)$. Notice that a is determined only modulo b, so we can always assume that $0 \le a \le b - 1$. We now identify this set of integers with an *arithmetic progression*:

Proposition 5.2. *Let U, p, γ, a, b be as above. There is an integer θ such that*

$$\mathcal{A}(U, p, \gamma, a, b) = \{\theta + \nu b : \nu \ge 0, \nu \in \mathbb{Z}\}. \tag{5.3}$$

Proof. Let $U = \mathbb{D}^*$. We claim that $\alpha \in \mathcal{S}_p(\mathbb{D}^*, \mu_\gamma)$ if and only if $p\alpha + \gamma + 2 > 0$. Indeed,

$$\|e_\alpha\|^p_{p,\mu_\gamma} = \int_{\mathbb{D}^*} |z|^{p\alpha+\gamma}\, dV = 2\pi \int_0^1 r^{p\alpha+\gamma+1}\, dr = \frac{2\pi}{p\alpha + \gamma + 2}, \tag{5.4}$$

as long as $p\alpha + \gamma + 2 > 0$; otherwise the integral diverges. Now let θ be the smallest integer such that (i) $\theta \equiv a \mod b$, and (ii) $p\theta + \gamma + 2 > 0$. Clearly (5.3) holds.

The case $U = \mathbb{D}$ is nearly identical, but the condition that e_α belongs to $L^p(\mathbb{D}, \mu_\gamma)$ now means that α must be nonnegative. If θ is the smallest integer in the set $\mathcal{S}_p(\mathbb{D}, \mu_\gamma)$, it is determined now by the following three conditions: (i) $\theta \equiv a \mod b$, (ii) $p\theta + \gamma + 2 > 0$, *and* (iii) $\theta \ge 0$. □

Remark 5.5. For U, p, γ, a, b as above (with $0 \le a \le b-1$), we can determine θ explicitly:

$$\theta = \begin{cases} a + b\ell, & U = \mathbb{D}^* \\ \max\{a+b\ell, a\}, & U = \mathbb{D}, \end{cases} \quad \text{where} \quad \ell = \left\lfloor -\frac{\gamma+2}{pb} - \frac{a}{b} + 1 \right\rfloor.$$

◊

Now define for $z, w \in U$ the kernel

$$k^U_{p,\gamma,a,b}(z,w) = \sum_{\alpha \in \mathcal{A}(U,p,\gamma,a,b)} \frac{e_\alpha(z) \cdot \overline{\chi_p^* e_\alpha(w)}}{\|e_\alpha\|^p_{p,\mu_\gamma}} = \sum_{\alpha \in \mathcal{A}(U,p,\gamma,a,b)} \frac{t^\alpha}{\|e_\alpha\|^p_{p,\mu_\gamma}}, \tag{5.6}$$

where χ_p^* is defined by (1.9) and $t = z\overline{w}\,|w|^{p-2}$. It is easily checked that in the terminology of (4.18), $k^U_{p,\gamma,a,b}$ is the subkernel of the Monomial Basis Kernel of $A^p(U, \mu_\gamma)$ corresponding to the subset of indices $\mathfrak{B} = \mathcal{A}(U, p, \gamma, a, b)$. We will refer to the kernels $k^U_{p,\gamma,a,b}$ as the *arithmetic progression subkernels.*

Proposition 5.7. *For $z, w \in U$ and notation as specified above, we have*

$$k^U_{p,\gamma,a,b}(z,w) = \frac{t^\theta}{2\pi} \cdot \frac{(p\theta+\gamma+2) - (\gamma+2+p(\theta-b))t^b}{(1-t^b)^2}. \tag{5.8}$$

Proof. The calculation in (5.4) shows that if $\alpha \in \mathcal{S}_p(U, \mu_\gamma)$, then

$$\|e_\alpha\|^p_{p,\mu_\gamma} = \frac{2\pi}{p\alpha+\gamma+2}.$$

Now combining (5.6) with Proposition 5.2, we see that

$$\begin{aligned} k^U_{p,\gamma,a,b}(z,w) = \sum_{\alpha \in \mathcal{A}(U,p,\gamma,a,b)} \frac{t^\alpha}{\|e_\alpha\|^p_{p,\mu_\gamma}} &= \frac{t^\theta}{2\pi} \sum_{\nu=0}^\infty (p(\theta+b\nu)+\gamma+2)t^{b\nu} \\ &= \frac{t^\theta}{2\pi}\left(p\sum_{\nu=0}^\infty (b\nu+1)t^{b\nu} + (p\theta+\gamma+2-p)\sum_{\nu=0}^\infty t^{b\nu}\right). \end{aligned}$$

Writing these sums in closed form gives

$$\begin{aligned} k^U_{p,\gamma,a,b}(z,w) &= \frac{t^\theta}{2\pi}\left(\frac{p+p(b-1)t^b}{(1-t^b)^2} + \frac{(p\theta+\gamma+2-p)}{1-t^b}\right) \\ &= \frac{t^\theta}{2\pi} \cdot \frac{(p\theta+\gamma+2) - (\gamma+2+p(\theta-b))t^b}{(1-t^b)^2}. \end{aligned}$$

□

Corollary 5.9. *The arithmetic progression kernel $k^U_{p,\gamma,a,b}$ admits the bound*

$$\left|k^U_{p,\gamma,a,b}(z,w)\right| \le C \frac{(|z||w|^{p-1})^\theta}{|1 - z^b\overline{w}^b|w|^{(p-2)b}|^2},$$

where $C > 0$ is independent of $z, w \in U$.

Proof. This follows from (5.8), on noting that $(p\theta+\gamma+2)$ is necessarily positive. □

Setting $a = 0$, $b = 1$ in Proposition 5.7 yields the full MBKs of $A^p(\mathbb{D}^*, \mu_\gamma)$ and $A^p(\mathbb{D}, \mu_\gamma)$:

Corollary 5.10. *Let $\gamma \in \mathbb{R}$, $\mu_\gamma(z) = |z|^\gamma$ and $t = z\overline{w}\,|w|^{p-2}$. The Monomial Basis Kernels of $A^p(\mathbb{D}^*, \mu_\gamma)$ and $A^p(\mathbb{D}, \mu_\gamma)$ are given by*

(1) $K^{\mathbb{D}^*}_{p,\mu_\gamma}(z,w) = \dfrac{1}{2\pi} \cdot \dfrac{(p\ell+\gamma+2)t^\ell - (\gamma+2+p(\ell-1))t^{\ell+1}}{(1-t)^2}$, *where* $\ell = \left\lfloor -\frac{\gamma+2}{p} + 1 \right\rfloor$.

(2) $K^{\mathbb{D}}_{p,\mu_\gamma}(z,w) = \dfrac{1}{2\pi}\cdot \dfrac{(pL+\gamma+2)t^L - (\gamma+2+p(L-1))t^{L+1}}{(1-t)^2}$, *where* $L = \max\{\ell, 0\}$.

5.2. **Two tools.** We now recall two important results.

Proposition 5.11 (Schur's test). *For $1 \le j \le N$, let D_j be a domain in $\mathbb{R}^{n_j}$, let $K_j : D_j \times D_j \to [0,\infty)$ be a kernel on D_j, and let λ^j be an a.e. positive weight on D_j. Suppose that for each j, there exist a.e. positive measurable functions ϕ_j, ψ_j on D_j and constants $C_1^j, C_2^j > 0$ such that the following two estimates hold:*

(1) *For $z \in D_j$,* $\displaystyle\int_{D_j} K_j(z,w)\psi_j(w)^q \lambda^j(w)\, dV(w) \le C_1^j \phi_j(z)^q$.

(2) *For $w \in D_j$,* $\displaystyle\int_{D_j} \phi_j(z)^p K_j(z,w)\lambda^j(z)\, dV(z) \le C_2^j \psi_j(w)^p$.

Let $D = D_1 \times \cdots \times D_N$ be the product of the domains, let $K(z,w) = \prod_{j=1}^N K_j(z_j,w_j)$, where $z_j, w_j \in D_j$, $z = (z_1,\dots,z_N) \in D$, $w = (w_1,\dots,w_N) \in D$, and let $\lambda(w) = \prod_{j=1}^N \lambda^j(w_j)$.

Then the operators $\boldsymbol{T}$ and $\boldsymbol{T}^\dagger$ defined by

$$\boldsymbol{T} f(z) = \int_D K(z,w) f(w)\lambda(w) dV(w), \qquad \boldsymbol{T}^\dagger g(z) = \int_D g(z) K(z,w)\lambda(z) dV(z),$$

are bounded on $L^p(D,\lambda)$ and $L^q(D,\lambda)$, respectively.

Proof. When $N = 1$, this is a form of the famous *Schur's test* for boundedness of integral operators on L^p-spaces defined by positive kernels (see [Zhu07, Theorem 3.6]). The case $N \ge 2$ is reduced to the case $N = 1$, if we let $\phi(z) = \prod_{j=1}^N \phi_j(z_j)$ and $\psi(z) = \prod_{j=1}^N \psi_j(z_j)$ ($z_j \in D_j$ and $z = (z_1,\dots,z_N) \in D$) and using the Tonelli-Fubini theorem to represent integrals over D as repeated integrals over the product representations. ☐

Proposition 5.12 (Lemma 3.4 of [EM16]; also see [FR74] for $\beta = 0$). *Let $U = \mathbb{D}$ or $\mathbb{D}^*$, $0 < \epsilon < 1$ and $\beta > -2$. There exists $C > 0$ such that*

$$\int_U \frac{(1-|w|^2)^{-\epsilon}}{|1-z\overline{w}|^2} |w|^\beta \, dV(w) \le C(1-|z|^2)^{-\epsilon}. \tag{5.13}$$

5.3. **L^p-boundedness of operators.** We now prove that arithmetic progression kernels represent operators bounded in the norm of $L^p(\Omega,\mu_\gamma)$. This proves the existence of the subprojections determined by the sets $\mathcal{A}(U,p,\gamma,a,b)$, and in particular, the existence of the full MBPs of $A^p(\mathbb{D}^*,\mu_\gamma)$ and $A^p(\mathbb{D},\mu_\gamma)$.

For $U = \mathbb{D}$ or $\mathbb{D}^*$, $\mu_\gamma(z) = |z|^\gamma$ with $\gamma \in \mathbb{R}$ and $1 < p < \infty$, define the integral operator

$$\boldsymbol{p}^U_{p,\gamma,a,b}(f)(z) = \int_U k^U_{p,\gamma,a,b}(z,w) f(w)\mu_\gamma(w) dV(w), \tag{5.14}$$

where $k^U_{p,\gamma,a,b}(z,w)$ is the kernel defined in (5.6). When this operator is bounded in the norm of $L^p(U,\mu_\gamma)$, Proposition 4.20 says that it is the subprojection $\boldsymbol{P}_{\mathfrak{B}} : L^p(U,\mu_\gamma) \to A^p_{\mathfrak{B}}(U,\mu_\gamma)$, with $\mathfrak{B} = \mathcal{A}(U,p,\gamma,a,b)$.

Following (4.24), we let $(\boldsymbol{p}^U_{p,\gamma,a,b})^+$ denote the "absolute operator"

$$(\boldsymbol{p}^U_{p,\gamma,a,b})^+(f)(z) = \int_U \left| k^U_{p,\gamma,a,b}(z,w) \right| f(w)\mu_\gamma(w) dV(w). \tag{5.15}$$

Proposition 5.16. *Define the following auxiliary functions on U:*

$$\phi(z) = |z|^{\frac{\theta}{q}} (1-|z|^{2b})^{-\frac{1}{pq}}, \qquad \psi(w) = |w|^{\frac{\theta}{q}} (1-|w|^{2b(p-1)})^{-\frac{1}{pq}}.$$

There exist constants $C_1, C_2 > 0$, such that the following estimates hold:

(1) For $z \in U$, $\displaystyle\int_U \left|k^U_{p,\gamma,a,b}(z,w)\right| \psi(w)^q \mu_\gamma(w)\, dV(w) \le C_1 \phi(z)^q.$

(2) For $w \in U$, $\displaystyle\int_U \phi(z)^p \left|k^U_{p,\gamma,a,b}(z,w)\right| \mu_\gamma(z)\, dV(z) \le C_2 \psi(w)^p.$

Proof. Throughout this proof, C will denote a positive number depending on p,γ,a,b but independent of $z,w \in U$. Its value will change from step to step.

From the kernel bound in Corollary 5.9, we obtain

$$\int_U \left|k^U_{p,\gamma,a,b}(z,w)\right| \psi(w)^q \mu_\gamma(w)\, dV(w) \le C \int_U \frac{(|z||w|^{p-1})^\theta}{|1 - z^b \overline{w}^b |w|^{(p-2)b}|^2} \psi(w)^q \mu_\gamma(w)\, dV(w)$$

$$= C|z|^\theta \int_U \frac{\big(1-|w|^{2b(p-1)}\big)^{-\frac{1}{p}}}{|1 - z^b \overline{w}^b |w|^{(p-2)b}|^2} |w|^{p\theta+\gamma}\, dV(w). \tag{5.17}$$

Set $\zeta = w^b|w|^{(p-2)b}$, so $|\zeta| = |w|^{(p-1)b}$, $|w| = |\zeta|^{\frac{q-1}{b}}$ and $dV(w) = \left(\frac{q-1}{b^2}\right)|\zeta|^{\frac{2(q-1)}{b}-2} dV(\zeta)$. This change of variable now shows

$$(5.17) \le C|z|^\theta \int_U \frac{(1-|\zeta|^2)^{-\frac{1}{p}}}{|1 - z^b \overline{\zeta}|^2} |\zeta|^{\frac{q\theta}{b} + \frac{(\gamma+2)(q-1)}{b} - 2}\, dV(\zeta). \tag{5.18}$$

This integral converges if and only if $q\theta + (\gamma+2)(q-1) > 0$. Multiplying by the positive number $\frac{p}{q}$, we see this condition is equivalent to requiring that $p\theta + \gamma + 2 > 0$, which is guaranteed to hold. Indeed, in the proof of Proposition 5.2, θ was shown to be the smallest integer such that (i) $\theta \equiv a \mod b$, and (ii) $p\theta + \gamma + 2 > 0$. Now apply Proposition 5.12:

$$(5.18) \le C|z|^\theta (1-|z|^{2b})^{-\frac{1}{p}} = C\left(|z|^{\frac{\theta}{q}}(1-|z|^{2b})^{-\frac{1}{pq}}\right)^q = C\phi(z)^q,$$

giving us estimate (1) upon setting this constant to be C_1. Now consider

$$\int_U \left|k^U_{p,\gamma,a,b}(z,w)\right| \phi(z)^p \mu_\gamma(z)\, dV(z) \le C \int_U \frac{(|z||w|^{p-1})^\theta}{|1 - z^b \overline{w}^b |w|^{(p-2)b}|^2} \phi(z)^p \mu_\gamma(z)\, dV(z)$$

$$= C|w|^{(p-1)\theta} \int_U \frac{(1-|z|^{2b})^{-\frac{1}{q}}}{|1 - w^b |w|^{(p-2)b} \overline{z}^b|^2} |z|^{(1+\frac{p}{q})\theta+\gamma}\, dV(z). \tag{5.19}$$

Set $\xi = z^b$, which says that $|z| = |\xi|^{\frac{1}{b}}$ and $dV(z) = b^{-2}|\xi|^{\frac{2}{b}-2} dV(\xi)$. This shows that

$$(5.19) \le C|w|^{(p-1)\theta} \int_U \frac{(1-|\xi|^2)^{-\frac{1}{q}}}{|1 - w^b|w|^{(p-2)b}\overline{\xi}|^2} |\xi|^{\frac{p\theta}{b} + \frac{\gamma+2}{b} - 2}\, dV(\xi), \tag{5.20}$$

This integral converges since $p\theta + \gamma + 2 > 0$ (this is the same condition as before). Now apply Proposition 5.12 again to see

$$(5.20) \le C|w|^{(p-1)\theta}\big(1-|w|^{2b(p-1)}\big)^{-\frac{1}{q}} = C\left(|w|^{\frac{\theta}{q}}(1-|w|^{2b(p-1)})^{-\frac{1}{pq}}\right)^p = C\psi(z)^p,$$

giving estimate (2) upon setting this constant to be C_2. □

Proposition 5.16 yields several immediate consequences.

Corollary 5.21. *The absolute operator $(\boldsymbol{p}^U_{p,\gamma,a,b})^+$ is bounded on $L^p(U,\mu_\gamma)$, and consequently the subprojection $\boldsymbol{p}^U_{p,\gamma,a,b} : L^p(U,\mu_\gamma) \to A^p_{\mathfrak{B}}(U,\mu_\gamma)$ exists, where $\mathfrak{B} = A(U,p,\gamma,a,b)$.*

Proof. From definition (5.15), the estimates in Proposition 5.16 allow for immediate application of Proposition 5.11 (Schur's test) with $N = 1$, proving the boundedness of the absolute operator. Since the existence of $L^p(U,\mu_\gamma)$ estimates on $(\boldsymbol{p}^U_{p,\gamma,a,b})^+$ immediately implies them for the integral operator (5.14), Proposition 4.20 finishes the proof. □

Corollary 5.22. *The Monomial Basis Projections of the spaces $A^p(\mathbb{D}, \mu_\gamma)$ and $A^p(\mathbb{D}^*, \mu_\gamma)$ exist and are absolutely bounded.*

Proof. This follows from Corollary 5.21 on noting that the MBP of $A^p(U, \mu_\gamma)$, $U = \mathbb{D}$ or $\mathbb{D}^*$, coincides with the operator $\boldsymbol{p}^U_{p,\gamma,0,1}$. □

Corollary 5.23. *Let $U = \mathbb{D}^*$ or $\mathbb{D}$. The dual space of $A^p(U)$ admits the identification*

$$A^p(U)' \simeq A^q(U, \eta_q), \qquad \eta_q(\zeta) = (q-1)|\zeta|^{2q-4},$$

via the pairing (4.30)*, sending $(f, g) \mapsto \{f, g\}_{p,1}$, where $f \in A^p(U)$, $g \in A^q(U, \eta_q)$.*

Proof. Recalling the definition of a Reinhart power in (3.9), it it clear that in our case $U^{(m)} = U$ for every $m > 0$. Proposition 4.36 now gives the result. □

Remark 5.24. The duality pairing in Corollary 5.23 should be contrasted with the usual Hölder duality pairing of L^p and L^q. On the disc $\mathbb{D}$, the Hölder pairing restricts to a duality pairing of the holomorphic subspaces, yielding the identification $A^p(\mathbb{D})' \simeq A^q(\mathbb{D})$. On the punctured disc $\mathbb{D}^*$, the Hölder pairing fails to restrict to a holomorphic duality pairing and any attempt to identify $A^p(\mathbb{D}^*)'$ with $A^q(\mathbb{D}^*)$ fails. This is discussed further in Section 9.3. ◊

6. Transformation laws under monomial maps

6.1. **The canonical-bundle pullback.** If $\phi : \Omega_1 \to \Omega_2$ is a finite-sheeted holomorphic map of domains in $\mathbb{C}^n$, and f is a function on Ω_2, we can define a function on Ω_1 by setting

$$\phi^\sharp(f) = f \circ \phi \cdot \det \phi', \tag{6.1}$$

where $\phi'(z) : \mathbb{C}^n \to \mathbb{C}^n$ is the complex derivative of the map ϕ at $z \in \Omega_1$. If we think of Ω_1, Ω_2 as subsets of $\mathbb{R}^{2n}$ and ϕ as a smooth mapping, we can also consider the $2n \times 2n$ real Jacobian $D\phi$ of ϕ. Using the well-known relation $\det D\phi = |\det \phi'|^2$ between the two types of Jacobians, we see that $\phi^\sharp$ defines a continuous linear mapping of Hilbert spaces $\phi^\sharp : L^2(\Omega_2) \to L^2(\Omega_1)$, and restricts to a map $A^2(\Omega_2) \to A^2(\Omega_1)$. We will refer to $\phi^\sharp$ as the *canonical-bundle pullback* induced by ϕ, or informally as the *♯-pullback*, in order to distinguish it from another notion of pullback to be introduced in Section 7. If ϕ is a biholomorphism, then $\phi^\sharp$ is an isometric isomorphism of Hilbert spaces $L^2(\Omega_2) \cong L^2(\Omega_1)$ that restricts to an isometric isomorphism $A^2(\Omega_2) \cong A^2(\Omega_1)$. This biholomorphic invariance of Bergman spaces can be understood intrinsically by interpreting the Bergman space as a space of top-degree holomorphic forms (see [Kob59] or [Kra13, pp. 178 ff.]), and the map $\phi^\sharp$ as the pullback map of forms induced by the holomorphic map ϕ.

6.2. **Proper maps of quotient type.** In the classical theory of holomorphic mappings, one considers proper holomorphic mappings, and extends the biholomorphic invariance of Bergman spaces to such mappings via Bell's transformation formula (see [Bel81, Bel82, DF82, BC82]). In our applications, we will be concerned with a specific class of proper holomorphic mappings. We begin with the following definition (see [BCEM22]):

Definition 6.2. Let $\Omega_1, \Omega_2 \subset \mathbb{C}^n$ be domains, let $\Phi : \Omega_1 \to \Omega_2$ be a proper holomorphic mapping and $\Gamma \subset \mathrm{Aut}(\Omega_1)$ a finite group of biholomorphic automorphisms of Ω_1. We say Φ is of *quotient type with respect to* Γ if

(1) there exist closed lower-dimensional complex-analytic subvarieties $Z_j \subset \Omega_j$, $j = 1, 2$, such that Φ restricts to a covering map $\Phi : \Omega_1 \setminus Z_1 \to \Omega_2 \setminus Z_2$, and
(2) for each $z \in \Omega_2 \setminus Z_2$, the action of Γ on Ω_1 restricts to a transitive action on the fiber $\Phi^{-1}(z)$.

The group Γ is called *the group of deck transformations of* Φ.

There are several other names current in the literature for this notion, for example, the name "ramified covering" is used in the paper [DM23]. We will continue to use the terminology and notation of [BCEM22].

It is straightforward to see that the restricted map $\Phi : \Omega_1 \setminus Z_1 \to \Omega_2 \setminus Z_2$ is a so called *regular* covering map (see [Mas91, page 135 ff.]); i.e., the covering map gives rise to a biholomorphism between $\Omega_2 \setminus Z_2$ and the quotient $(\Omega_1 \setminus Z_1)/\Gamma$, where it can be shown that Γ acts properly and discontinuously on $\Omega_1 \setminus Z_1$. Further, it follows that Γ is in fact the full group of deck transformations of the covering map $\Phi : \Omega_1 \setminus Z_1 \to \Omega_2 \setminus Z_2$, and that this covering map has exactly $|\Gamma|$ sheets, where $|\Gamma|$ is the size of the group Γ. Notice that by analytic continuation, the relation $\Phi \circ \sigma = \Phi$ holds for each σ in Γ on all of Ω_1.

Definition 6.3. Given a domain $\Omega \subset \mathbb{C}^n$, a group $\Gamma \subset \mathrm{Aut}(\Omega)$ and a space $\mathfrak{F}$ of functions on Ω, we define

$$[\mathfrak{F}]^\Gamma = \{f \in \mathfrak{F} : f = \sigma^\sharp(f) \text{ for all } \sigma \in \Gamma\}, \tag{6.4}$$

where $\sigma^\sharp$ is the canonical-bundle pullback induced by σ as in (6.1). We say that functions in this space are Γ*-invariant in the* $\sharp$ *sense*.

Fix $1 < p < \infty$ and consider a proper holomorphic mapping $\Phi : \Omega_1 \to \Omega_2$ of quotient type with respect to group Γ. Define the function

$$\lambda_p = |\det \Phi'|^{2-p}. \tag{6.5}$$

This function arises as a weight in naturally occuring L^p-spaces. Indeed, in Proposition 4.5 of [BCEM22] it was shown that the map

$$\Phi^\sharp : L^p(\Omega_2) \to [L^p(\Omega_1, \lambda_p)]^\Gamma \tag{6.6}$$

is a homothetic isomorphism in the sense of (2.12) with

$$\left\|\Phi^\sharp(f)\right\|^p_{L^p(\Omega_1,\lambda_p)} = |\Gamma| \cdot \|f\|^p_{L^p(\Omega_2)}, \tag{6.7}$$

which restricts to a homothetic isomorphism of the holomorphic subspaces

$$\Phi^\sharp : A^p(\Omega_2) \to [A^p(\Omega_1, \lambda_p)]^\Gamma. \tag{6.8}$$

6.3. **Monomial maps.** Consider an $n \times n$ integer matrix A whose element in the j-th row and k-th column of A is a^j_k. Let a^j denote the j-th row of A, and a_k the k-th column. Letting the rows of A correspond to monomials e_{a^j}, define for $z \in \mathbb{C}^n$ the *matrix power*

$$z^A = \begin{pmatrix} e_{a^1}(z) \\ \vdots \\ e_{a^n}(z) \end{pmatrix} = \begin{pmatrix} z^{a^1} \\ \vdots \\ z^{a^n} \end{pmatrix} = \begin{pmatrix} z_1^{a^1_1} z_2^{a^1_2} \cdots z_n^{a^1_n} \\ \vdots \\ z_1^{a^n_1} z_2^{a^n_2} \cdots z_n^{a^n_n} \end{pmatrix}, \tag{6.9}$$

provided each component is defined. Define the *monomial map* Φ_A to be the rational map on $\mathbb{C}^n$ given by

$$\Phi_A(z) = z^A. \tag{6.10}$$

The following properties of monomial maps are known in the literature and references to their proofs are given at the end of the list. Three pieces of notation must first be explained: The element-wise exponential map $\exp : \mathbb{C}^n \to (\mathbb{C}^*)^n$ is given by $\exp(z) = (e^{z_1}, \dots, e^{z_n})$; if $z = (z_1, \dots, z_n)$, $w = (w_1, \dots, w_n)$ are points in $\mathbb{C}^n$, define their component-wise product to be $z \odot w = (z_1w_1, z_2w_2, \dots, z_nw_n)$; $\mathbb{1} \in \mathbb{Z}^{1\times n}$ is a row vector with 1 in each component.

(1) The following formula generalizes the familiar power-rule:

$$\det \Phi'_A = \det A \cdot e_{\mathbb{1}A-\mathbb{1}}. \tag{6.11a}$$

(2) If A is an invertible $n \times n$ matrix of nonnegative integers, then $\Phi_A : \mathbb{C}^n \to \mathbb{C}^n$ is a proper holomorphic map of quotient type with respect to the group

$$\Gamma_A = \{\sigma_\nu : \sigma_\nu(z) = \exp\left(2\pi i A^{-1}\nu\right) \odot z,\ \nu \in \mathbb{Z}^{n\times 1}\}. \tag{6.11b}$$

(3) The group Γ_A has exactly $|\det A|$ elements.

(4) The canonical-bundle pullback of the monomial e_α via the element $\sigma_\nu \in \Gamma_A$ is

$$\sigma_\nu^\sharp(e_\alpha) = e^{2\pi i(\alpha+\mathbb{1})A^{-1}\nu} \cdot e_\alpha. \tag{6.11c}$$

(5) The set of monomials that are Γ_A-invariant in the $\sharp$ sense as defined by (6.4) is

$$\{e_\alpha : \alpha = \beta A - \mathbb{1},\ \beta \in \mathbb{Z}^{1\times n}\}. \tag{6.11d}$$

Proof. Property (1) is proved in both [NP09, Lemma 4.2] and [BCEM22, Lemma 3.8]. Properties (2) and (3) can be found in [BCEM22, Theorem 3.12]. See also [Zwo00, NP21] for related results. Properties (4) and (5) are found in [BCEM22, Proposition 6.12]. □

6.4. **A hypothesis and some immediate consequences.** Throughout much of Sections 6 and 7, we will assume this general hypothesis in the statements of our results:

Hypothesis ★: *The domain $\Omega_2 \subset \mathbb{C}^n$ is pseudoconvex and Reinhardt, A is an $n \times n$ matrix of nonnegative integers such that $\det A \neq 0$, and $\Omega_1 = \Phi_A^{-1}(\Omega_2)$, the inverse image of Ω_2 under the monomial map $\Phi_A : \mathbb{C}^n \to \mathbb{C}^n$ defined in (6.10).*

This hypothesis has several elementary consequences. In statements using ★, we will assume the knowledge of the following:

(1) We obtain by restriction a proper holomorphic map

$$\Phi_A : \Omega_1 \to \Omega_2,$$

which is of quotient type with respect to the group Γ_A defined in (6.11b). We will also often omit explicit reference to the matrix A and simply write, e.g., $\Phi_A = \Phi$, $\Gamma_A = \Gamma$, etc.

(2) The domain Ω_1 is pseudoconvex and Reinhardt.

(3) The weight λ_p from (6.5) is given by

$$\lambda_p(\zeta) = |\det \Phi_A'(\zeta)|^{2-p} = |\det A|^{2-p} \prod_{k=1}^{n} |\zeta_k|^{(\mathbb{1}\cdot a_k - 1)(2-p)}, \tag{6.12}$$

where as before $\mathbb{1} \in \mathbb{Z}^{1\times n}$ has 1 in each component and a_k is the k-th column of A.

(4) By Proposition 3.14, the weight λ_p is admissible in the sense of Section 1.4.

(5) By (6.6) the canonical-bundle pullback gives a homothetic isomorphism

$$\Phi_A^\sharp : L^p(\Omega_2) \to [L^p(\Omega_1, \lambda_p)]^{\Gamma_A},$$

which by (6.8) restricts to a homothetic isomorphism of the holomorphic subspaces

$$\Phi_A^\sharp : A^p(\Omega_2) \to [A^p(\Omega_1, \lambda_p)]^{\Gamma_A}.$$

6.5. **Group invariant kernels and projections.** Throughout this section we assume Hypothesis ★ in all statements. Define the set of *Γ-invariant, p-allowable indices*:

$$\mathcal{S}_p^\Gamma(\Omega_1, \lambda_p) = \{\alpha \in \mathcal{S}_p(\Omega_1, \lambda_p) : \sigma^\sharp(e_\alpha) = e_\alpha \text{ for all } \sigma \in \Gamma\}. \tag{6.13}$$

This is by definition a subset of the p-allowable multi-indices and will underlie our transformation laws for the MBK and MBP.

Proposition 6.14. *We have equality of the following sets*

$$\left\{e_\beta : \beta \in \mathcal{S}_p^\Gamma(\Omega_1, \lambda_p)\right\} = \left\{\tfrac{1}{\det A}\Phi^\sharp(e_\alpha) : \alpha \in \mathcal{S}_p(\Omega_2)\right\}.$$

Proof. Thinking of α as an element of $\mathbb{Z}^{1\times n}$, computation shows that $e_\alpha \circ \Phi_A = e_{\alpha A}$. Thus $\Phi^\sharp(e_\alpha) = (\det A) e_{(\alpha+\mathbb{1})A-1}$, so we have

$$\left\{ \tfrac{1}{\det A} \Phi^\sharp(e_\alpha) : \alpha \in \mathcal{S}_p(\Omega_2) \right\} = \{ e_{(\alpha+\mathbb{1})A-\mathbb{1}} : \alpha \in \mathcal{S}_p(\Omega_2) \}. \tag{6.15}$$

Since the image of $A^p(\Omega_2)$ under $\Phi^\sharp$ is the space $[A^p(\Omega_1, \lambda_p)]^\Gamma$, we see

$$\{ e_{(\alpha+\mathbb{1})A-\mathbb{1}} : \alpha \in \mathcal{S}_p(\Omega_2) \} \subset \{ e_\beta : \beta \in \mathcal{S}_p(\Omega_1, \lambda_p),\ \sigma^\sharp(e_\beta) = e_\beta \text{ for all } \sigma \in \Gamma \}.$$

But since the map $\Phi^\sharp : A^p(\Omega_2) \to [A^p(\Omega_1, \lambda_p)]^\Gamma$ is linear, $\Phi^\sharp(f)$ must have more than one term in its Laurent expansion if f has more than one term in its Laurent expansion. Thus

$$\begin{aligned} \{ e_{(\alpha+\mathbb{1})A-\mathbb{1}} : \alpha \in \mathcal{S}_p(\Omega_2) \} &= \{ e_\beta : \beta \in \mathcal{S}_p(\Omega_1, \lambda_p),\ \sigma^\sharp(e_\beta) = e_\beta \text{ for all } \sigma \in \Gamma \} \\ &= \left\{ e_\beta : \beta \in \mathcal{S}_p^\Gamma(\Omega_1, \lambda_p) \right\}. \end{aligned}$$

□

Proposition 6.16. *The collection $\left\{ e_\alpha : \alpha \in \mathcal{S}_p^\Gamma(\Omega_1, \lambda_p) \right\}$ is a Schauder basis of the Banach space $[A^p(\Omega_1, \lambda_p)]^\Gamma$ in the sense of Definition 2.1, that is, with respect to the Schauder exhaustion by square partial sums.*

Proof. We need to show that for $f \in [A^p(\Omega_1, \lambda_p)]^\Gamma$, the partial sums

$$\sum_{\substack{|\alpha|_\infty \le N \\ \alpha \in \mathcal{S}_p^\Gamma(\Omega_1, \lambda_p)}} a_\alpha(f) e_\alpha$$

converge to f in the norm of $L^p(\Omega_1, \lambda_p)$. In view of Theorem 2.16, we need to show that the Laurent expansion of f contains only those monomials e_α with $\alpha \in \mathcal{S}_p^\Gamma(\Omega_1, \lambda_p)$. Now by Theorem 2.16, we have

$$f = \lim_{N\to\infty} \sum_{\substack{|\alpha|_\infty \le N \\ \alpha \in \mathcal{S}_p(\Omega_1, \lambda_p)}} a_\alpha(f) e_\alpha,$$

so for $\sigma_\nu \in \Gamma$ as in (6.11b) for some $\nu \in \mathbb{Z}^{n\times 1}$, applying the isometric automorphism $\sigma_\nu^\sharp$ of $A^p(\Omega, \lambda_p)$ to both sides, we have

$$\sigma_\nu^\sharp(f) = \lim_{N\to\infty} \sum_{\substack{|\alpha|_\infty \le N \\ \alpha \in \mathcal{S}_p(\Omega_1, \lambda_p)}} a_\alpha(f) \sigma_\nu^\sharp(e_\alpha) = \lim_{N\to\infty} \sum_{\substack{|\alpha|_\infty \le N \\ \alpha \in \mathcal{S}_p(\Omega_1, \lambda_p)}} e^{2\pi i (\alpha+\mathbb{1})A^{-1}\nu} a_\alpha(f) e_\alpha,$$

using the formula (6.11c). Since $\sigma_\nu^\sharp(f) = f$, the uniqueness of Laurent expansions now gives that for each α, either we have $a_\alpha(f) = 0$ or $\sigma_\nu^\sharp(e_\alpha) = e_\alpha$. Since this is true for each ν, the result follows. □

Definition 6.17. A bounded linear projection $\boldsymbol{P}^{\Omega_1}_{p,\lambda_p,\Gamma}$ from $[L^p(\Omega_1, \lambda_p)]^\Gamma$ onto $[A^p(\Omega_1, \lambda_p)]^\Gamma$ is called the Γ-*Invariant Monomial Basis Projection* of $A^p(\Omega_1, \lambda_p)$ if for $f \in [L^p(\Omega, \lambda)]^\Gamma$,

$$\boldsymbol{P}^{\Omega_1}_{p,\lambda_p,\Gamma}(f) = \lim_{N\to\infty} \sum_{\substack{|\alpha|_\infty \le N \\ \alpha \in \mathcal{S}_p^\Gamma(\Omega_1, \lambda_p)}} \widehat{a}_\alpha(f) e_\alpha, \tag{6.18}$$

where the series converges in the norm of $L^p(\Omega, \lambda)$. Here $\widehat{a}_\alpha : [L^p(\Omega_1, \lambda_p)]^\Gamma \to \mathbb{C}$ is the Hahn-Banach extension of the coefficient functional $a_\alpha : [A^p(\Omega_1, \lambda_p)]^\Gamma \to \mathbb{C}$.

Remark 6.19. By taking $A = [A^p(\Omega_1, \lambda_p)]^\Gamma$ and $E = [L^p(\Omega_1, \lambda_p)]^\Gamma$ in Corollary 2.5, we see that the Hahn-Banach extension $\widehat{a}_\alpha$ is unique. ◊

Remark 6.20. By taking $A = [A^p(\Omega_1, \lambda_p)]^\Gamma$ and $L = [L^p(\Omega_1, \lambda_p)]^\Gamma$ in Definition 2.6, we see that $\boldsymbol{P}^{\Omega_1}_{p,\lambda_p,\Gamma}$ is the basis projection from A onto L determined by $\{e_\alpha : \alpha \in \mathcal{S}^\Gamma_p(\Omega, \lambda_p)\}$. $\diamondsuit$

Along with the Γ-invariant MBP defined in (6.18), we can also consider the basis projection from $L^p(\Omega_1, \lambda_p)$ onto $[A^p(\Omega_1, \lambda_p)]^\Gamma$. In the notation of (4.19), this is the subprojection $\boldsymbol{P}_{\mathfrak{B}}$ corresponding to $\mathfrak{B} = \mathcal{S}^\Gamma_p(\Omega, \lambda)$. Denote this operator by $\boldsymbol{P}_{\mathcal{S}^\Gamma_p(\Omega,\lambda)}$.

Proposition 6.21. *The following statements are equivalent.*

(1) The subprojection $\boldsymbol{P}_{\mathcal{S}^\Gamma_p(\Omega,\lambda_p)} : L^p(\Omega_1, \lambda_p) \to [A^p(\Omega_1, \lambda_p)]^\Gamma$ *exists.*
(2) The Γ*-invariant MBP* $\boldsymbol{P}^{\Omega_1}_{p,\lambda_p,\Gamma} : [L^p(\Omega_1, \lambda_p)]^\Gamma \to [A^p(\Omega_1, \lambda_p)]^\Gamma$ *exists.*

Proof. To show (1) $\implies$ (2), it is clear that if $\boldsymbol{P}_{\mathcal{S}^\Gamma_p(\Omega,\lambda_p)}$ exists, the Γ-invariant MBP $\boldsymbol{P}^{\Omega_1}_{p,\lambda_p,\Gamma}$ also exists, since by Corollary 2.5 it is the restriction

$$\boldsymbol{P}_{\mathcal{S}^\Gamma_p(\Omega,\lambda)}\big|_{[L^p(\Omega_1,\lambda_p)]^\Gamma} = \boldsymbol{P}^{\Omega_1}_{p,\lambda_p,\Gamma}. \tag{6.22}$$

To show (2) $\implies$ (1), consider the operator Π on $L^p(\Omega_1, \lambda_p)$ given by

$$\Pi f = \frac{1}{|\Gamma|} \sum_{\sigma \in \Gamma} \sigma^\sharp f, \tag{6.23}$$

where $\sigma^\sharp$ is as in (6.1). It is easy to see that $\sigma^\sharp$ is an isometric self-isomorphism of the Banach space $L^p(\Omega_1, \lambda_p)$, and therefore we have

$$\|\Pi f\|_{L^p(\Omega_1,\lambda_p)} \le \frac{1}{|\Gamma|} \sum_{\sigma\in\Gamma} \left\|\sigma^\sharp f\right\|_{L^p(\Omega_1,\lambda_p)} = \frac{1}{|\Gamma|} \sum_{\sigma\in\Gamma} \|f\|_{L^p(\Omega_1,\lambda_p)} = \|f\|_{L^p(\Omega_1,\lambda_p)},$$

so that the operator norm $\|\Pi\|_{\mathrm{op}} \le 1$. On the other hand, it is easy to see that $\Pi f = f$ if and only if $f \in [L^p(\Omega_1, \lambda_p)]^\Gamma$, so $\Pi : L^p(\Omega_1, \lambda_p) \to [L^p(\Omega_1, \lambda_p)]^\Gamma$ is a projection operator of norm 1.

Now consider the composition of the two bounded projections

$$\boldsymbol{T} = \boldsymbol{P}^{\Omega_1}_{p,\lambda_p,\Gamma} \circ \Pi : L^p(\Omega_1, \lambda_p) \to [A^p(\Omega_1, \lambda_p)]^\Gamma,$$

which is itself a projection operator bounded in the norm of $L^p(\Omega_1, \lambda_p)$. Since the operator $\boldsymbol{P}^{\Omega_1}_{p,\lambda_p,\Gamma}$ takes the form as described by (6.18), we have

$$\boldsymbol{T} f = \sum_{\alpha\in\mathcal{S}^\Gamma_p(\Omega_1,\lambda_p)} (\widehat{a}_\alpha \circ \Pi)(f) e_\alpha.$$

Notice however that, the norms of the functionals $\widehat{a}_\alpha \circ \Pi : L^p(\Omega_1, \lambda_p) \to \mathbb{C}$ are given by

$$\|\widehat{a}_\alpha \circ \Pi\|_{L^p(\Omega_1,\lambda_p)'} \le \|\widehat{a}_\alpha\|_{([L^p(\Omega_1,\lambda_p)]^\Gamma)'} \|\Pi\|_{\mathrm{op}} = \|\widehat{a}_\alpha\|_{([L^p(\Omega_1,\lambda_p)]^\Gamma)'} = \|a_\alpha\|_{([A^p(\Omega_1,\lambda_p)]^\Gamma)'},$$

where the last equality holds since $\widehat{a}_\alpha$ is the Hahn-Banach extension of a_α to $[L^p(\Omega_1, \lambda_p)]^\Gamma$. It follows that $\widehat{a}_\alpha \circ \Pi = \widetilde{a}_\alpha$, where $\widetilde{a}_\alpha$ is the Hahn-Banach extension of a_α to $L^p(\Omega_1, \lambda_p)$ and consequently $\boldsymbol{T}$ is the basis projection $\boldsymbol{P}_{\mathcal{S}^\Gamma_p(\Omega,\lambda)}$, which is therefore bounded. $\square$

This leads to the following integral representation of the Γ-invariant MBP:

Proposition 6.24. *The* Γ*-invariant MBP* $\boldsymbol{P}^{\Omega_1}_{p,\lambda_p,\Gamma} : [L^p(\Omega_1, \lambda_p)]^\Gamma \to [A^p(\Omega_1, \lambda_p)]^\Gamma$ *admits the integral representation*

$$\boldsymbol{P}^{\Omega_1}_{p,\lambda_p,\Gamma} f(z) = \int_{\Omega_1} K^{\Omega_1}_{p,\lambda_p,\Gamma}(z, w) f(w) \lambda_p(w)\, dV(w), \qquad f \in [L^p(\Omega_1, \lambda_p)]^\Gamma, \tag{6.25}$$

where $K^{\Omega_1}_{p,\lambda_p,\Gamma}$ *is the subkernel determined by* $\mathfrak{B} = \mathcal{S}_p^\Gamma(\Omega_1, \lambda_p)$ *as in* (4.18)*:*

$$K^{\Omega_1}_{p,\lambda_p,\Gamma}(z,w) = \sum_{\alpha\in\mathcal{S}_p^\Gamma(\Omega_1,\lambda_p)} \frac{e_\alpha(z)\overline{\chi_p^* e_\alpha(w)}}{\|e_\alpha\|^p_{p,\lambda_p}} = \sum_{\alpha\in\mathcal{S}_p^\Gamma(\Omega_1,\lambda_p)} \frac{e_\alpha(z)\overline{e_\alpha(w)}|e_\alpha(w)|^{p-2}}{\|e_\alpha\|^p_{p,\lambda_p}}. \tag{6.26}$$

We call $K^{\Omega_1}_{p,\lambda_p,\Gamma}$ *the* Γ-invariant Monomial Basis Kernel *of* $A^p(\Omega_1, \lambda_p)$.

Proof. Proposition 4.20 shows that for $f \in L^p(\Omega_1, \lambda_p)$ the integral operator in (6.25) represents the subprojection $\boldsymbol{P}_{\mathcal{S}_p^\Gamma(\Omega,\lambda)} : L^p(\Omega_1, \lambda_p) \to [A^p(\Omega_1, \lambda_p)]^\Gamma$, which is known to exist by Proposition 6.21 above. Restricting f to the subspace $[L^p(\Omega_1, \lambda_p)]^\Gamma$ and using the relation (6.22) the result follows. □

6.6. **A transformation law.** Assuming the conditions of Hypothesis ★, we obtain a transformation law for the Monomial Basis Projection under monomial maps.

Theorem 6.27. *The following statements are equivalent.*

(1) The MBP $\boldsymbol{P}^{\Omega_2}_{p,1} : L^p(\Omega_2) \to A^p(\Omega_2)$ *exists.*
(2) The Γ*-invariant MBP* $\boldsymbol{P}^{\Omega_1}_{p,\lambda_p,\Gamma} : [L^p(\Omega_1, \lambda_p)]^\Gamma \to [A^p(\Omega_1, \lambda_p)]^\Gamma$ *exists.*

When these equivalent statements hold,

$$\Phi^\sharp \circ \boldsymbol{P}^{\Omega_2}_{p,1} = \boldsymbol{P}^{\Omega_1}_{p,\lambda_p,\Gamma} \circ \Phi^\sharp \tag{6.28}$$

as bounded operators on $L^p(\Omega_2)$*. Additionally, if the MBP* $\boldsymbol{P}^{\Omega_1}_{p,\lambda_p} : L^p(\Omega_1, \lambda_p) \to A^p(\Omega_1, \lambda_p)$ *exists, the following diagram commutes (where* ↪ *denotes inclusion, and* ≅ *denotes homothetic isomorphim):*

$$\begin{array}{ccccc} L^p(\Omega_2) & \xrightarrow[\cong]{\Phi^\sharp} & [L^p(\Omega_1,\lambda_p)]^\Gamma & \hookrightarrow & L^p(\Omega_1,\lambda_p) \\ \downarrow{\scriptstyle \boldsymbol{P}^{\Omega_2}_{p,1}} & & \downarrow{\scriptstyle \boldsymbol{P}^{\Omega_1}_{p,\lambda_p,\Gamma}} & & \downarrow{\scriptstyle \boldsymbol{P}^{\Omega_1}_{p,\lambda_p}} \\ A^p(\Omega_2) & \xrightarrow[\cong]{\Phi^\sharp} & [A^p(\Omega_1,\lambda_p)]^\Gamma & \hookrightarrow & A^p(\Omega_1,\lambda_p). \end{array} \tag{6.29}$$

Remark 6.30. We can relax the last hypothesis on the existence of $\boldsymbol{P}^{\Omega_1}_{p,\lambda_p}$ and give an analogous diagram where the rightmost vertical arrow is replaced by the integral operator below. Assuming (1) and (2), the operator $\boldsymbol{P}^{\Omega_1}_{p,\lambda_p,\Gamma}$ admits an integral representation involving the full MBK of the Bergman space $A^p(\Omega_1, \lambda_p)$:

$$\boldsymbol{P}^{\Omega_1}_{p,\lambda_p,\Gamma} f(z) = \int_{\Omega_1} K^{\Omega_1}_{p,\lambda_p}(z,w) f(w) \lambda_p(w)\, dV(w), \qquad f \in [L^p(\Omega_1, \lambda_p)]^\Gamma.$$

This lets us sidestep direct reference to the Γ-invariant MBK. Of course, the operator represented by this integral coincides with $\boldsymbol{P}^{\Omega_1}_{p,\lambda_p}$ when the latter operator exists. ◊

Proof. By (6.6), the map $\Phi^\sharp$ is a homothetic isomorphism from $L^p(\Omega_2)$ onto $[L^p(\Omega_1, \lambda_p)]^\Gamma$ which by (6.8) restricts to homothetic isomorphism of the holomorphic subspaces $A^p(\Omega_2)$ and $[A^p(\Omega_1, \lambda_p)]^\Gamma$. Proposition 2.13 therefore says that $\boldsymbol{P}^{\Omega_2}_{p,1}$ exists if and only if the induced basis projection $\boldsymbol{Q}$ from $[L^p(\Omega_1, \lambda_p)]^\Gamma$ onto $[A^p(\Omega_1, \lambda_p)]^\Gamma$ exists, where $\boldsymbol{Q}$ is determined by the induced Schauder basis of $[A^p(\Omega_1, \lambda_p)]^\Gamma$ consisting of the set

$$\{\Phi^\sharp(e_\alpha) : \alpha \in \mathcal{S}_p(\Omega_2)\}, \tag{6.31}$$

together with its *Schauder exhaustion by square partial sums in the index* α. When these equivalent operators exist, the following diagram commutes:

$$\begin{tikzcd} L^p(\Omega_2) \arrow[r, "\Phi^\sharp", "\cong"'] \arrow[d, "\boldsymbol{P}^{\Omega_2}_{p,1}"] & {[L^p(\Omega_1,\lambda_p)]^\Gamma} \arrow[d, "\boldsymbol{Q}"] \\ A^p(\Omega_2) \arrow[r, "\Phi^\sharp", "\cong"'] & {[A^p(\Omega_1,\lambda_p)]^\Gamma} . \end{tikzcd}$$

Since Proposition 2.8 allows for a re-scaling of the basis elements in (6.31) without changing the operator, $\boldsymbol{Q}$ can be equivalently viewed as the basis projection from $[L^p(\Omega_1,\lambda_p)]^\Gamma$ onto $[A^p(\Omega_1,\lambda_p)]^\Gamma$ corresponding to the Schauder basis elements

$$\left\{\tfrac{1}{\det A}\Phi^\sharp(e_\alpha) : \alpha \in \mathcal{S}_p(\Omega_2)\right\} = \{e_{(\alpha+\mathbb{1})A-\mathbb{1}} : \alpha \in \mathcal{S}_p(\Omega_2)\}, \tag{6.32}$$

still under the exhaustion by square partial sums in α. Recall that the equality of sets in (6.32) was previously noted in (6.15). The Schauder exhaustion corresponding to the set of monomials on the right hand side of (6.32) can be expressed in more explicit terms by

$$\mathfrak{B}_N = \{(\alpha+\mathbb{1})A - \mathbb{1} : |\alpha|_\infty \le N\} \cap \mathcal{S}_p(\Omega_1,\lambda_p). \tag{6.33}$$

Proposition 6.14 shows that, by setting $\beta = (\alpha+\mathbb{1})A-\mathbb{1}$, the Schauder basis in (6.32) determined by the exhausting sequence $\{\mathfrak{B}_N\}_{N=1}^\infty$ can re-expressed as the set of monomials

$$\{e_\beta : \beta \in \mathcal{S}_p^\Gamma(\Omega_1,\lambda_p)\},$$

together with its *Schauder exhaustion by the sets*

$$\widetilde{\mathfrak{B}}_N = \left\{\beta : \left|(\beta+\mathbb{1})A^{-1} - \mathbb{1}\right|_\infty \le N\right\} \cap \mathcal{S}_p(\Omega_1,\lambda_p).$$

Consequently, letting $f \in [L^p(\Omega_1,\lambda_p)]^\Gamma$, the basis projection $\boldsymbol{Q}$ is given by

$$\boldsymbol{Q}f = \lim_{N\to\infty} \sum_{\beta\in\widetilde{\mathfrak{B}}_N} \widehat{a}_\beta(f) e_\beta, \tag{6.34}$$

where $\widehat{a}_\beta$ is the unique Hahn-Banach extension of the functional a_β to $[L^p(\Omega_1,\lambda_p)]^\Gamma$.

We now claim that the indexing sequence $\{\widetilde{\mathfrak{B}}_N\}_{N=1}^\infty$ appearing in the sum (6.34) can be replaced by the standard Schauder exhaustion by square partial sums

$$\mathfrak{C}_N = \{\beta \in \mathcal{S}_p^\Gamma(\Omega_1,\lambda_p) : |\beta|_\infty \le N\}$$

without changing the operator $\boldsymbol{Q}$.

To see this, consider the operator $\boldsymbol{Q}\circ\Pi$, where $\Pi : L^p(\Omega_1,\lambda_p) \to [L^p(\Omega_1,\lambda_p)]^\Gamma$ is the "norm 1" linear projection defined in (6.23). By giving an argument identical to the one in the last paragraph of Proposition 6.21, we see that for $\beta \in \mathcal{S}_p^\Gamma(\Omega_1,\lambda_p)$, $\widehat{a}_\beta\circ\Pi = \widetilde{a}_\beta$, where $\widetilde{a}_\beta : L^p(\Omega_1,\lambda_p) \to \mathbb{C}$ is the unique Hahn-Banach extension of the (already partially extended) functional $\widehat{\alpha}_\beta : [L^p(\Omega_1,\lambda_p)]^\Gamma \to \mathbb{C}$. This implies that $\boldsymbol{Q}\circ\Pi$ is the subprojection (see (4.19)) of $L^p(\Omega_1,\lambda_p) \to [A^p(\Omega_1,\lambda_p)]^\Gamma$ determined by the Schauder exhaustion $\{\widetilde{\mathfrak{B}}_N\}_{N=1}^\infty$.

Recall now the Γ-invariant MBP from Definition 6.17, which can be written as

$$\boldsymbol{P}^{\Omega_1}_{p,\lambda_p,\Gamma} f = \lim_{N\to\infty} \sum_{\beta\in\mathfrak{C}_N} \widehat{a}_\alpha(f) e_\alpha.$$

In the proof of Proposition 6.21, we have seen that $\boldsymbol{P}^{\Omega_1}_{p,\lambda_p,\Gamma}\circ\Pi$ is the subprojection in the sense of (4.19) from $L^p(\Omega_1,\lambda_p) \to [A^p(\Omega_1,\lambda_p)]^\Gamma$ determined by the exhaustion $\{\mathfrak{C}_N\}_{N=1}^\infty$.

Corollary 4.22 equates these two subprojections, saying that the subprojections are independent of the choices of Schauder exhaustions. In other words, for $g \in L^p(\Omega_1, \lambda_p)$,

$$\boldsymbol{Q} \circ \Pi(g) = \lim_{N\to\infty} \sum_{\beta \in \widetilde{\mathfrak{B}}_N} (\widehat{a}_\beta \circ \Pi)(g) e_\beta = \lim_{N\to\infty} \sum_{\beta \in \mathfrak{C}_N} (\widehat{a}_\beta \circ \Pi)(g) e_\beta = \boldsymbol{P}^{\Omega_1}_{p,\lambda_p,\Gamma} \circ \Pi(g).$$

Now by restricting both sides to $f \in [L^p(\Omega_1, \lambda_p)]^\Gamma$, we see that

$$\boldsymbol{Q}(f) = \lim_{N\to\infty} \sum_{\beta \in \widetilde{\mathfrak{B}}_N} \widehat{a}_\beta(f) e_\beta = \lim_{N\to\infty} \sum_{\beta \in \mathfrak{C}_N} \widehat{a}_\beta(f) e_\beta = \boldsymbol{P}^{\Omega_1}_{p,\lambda_p,\Gamma}(f), \tag{6.35}$$

and so as operators $\boldsymbol{Q} = \boldsymbol{P}^{\Omega_1}_{p,\lambda_p,\Gamma}$. This completes the proof that the left quadrilateral in the diagram (6.29) commutes and (6.28) holds.

We see from (6.11b) that each $\sigma \in \Gamma$ is a biholomorphic automorphism of the Reinhardt domain Ω_1 which multiplies each coordinate by a complex number of absolute value 1. This means the complex Jacobian matrix σ' is a diagonal matrix whose entries are complex numbers of absolute value 1, which implies $|\det \sigma'| = 1$. It follows that we have an isometric isomorphism of Banach spaces induced by σ

$$\sigma^\sharp : L^p(\Omega_1, \lambda_p) \to L^p(\Omega_1, \lambda_p),$$

which restricts to an isometric isomorphism of the holomorphic subspaces. Therefore by Proposition 2.13, if the MBP $\boldsymbol{P}^{\Omega_1}_{p,\lambda_p}$ exists, we have a commutative diagram:

$$\begin{array}{ccc} L^p(\Omega_1, \lambda_p) & \xrightarrow[\cong]{\sigma^\sharp} & L^p(\Omega_1, \lambda_p) \\ \Big\downarrow \boldsymbol{P}^{\Omega_1}_{p,\lambda_p} & & \Big\downarrow \boldsymbol{R} \\ A^p(\Omega_1, \lambda_p) & \xrightarrow[\cong]{\sigma^\sharp} & A^p(\Omega_1, \lambda_p), \end{array}$$

where $\boldsymbol{R}$ is the basis projection from $L^p(\Omega_1, \lambda_p)$ onto $A^p(\Omega_1, \lambda_p)$ with respect to the induced basis $\{\sigma^\sharp(e_\alpha) : \alpha \in \mathcal{S}_p(\Omega_1, \lambda_p)\}$. But by (6.11c), we have that $\sigma^\sharp(e_\alpha) = c_\alpha e_\alpha$ for some complex number c_α of absolute value 1, so it follows by Proposition 2.8 that in fact $\boldsymbol{R} = \boldsymbol{P}^{\Omega_1}_{p,\lambda_p}$. Therefore,

$$\sigma^\sharp \circ \boldsymbol{P}^{\Omega_1}_{p,\lambda_p} = \boldsymbol{P}^{\Omega_1}_{p,\lambda_p} \circ \sigma^\sharp$$

as operators on $L^p(\Omega_1, \lambda_p)$. Now suppose that $f \in [L^p(\Omega_1, \lambda_p)]^\Gamma$, so that for each $\sigma \in \Gamma$ we have $\sigma^\sharp f = f$. Therefore for each $\sigma \in \Gamma$:

$$\boldsymbol{P}^{\Omega_1}_{p,\lambda_p} f = \boldsymbol{P}^{\Omega_1}_{p,\lambda_p}(\sigma^\sharp f) = \sigma^\sharp(\boldsymbol{P}^{\Omega_1}_{p,\lambda_p} f), \tag{6.36}$$

which shows that $\boldsymbol{P}^{\Omega_1}_{p,\lambda_p}$ maps $[L^p(\Omega_1, \lambda_p)]^\Gamma$ into the subspace $[A^p(\Omega_1, \lambda_p)]^\Gamma$.

Using the definition of the MBP together with the fact that $\sigma^\sharp$ is an isometric isomorphism, we can write the first and last terms in (6.36) as

$$\boldsymbol{P}^{\Omega_1}_{p,\lambda_p} f = \sum_{\alpha \in \mathcal{S}_p(\Omega_1,\lambda_p)} \widetilde{a}_\alpha(f) e_\alpha, \qquad \text{and} \qquad \sigma^\sharp(\boldsymbol{P}^{\Omega_1}_{p,\lambda_p} f) = \sum_{\alpha \in \mathcal{S}_p(\Omega_1,\lambda_p)} \widetilde{a}_\alpha(f) \sigma^\sharp(e_\alpha),$$

where convergence of square partial sums in the norm of $L^p(\Omega_1, \lambda_p)$ is implied. Since these are equal, by the uniqueness of expansions with respect to Schauder bases, we see that the only nonzero terms correspond to those monomials e_α for which $e_\alpha = \sigma^\sharp(e_\alpha)$. Since this holds for each $\sigma \in \Gamma$, only those monomials e_α with $\alpha \in \mathcal{S}^\Gamma_p(\Omega_1, \lambda_p)$ occur in the series, and we conclude that for each $f \in [L^p(\Omega_1, \lambda_p)]^\Gamma$ we have

$$\boldsymbol{P}^{\Omega_1}_{p,\lambda_p} f = \sum_{\alpha \in \mathcal{S}^\Gamma_p(\Omega_1,\lambda_p)} \widetilde{a}_\alpha(f) e_\alpha = \boldsymbol{P}^{\Omega_1}_{p,\lambda_p,\Gamma} f.$$

This completes the proof of the commutativity of the right rectangle in (6.29). □

7. Transforming the absolute monomial basis operator

7.1. **Density-bundle pullbacks.** In this section we consider a more general description of the pullback operation for real maps. This arises naturally by thinking of functions as coefficients of *densities* rather than of forms. Let Ω_1, Ω_2 be open sets in $\mathbb{R}^d$, and $\phi : \Omega_1 \to \Omega_2$ a smooth map. For a function f on Ω_2, define the *(one-half) density-bundle pullback* of f, denoted by $\phi_\flat f$, to be the function on Ω_1 given by

$$\phi_\flat f = f \circ \phi \cdot |\det D\phi|^{\frac{1}{2}}\,, \tag{7.1}$$

where as before, $D\phi$ denotes the $d \times d$ Jacobian matrix of ϕ. One can think of $\phi_\flat$ invariantly as the pullback operation on the space of $\frac{1}{2}$-densities (see [Nic07, pp. 113 ff.]). From the change of variables formula, it follows that if $\phi : \Omega_1 \to \Omega_2$ is a diffeomorphism, then the induced map $\phi_\flat : L^2(\Omega_2) \to L^2(\Omega_1)$ is an isometric isomorphism of Hilbert spaces. When Ω_1, Ω_2 are domains in a complex Euclidean space $\mathbb{C}^n$ and the map $\phi : \Omega_1 \to \Omega_2$ is holomorphic, then

$$\phi_\flat f = f \circ \phi \cdot \left|\det \phi'\right|, \tag{7.2}$$

where as before, ϕ' denotes the complex derivative.

The density-bundle pullback leads to a new notion of invariant functions (note carefully the location of the symbol Γ in (7.4), as compared to (6.4)):

Definition 7.3. Given a domain $\Omega \subset \mathbb{C}^n$, group $\Gamma \subset \mathrm{Aut}(\Omega)$ and function space $\mathfrak{F}$ consisting of functions on Ω, define the subspace

$$[\mathfrak{F}]_\Gamma = \{f \in \mathfrak{F} : f = \sigma_\flat(f) \text{ for all } \sigma \in \Gamma\}, \tag{7.4}$$

where $\sigma_\flat$ is the density-bundle pullback in (7.2). Functions in $[\mathfrak{F}]_\Gamma$ are said to be *"Γ-invariant in the $\flat$ sense"*, or simply *"$\flat$-invariant"* when Γ is clear from context.

Under proper holomorphic mappings, the behavior of the $\flat$-pullback regarding L^p-spaces and $\flat$-invariant functions is directly analogous to the behavior of the $\sharp$-pullback regarding L^p-spaces and $\sharp$-invariant functions (as described in (6.6) above):

Proposition 7.5. *Let $1 < p < \infty$, Ω_1, Ω_2 be domains in $\mathbb{C}^n$ and $\Phi : \Omega_1 \to \Omega_2$ be a proper holomorphic map of quotient type with respect to the group $\Gamma \subset \mathrm{Aut}(\Omega_1)$. Then*

$$\Phi_\flat : L^p(\Omega_2) \to [L^p(\Omega_1, \lambda_p)]_\Gamma \tag{7.6}$$

is a homothetic isomorphism.

Proof. Let $f \in L^p(\Omega_2)$. By Definition 6.2, there exist varieties $Z_1 \subset \Omega_1$, $Z_2 \subset \Omega_2$ such that $\Phi : \Omega_1 \backslash Z_1 \to \Omega_2 \backslash Z_2$ is a regular covering map of order $|\Gamma|$. Using the change of variables formula (accounting for the fact that Φ is a $|\Gamma|$-to-one mapping), we see

$$|\Gamma|\, \|f\|^p_{L^p(\Omega_2)} = |\Gamma| \int_{\Omega_2 \backslash Z_2} |f|^p\, dV = \int_{\Omega_1 \backslash Z_1} |f \circ \Phi|^p |\det \Phi'|^2\, dV = \|\Phi_\flat(f)\|^p_{L^p(\Omega_1, \lambda_p)}\,, \tag{7.7}$$

which shows $\Phi_\flat(f) \in L^p(\Omega_1, \lambda_p)$. Observe also that for any $\sigma \in \Gamma$,

$$\begin{aligned}\sigma_\flat(f \circ \Phi \cdot |\det \Phi'|) = (f \circ \Phi) \circ \sigma \cdot |\det(\Phi' \circ \sigma)| \cdot |\det \sigma'| &= f \circ (\Phi \circ \sigma) \cdot |\det(\Phi \circ \sigma)'| \\ &= f \circ \Phi \cdot |\det \Phi'|,\end{aligned}$$

showing that $\Phi_\flat(f) \in [L^p(\Omega_1, \lambda_p)]_\Gamma$. This shows $\Phi_\flat$ is a homothetic isomorphism of $L^p(\Omega_2)$ onto a subspace of $[L^p(\Omega_1, \lambda_p)]_\Gamma$.

It remains to show that this image is the full space. By a partition of unity argument, it is sufficient to show that a function $g \in [L^p(\Omega_1, \lambda_p)]_\Gamma$ is in the range of $\Phi_\flat$, provided the support of g is contained in a set of the form $\Phi^{-1}(U)$, where U is an connected open subset of $\Omega_2 \setminus Z_2$ evenly covered by the covering map Φ. Notice that $\Phi^{-1}(U)$ is a disjoint collection of connected open components each biholomorphic to U, and if U_0 is one of them, $\Phi^{-1}(U)$ is the disjoint union $\bigcup_{\sigma\in\Gamma} \sigma(U_0)$. Let $\Psi : U \to U_0$ be the local inverse of Φ onto U_0. Define f_0 on U by

$$f_0 = \Psi_\flat \left(g|_{U_0}\right). \tag{7.8}$$

We claim that f_0 is defined independently of the choice of the component U_0 of $\Phi^{-1}(U)$. Indeed, any other choice is of the form $\sigma(U_0)$ for some $\sigma \in \Gamma$ and the corresponding local inverse is $\sigma \circ \Psi$. But we have

$$(\sigma \circ \Psi)_\flat \left(g|_{\sigma(U_0)}\right) = \Psi_\flat \circ \sigma_\flat \left(g|_{\sigma(U_0)}\right) = \Psi_\flat \left(g|_{U_0}\right) = f_0,$$

where we have used the fact that $\sigma_\flat g = g$ since $g \in [L^p(\Omega_1, \lambda_p)]_\Gamma$. A partition of unity argument completes the proof. □

7.2. **Transforming the Absolute Monomial Basis Operator.** All statements in this section assume Hypothesis ★. See Section 6.4 to recall the assumptions on the proper holomorphic monomial map $\Phi : \Omega_1 \to \Omega_2$, the group Γ, the weight λ_p, etc.

We prove a transformation law for *Absolute Monomial Basis Operator (AMBO)*, the absolute operator in the sense of (4.24) associated to the MBK $K^{\Omega_2}_{p,1}$:

$$(\boldsymbol{P}^{\Omega_2}_{p,1})^+ f(z) = \int_{\Omega_2} \left|K^{\Omega_2}_{p,1}(z, w)\right| f(w)\, dV(w), \qquad f \in C_c(\Omega_2). \tag{7.9}$$

This operator is defined on the subspace $C_c(\Omega_2)$, and exists as a bounded operator if and only if it satisfies L^p-estimates on $C_c(\Omega_2)$. We relate it to a certain "Γ-invariant AMBO" on Ω_1, namely, the one obtained by integration against the absolute value of the Γ-invariant MBK from (6.26):

$$(\boldsymbol{P}^{\Omega_1}_{p,\lambda_p,\Gamma})^+ f(z) = \int_{\Omega_1} \left|K^{\Omega_1}_{p,\lambda_p,\Gamma}(z, w)\right| f(w)\lambda_p(w)dV(w), \qquad f \in C_c(\Omega_1), \tag{7.10}$$

where λ_p is the multi-radial weight given in (6.12) arising from Φ. These operators are closely related via the density-bundle pullback of Section 7.1:

Theorem 7.11. *The following statements are equivalent:*

(1) $(\boldsymbol{P}^{\Omega_2}_{p,1})^+$ *extends to a bounded operator* $(\boldsymbol{P}^{\Omega_2}_{p,1})^+ : L^p(\Omega_2) \to L^p(\Omega_2)$.

(2) $(\boldsymbol{P}^{\Omega_1}_{p,\lambda_p,\Gamma})^+$ *extends to a bounded operator* $(\boldsymbol{P}^{\Omega_1}_{p,\lambda_p,\Gamma})^+ : [L^p(\Omega_1, \lambda_p)]_\Gamma \to [L^p(\Omega_1, \lambda_p)]_\Gamma$.

When these equivalent statements hold,

$$\Phi_\flat \circ (\boldsymbol{P}^{\Omega_2}_{p,1})^+ = (\boldsymbol{P}^{\Omega_1}_{p,\lambda_p,\Gamma})^+ \circ \Phi_\flat \tag{7.12}$$

as operators on $L^p(\Omega_2)$*, which is to say that the following diagram commutes*

$$\begin{array}{ccc} L^p(\Omega_2) & \xrightarrow[\cong]{\Phi_\flat} & L^p(\Omega_1, \lambda_p)_\Gamma \\ \Big\downarrow (\boldsymbol{P}^{\Omega_2}_{p,1})^+ & & \Big\downarrow (\boldsymbol{P}^{\Omega_1}_{p,\lambda_p,\Gamma})^+ \\ L^p(\Omega_2) & \xrightarrow{\Phi_\flat} & L^p(\Omega_1, \lambda_p)_\Gamma. \end{array} \tag{7.13}$$

The key idea in the proof of Theorem 7.11 is a kernel transformation law reminiscent of Bell's law [Bel81, Bel82] for the Bergman kernel under proper holomorphic maps:

Proposition 7.14. *The Monomial Basis Kernel admits the transformation law*

$$K^{\Omega_1}_{p,\lambda_p,\Gamma}(z, w) = \frac{1}{|\Gamma|} \det \Phi'(z) \cdot K^{\Omega_2}_{p,1}(\Phi(z), \Phi(w)) \cdot \frac{|\det \Phi'(w)|^p}{\det \Phi'(w)}. \tag{7.15}$$

Proof. Starting from the series representation for $K^{\Omega_2}_{p,1}(z,w)$ in (3.5), we have

$$\begin{aligned}
K^{\Omega_2}_{p,1}(\Phi(z),\Phi(w)) &= \sum_{\alpha\in\mathcal{S}_p(\Omega_2)} \frac{e_\alpha(\Phi(z))\overline{e_\alpha(\Phi(w))}|e_\alpha(\Phi(w))|^{p-2}}{\|e_\alpha\|^p_{L^p(\Omega_2)}} \\
&= |\Gamma| \sum_{\alpha\in\mathcal{S}_p(\Omega_2)} \frac{e_\alpha(\Phi(z))\overline{e_\alpha(\Phi(w))}|e_\alpha(\Phi(w))|^{p-2}}{\|\Phi^\sharp(e_\alpha)\|^p_{L^p(\Omega_1,\lambda_p)}}, \qquad (7.16)
\end{aligned}$$

since by (6.7), the homothetic isomorphism $\Phi^\sharp$ scales norms uniformly for each $f\in L^p(\Omega_2)$ by $|\Gamma|\cdot\|f\|^p_{L^p(\Omega_2)} = \left\|\Phi^\sharp(f)\right\|^p_{L^p(\Omega_1,\lambda_p)}$. Now use the definition of $\Phi^\sharp$ to write

$$\begin{aligned}
(7.16) &= |\Gamma|\frac{\det\Phi'(w)}{\det\Phi'(z)|\det\Phi'(w)|^p} \sum_{\alpha\in\mathcal{S}_p(\Omega_2)} \frac{\Phi^\sharp(e_\alpha)(z)\overline{\Phi^\sharp(e_\alpha)(w)}|\Phi^\sharp(e_\alpha)(w)|^{p-2}}{\|\Phi^\sharp(e_\alpha)\|^p_{L^p(\Omega_1,\lambda_p)}} \\
&= |\Gamma|\frac{\det\Phi'(w)}{\det\Phi'(z)|\det\Phi'(w)|^p} \sum_{\beta\in\mathcal{S}^\Gamma_p(\Omega_1,\lambda_p)} \frac{e_\beta(z)\overline{e_\beta(w)}|e_\beta(w)|^{p-2}}{\|e_\beta\|^p_{L^p(\Omega_1,\lambda_p)}} \qquad (7.17) \\
&= |\Gamma|\frac{\det\Phi'(w)}{\det\Phi'(z)|\det\Phi'(w)|^p}\cdot K^{\Omega_1}_{p,\lambda_p,\Gamma}(z,w). \qquad (7.18)
\end{aligned}$$

Equation (7.17) follows from Proposition 6.14 and (7.18) follows from the definition of the Γ-invariant MBK given in (6.26). This completes the proof. □

Proof of Theorem 7.11. Proposition 7.5 and (7.7) show that $\Phi_\flat : L^p(\Omega_2)\to L^p(\Omega_1,\lambda_p)_\Gamma$ is a homothetic isomorphism and that $\|\Phi_\flat f\|^p_{L^p(\Omega_1,\lambda_p)} = |\Gamma|\,\|f\|^p_{L^p(\Omega_2)}$. For $f\in C_c(\Omega_2)$,

$$\begin{aligned}
\Phi_\flat\circ(\boldsymbol{P}^{\Omega_2}_{p,1})^+ f(z) &= |\det\Phi'(z)|\int_{\Omega_2}\left|K^{\Omega_2}_{p,1}(\Phi(z),w)\right| f(w)\,dV(w) \\
&= \frac{|\det\Phi'(z)|}{|\Gamma|}\int_{\Omega_1}\left|K^{\Omega_2}_{p,1}(\Phi(z),\Phi(w))\right| f(\Phi(w))\cdot|\det\Phi'(w)|^2\,dV(w) \\
&= \int_{\Omega_1}\left|K^{\Omega_1}_{p,\lambda_p,\Gamma}(z,w)\right|\Phi_\flat f(w)\lambda_p(w)\,dV(w) \qquad (7.19) \\
&= (\boldsymbol{P}^{\Omega_1}_{p,\lambda_p,\Gamma})^+\circ\Phi_\flat f(z).
\end{aligned}$$

Equality in (7.19) uses the kernel transformation law in (7.15), and the final line makes sense since the properness of Φ guarantees $\Phi_\flat f\in[C_c(\Omega_1)]_\Gamma$.

The fact that $C_c(\Omega_2)$ is dense in $L^p(\Omega_2)$, along with the fact that its image $\Phi_\flat\left(C_c(\Omega_2)\right) = [C_c(\Omega_1)]_\Gamma$ is dense in $[L^p(\Omega_1,\lambda_p)]_\Gamma$ shows that statements (1) and (2) are equivalent. When these statements hold, equation (7.12) and Diagram (7.13) follow immediately. □

Remark 7.20. The crucial difference between the MBP transformation law (Theorem 6.27) and its AMBO counterpart (Theorem 7.11) can be seen from their diagrams. The rightmost column in Diagram (6.29) lets us work directly with the full MBP – or more precisely, the integral operator with $K^{\Omega_1}_{p,\lambda_p}$ as its kernel, which coincides with $\boldsymbol{P}^{\Omega_1}_{p,\lambda_p}$ when the MBP exists – sidestepping any need to consider the Γ-invariant MBP. No such luxury is afforded us when working with the AMBO, where there is no choice but to consider $\boldsymbol{P}^{\Omega_1}_{p,\lambda_p,\Gamma}$ directly. In Section 8.2 we meet this in a concrete setting. ◊

8. The Monomial Basis Projection on Monomial Polyhedra

In this section we show that if $\mathscr{U}$ is a monomial polyhedron (defined in Section 1.7 of the introduction) and $1 < p < \infty$, the Monomial Basis Projection $\boldsymbol{P}^{\mathscr{U}}_{p,1}$ of the space $A^p(\mathscr{U})$ both exists and is absolutely bounded in the sense of Definition 4.23:

Theorem 8.1. *The Absolute Monomial Basis Operator*

$$(\boldsymbol{P}^{\mathscr{U}}_{p,1})^{+}(f)(z) = \int_{\mathscr{U}} \left| K^{\mathscr{U}}_{p,1}(z,w) \right| f(w)\, dV(w) \tag{8.2}$$

is bounded from $L^p(\mathscr{U}) \to L^p(\mathscr{U})$.

Using this result we obtain a representation of the dual space $A^p(\mathscr{U})'$ as a weighted Bergman space on the same domain in Theorem 8.24.

8.1. **Monomial polyhedra.** We denote the spaces of row and column vectors with integer entries by $\mathbb{Z}^{1\times n}$ and $\mathbb{Z}^{n\times 1}$, respectively. Suppose $B = (b^j_k) \in M_{n\times n}(\mathbb{Z})$ is a matrix of integers with $\det B \neq 0$, whose rows are written as $b^j = (b^j_1, \dots, b^j_n) \in \mathbb{Z}^{1\times n}$. We define the *monomial polyhedron* (associated to the matrix B) to be the domain,

$$\mathscr{U}_B = \{ z \in \mathbb{C}^n : |e_{b^j}(z)| < 1, \quad 1 \le j \le n \}, \tag{8.3}$$

provided it is bounded. The matrix B in (8.3) is far from unique. If B' is obtained from B by permuting the rows, or by multiplying any row by a positive integer, then $\mathscr{U}_B = \mathscr{U}_{B'}$. We record the following observation, originally proved in [BCEM22, Proposition 3.2]:

Proposition 8.4. *Suppose that $\mathscr{U}_B$ is a bounded monomial polyhedron of form* (8.3)*, with $B \in M_{n\times n}(\mathbb{Z})$ an invertible matrix. Without loss of generality we may assume that*

(1) $\det B > 0$.
(2) each entry in the inverse matrix B^{-1} is nonnegative.

General Hypothesis for Section 8. *We assume that B is an integer matrix satisfying properties* (1) *and* (2) *from Proposition 8.4 and $\mathscr{U}_B$ is a monomial polyhedron as in* (8.3).

The following representation of monomial polyhedra as quotients was first proved in [BCEM22, Theorem 3.12]; it can be viewed as a resolution of the singularities of monomial polyhedra via monomial maps.

Proposition 8.5. *Let $A = \operatorname{adj} B = (\det B)B^{-1} \in M_{n\times n}(\mathbb{Z})$ be the adjugate of B. There exists a product domain*

$$\Omega = U_1 \times \cdots \times U_n \subset \mathbb{C}^n, \tag{8.6}$$

with each factor U_j equal to either a unit disc $\mathbb{D}$ or punctured disc $\mathbb{D}^*$, such that the monomial map $\Phi_A : \mathbb{C}^n \to \mathbb{C}^n$ of (6.10) *restricts to a proper holomorphic map $\Phi_A : \Omega \to \mathscr{U}_B$. This map is of quotient type with respect to group Γ_A, which consists of automorphisms $\sigma_\nu : \Omega \to \Omega$ given by* (6.11b).

Consequently, the conditions of Hypothesis ★ introduced in Section 6.4 are satisfied, if we take $\Omega_1 = \Omega$, $\Omega_2 = \mathscr{U}_B$, and A, Φ_A, Γ_A as above in Proposition 8.5. The consequences of Hypothesis ★ noted in Section 6.4 also apply, with appropriate changes of notation, as do other results in the two previous sections assuming Hypothesis ★. Notice that in the special situation of Proposition 8.5, the source $\Omega_1 = \Omega$ is a product domain, and the weight $\lambda_p = |\Phi'_A|^{2-p}$ of (6.12) that appears in the transformation formulas has a tensor product structure up to a constant:

$$\lambda_p(\zeta) = \left|\det \Phi'_A(\zeta)\right|^{2-p} = (\det A)^{2-p} \prod_{j=1}^{n} \mu_{\gamma_j}(\zeta_j), \tag{8.7}$$

where μ_{γ_j} is the weight on the factor U_j (which is $\mathbb{D}$ or $\mathbb{D}^*$) given by

$$\mu_{\gamma_j}(z) = |z|^{\gamma_j} \qquad \text{and} \qquad \gamma_j = (\mathbb{1} \cdot a_j - 1)(2-p), \tag{8.8}$$

with $\mathbb{1} \in \mathbb{Z}^{1\times n}$ a row vector with 1 in each component and $a_j \in \mathbb{Z}^{n\times 1}$ the j-th column of A. We can remove the absolute value from $\det A$ since $\det A = (\det B)^n \cdot \frac{1}{\det B} = \det B^{n-1} > 0$, thanks to Proposition 8.4 and the General Hypothesis for Section 8.

8.2. **The Monomial Basis Projection on monomial polyhedra.** As a consequence of the above we have the following.

Proposition 8.9. *Let $\mathscr{U}$ be a monomial polyhedron of form* (8.3)*. The Monomial Basis Projection $\boldsymbol{P}^{\mathscr{U}}_{p,1}$ of the space $A^p(\mathscr{U})$ exists.*

Proof. Theorem 6.27 says the existence of the MBP $\boldsymbol{P}^{\mathscr{U}}_{p,1} : L^p(\mathscr{U}) \to A^p(\mathscr{U})$ would follow from verification of the stronger statement that the MBP $\boldsymbol{P}^{\Omega}_{p,\lambda_p} : L^p(\Omega, \lambda_p) \to A^p(\Omega, \lambda_p)$ exists. Since Ω is a product domain and λ_p decomposes as a tensor product of weights on the factors U_j up to a constant factor (see (8.7)), we see by (3.18) that the MBK of $A^p(\Omega, \lambda_p)$ decomposes as (a constant multiple of) the tensor product of the MBKs on the spaces $A^p(U_j, \mu_{\gamma_j})$:

$$K^{\Omega}_{p,\lambda_p}(z,w) = (\det A)^{p-2} \prod_{j=1}^{n} K^{U_j}_{p,\mu_{\gamma_j}}(z_j, w_j). \tag{8.10}$$

The MBK $K^{U_j}_{p,\mu_{\gamma_j}}$ is the arithmetic progression subkernel $k^{U_j}_{p,\gamma_j,0,1}$. Therefore by Proposition 5.16, for each $1 \le j \le n$, there exist functions ϕ_j, ψ_j and constants C_1^j, C_2^j such that

$$\int_{U_j} \left| K^{U_j}_{p,\mu_{\gamma_j}}(z,w) \right| \psi_j(w)^q \mu_{\gamma_j}(w)\, dV(w) \le C_1^j \phi_j(z)^q,$$
$$\int_{U_j} \phi_j(z)^p \left| K^{U_j}_{p,\mu_{\gamma_j}}(z,w) \right| \mu_{\gamma_j}(z)\, dV(z) \le C_2^j \psi_j(w)^p.$$

The result follows by Proposition 5.11. □

8.3. **Boundedness of the Absolute Monomial Basis Operator.** In Proposition 8.9 we showed that the MBP of $A^p(\mathscr{U})$ exists. In this section we show that it is absolutely bounded on $L^p(\mathscr{U})$, i.e., the AMBO $(\boldsymbol{P}^{\mathscr{U}}_{p,1})^+$ is bounded on $L^p(\mathscr{U})$. For this, we must better understand the Γ-invariant kernel $K^{\Omega}_{p,\lambda_p,\Gamma}$ defined by (6.26), which unlike the full MBK K^{Ω}_{p,λ_p} does not simply factor as a tensor product of lower-dimensional MBKs. A useful decomposition of $K^{\Omega}_{p,\lambda_p,\Gamma}$ can still be given, upon viewing the set of Γ-invariant monomials through the lens of lattice geometry. The idea behind the decomposition has interesting parallels with some recent work on decomposition of kernels associated to some nonabelian groups (see [DM23]).

Proposition 8.11. *Let the notation and hypotheses be as in Proposition 8.5. Letting $d = \det A$ (which is a positive integer), the Γ-invariant Monomial Basis Kernel of the space $A^p(\Omega, \lambda_p)$ defined in* (6.26) *admits the decomposition*

$$K^{\Omega}_{p,\lambda_p,\Gamma}(z,w) = \sum_{i=1}^{d^{n-1}} K_i(z,w), \tag{8.12}$$

where each K_i a tensor product of n arithmetic progression subkernels defined in (5.6)*:*

$$K_i(z,w) = (\det A)^{p-2} \prod_{j=1}^{n} k^{U_j}_{p,\gamma_j,\alpha_{i,j},d}(z_j, w_j), \tag{8.13}$$

where as above, $d = \det A$, the exponent γ_j is determined by (8.8) *and $\alpha_{i,j} \in \mathbb{Z}/d\mathbb{Z}$ is determined by the group Γ.*

Proof. Recall from (6.26) that Γ-invariant MBK is given by

$$K^{\Omega}_{p,\lambda_p,\Gamma}(z,w) = \sum_{\alpha \in \mathcal{S}_p^{\Gamma}(\Omega,\lambda_p)} \frac{e_\alpha(z)\overline{\chi_p^* e_\alpha(w)}}{\|e_\alpha\|^p_{p,\lambda_p}},$$

where, as in (6.13), we have

$$\begin{aligned} \mathcal{S}_p^{\Gamma}(\Omega,\lambda_p) &= \{\alpha \in \mathcal{S}_p(\Omega,\lambda_p) : \sigma^\sharp(e_\alpha) = e_\alpha \text{ for all } \sigma \in \Gamma\} \\ &= \mathcal{S}_p(\Omega,\lambda_p) \cap [\mathbb{Z}^n]^\Gamma \end{aligned} \tag{8.14}$$

where $[\mathbb{Z}^n]^\Gamma$ is defined to be the subset of $\mathbb{Z}^{1\times n}$ consisting of those multi-indices for which the corresponding monomials are Γ-invariant, i.e.

$$[\mathbb{Z}^n]^\Gamma = \{\alpha \in \mathbb{Z}^{1\times n} : \sigma^\sharp(e_\alpha) = e_\alpha \text{ for all } \sigma \in \Gamma\}.$$

Thanks to (6.11d), we have $[\mathbb{Z}^n]^\Gamma = \{\alpha \in \mathbb{Z}^{1\times n} : \alpha = \beta A - \mathbb{1}, \beta \in \mathbb{Z}^{1\times n}\}$, so after translating by $\mathbb{1}$, we have

$$[\mathbb{Z}^n]^\Gamma + \mathbb{1} = \mathbb{Z}^{1\times n} A = \{\beta A : \beta \in \mathbb{Z}^{1\times n}\} \subset \mathbb{Z}^{1\times n}.$$

We make two observations: first, it is known (see Lemma 3.3 of [NP21]) that $\mathbb{Z}^{1\times n}A$ is a sublattice of $\mathbb{Z}^{1\times n}$ with index

$$\left|\mathbb{Z}^{1\times n}/(\mathbb{Z}^{1\times n}A)\right| = \det A = d.$$

Second, we claim that $\mathbb{Z}^{1\times n}A$ contains $d\,\mathbb{Z}^{1\times n}$ as a sublattice, where

$$d\,\mathbb{Z}^{1\times n} = \{d\beta : \beta \in \mathbb{Z}^{1\times n}\}.$$

Indeed, consider a vector $v = dy$, for some $y \in \mathbb{Z}^{1\times n}$ and check that $v \in \mathbb{Z}^{1\times n}A$. Since A is invertible, there is a solution $x \in \mathbb{Q}^{1\times n}$ with $v = dy = xA$. Write the matrix A in terms of its rows $a^1, \cdots, a^n \in \mathbb{Z}^{1\times n}$ as $A = [a^1, \cdots, a^n]^T$. By Cramer's rule, we see the j-th component of x is

$$x_j = \frac{\det\left([a^1, \cdots, a^{j-1}, dy, a^{j+1}, \cdots, a^n]^T\right)}{\det A} = \det\left([a^1, \cdots, a^{j-1}, y, a^{j+1}, \cdots, a^n]^T\right) \in \mathbb{Z},$$

confirming that $x \in \mathbb{Z}^{1\times n}$, and therefore that $d\,\mathbb{Z}^{1\times n}$ is a sublattice of $\mathbb{Z}^{1\times n}A$.

Since it is clear that the index $\left|\mathbb{Z}^{1\times n}/d\,\mathbb{Z}^{1\times n}\right| = d^n$, by the Third Isomorphism Theorem for groups (see Section 3.3 in [DF04]) we see that

$$\left|\mathbb{Z}^{1\times n}A/d\,\mathbb{Z}^{1\times n}\right| = \frac{\left|\mathbb{Z}^{1\times n}/d\,\mathbb{Z}^{1\times n}\right|}{\left|\mathbb{Z}^{1\times n}/\mathbb{Z}^{1\times n}A\right|} = d^{n-1}.$$

It now follows that we have a representation of the group $\mathbb{Z}^{1\times n}A$ as a disjoint union of d^{n-1} cosets of the subgroup $d\,\mathbb{Z}^{1\times n}$, i.e., there are $\ell^i \in \mathbb{Z}^{1\times n}A$, such that we have

$$\mathbb{Z}^{1\times n}A = [\mathbb{Z}^n]^\Gamma + \mathbb{1} = \bigsqcup_{i=1}^{d^{n-1}} (d\,\mathbb{Z}^{1\times n} + \ell^i),$$

where $\bigsqcup$ denotes disjoint union. Therefore, we have

$$[\mathbb{Z}^n]^\Gamma = \left(\bigsqcup_{i=1}^{d^{n-1}} (d\,\mathbb{Z}^{1\times n} + \ell^i)\right) - \mathbb{1} = \bigsqcup_{i=1}^{d^{n-1}} \left(d\,\mathbb{Z}^{1\times n} + (\ell^i - \mathbb{1})\right).$$

Fix an $i, 1 \leq i \leq d^{n-1}$ and write $\ell^i = (\ell^i_1, \ldots, \ell^i_n)$ with $\ell^i_j \in \mathbb{Z}$. Then we have

$$\begin{aligned} d\,\mathbb{Z}^{1\times n} + (\ell^i - \mathbb{1}) &= \{(d\cdot\nu_1 + \ell^i_1 - 1, \ldots, d\cdot\nu_n + \ell^i_n - 1),\, \nu_1, \ldots, \nu_n \in \mathbb{Z}\} \\ &= \prod_{j=1}^n \{\alpha \in \mathbb{Z} : \alpha \equiv \ell^i_j - 1 \mod d\}, \end{aligned} \tag{8.15}$$

where in the last line we have the cartesian product of n sets of integers.

We now analyze the other intersecting set $\mathcal{S}_p(\Omega, \lambda_p)$ in (8.14). Let $\alpha \in \mathbb{Z}^n$. Since we have $e_\alpha(z) = \prod_{j=1}^n e_{\alpha_j}(z_j)$, by the tensor product representation (up to a constant) (8.7) of the weight λ_p and the Tonelli-Fubini theorem we obtain the following representation of the norm of the monomial e_α on Ω in terms of the norms of the e_{α_j} on the factors U_j:

$$\|e_\alpha\|^p_{L^p(\Omega,\lambda_p)} = (\det A)^{2-p} \cdot \prod_{j=1}^n \left\|e_{\alpha_j}\right\|^p_{L^p(U_j \mu_{\gamma_j})}, \tag{8.16}$$

where it is possible that both sides are infinite. The left-hand side is finite, i.e., $\alpha \in \mathcal{S}_p(\Omega, \lambda_p)$, if and only if each factor on the right-hand side is finite, i.e., for each $1 \leq j \leq n$ we have $\alpha_j \in \mathcal{S}_p(U_j, \mu_{\gamma_j})$. Consequently we obtain a cartesian product representation of sets

$$\mathcal{S}_p(\Omega, \lambda_p) = \prod_{j=1}^n \mathcal{S}_p(U_j, \mu_{\gamma_j}). \tag{8.17}$$

Therefore, by (8.14), we have

$$\mathcal{S}^\Gamma_p(\Omega, \lambda_p) = \mathcal{S}_p(\Omega, \lambda_p) \cap \left(\bigsqcup_{i=1}^{d^{n-1}} \big((d\,\mathbb{Z}^{1\times n} + \ell_i) - \mathbb{1}\big) \right) = \bigsqcup_{i=1}^{d^{n-1}} \mathscr{L}_i,$$

where

$$\begin{aligned} \mathscr{L}_i &= \mathcal{S}_p(\Omega, \lambda_p) \cap \big((d\,\mathbb{Z}^{1\times n} + \ell_i) - \mathbb{1}\big) && \text{by definition} \\ &= \left(\prod_{j=1}^n \mathcal{S}_p(U_j, \mu_{\gamma_j}) \right) \bigcap \left(\prod_{j=1}^n \{\alpha \in \mathbb{Z} : \alpha \equiv \ell^i_j - 1 \mod d\} \right) && \text{by (8.15) and (8.17)} \\ &= \prod_{j=1}^n \big(\mathcal{S}_p(U_j, \mu_{\gamma_j}) \cap \{\alpha \in \mathbb{Z} : \alpha \equiv \ell^i_j - 1 \mod d\}\big) \\ &= \prod_{j=1}^n \mathcal{A}(U_j, p, \gamma_j, \ell^i_j - 1, d), \end{aligned} \tag{8.18}$$

where the last equality follows from the definition (5.1). We now define

$$K_i(z, w) = \sum_{\alpha \in \mathscr{L}_i} \frac{e_\alpha(z) \cdot \overline{\chi^*_p e_\alpha(w)}}{\|e_\alpha\|^p_{p,\lambda_p}}, \tag{8.19}$$

we certainly have (8.12), since by absolute convergence we can rearrange the series defining $K^{\Omega_1}_{p,\lambda_p,\Gamma}$. Now, using (8.16), and the tensor-product representations of e_α and $\chi^*_p e_\alpha$, we see that for $\alpha \in \mathscr{L}_i$ we have

$$\frac{e_\alpha(z) \cdot \overline{\chi^*_p e_\alpha(w)}}{\|e_\alpha\|^p_{p,\lambda_p}} = (\det A)^{p-2} \cdot \prod_{j=1}^n \frac{e_{\alpha_j}(z_j) \cdot \overline{\chi^*_p e_{\alpha_j}(w_j)}}{\left\|e_{\alpha_j}\right\|^p_{p,\mu_{\gamma_j}}}, \tag{8.20}$$

where for each j, we have $\alpha_j \in \mathcal{A}(U_j, p, \gamma_j, \ell^i_j - 1, d)$, and on the right hand side $\chi_p : \mathbb{C} \to \mathbb{C}$ is the one-dimensinal version of the map (1.8). Using (8.18) and (8.20), we can rearrange the sum (8.19) as

$$\begin{aligned} K_i(z,w) &= (\det A)^{p-2} \cdot \prod_{j=1}^{n} \left(\sum_{\alpha_j \in \mathcal{A}(U_j, p, \gamma_j, \ell^i_j - 1, d)} \frac{e_{\alpha_j}(z_j) \cdot \overline{\chi_p^* e_{\alpha_j}(w_j)}}{\left\| e_{\alpha_j} \right\|^p_{p, \mu_{\gamma_j}}} \right) \\ &= (\det A)^{p-2} \cdot \prod_{j=1}^{n} k^{U_j}_{p, \gamma_j, \ell^i_j - 1, d}(z_j, w_j) \end{aligned} \tag{8.21}$$

where the rearrangement in (8.21) is justified since each of the n factor series on the right hand side is absolutely convergent. This completes the proof. $\square$

Proof of Theorem 8.1. Theorem 7.11 shows that $(\boldsymbol{P}^{\mathscr{U}}_{p,1})^+ : L^p(\mathscr{U}) \to L^p(\mathscr{U})$ is a bounded operator if and only if $(\boldsymbol{P}^{\Omega}_{p,\lambda_p,\Gamma})^+ : [L^p(\Omega, \lambda_p)]_\Gamma \to [L^p(\Omega, \lambda_p)]_\Gamma$ is bounded. From formula (8.12) we see that

$$\left| K^{\Omega}_{p,\lambda_p,\Gamma}(z,w) \right| \leq \sum_{i=1}^{d^{n-1}} |K_i(z,w)| \,. \tag{8.22}$$

Formula (7.10) defining the operator $(\boldsymbol{P}^{\Omega}_{p,\lambda_p,\Gamma})^+$ therefore shows that it suffices to prove that for each $1 \leq i \leq n$, the operator

$$f \mapsto \int_{\Omega} |K_i(\cdot, w)| \, f(w) \lambda_p(w) dV(w)$$

is bounded on $L^p(\Omega, \lambda_p)$. Formula (8.13) now gives

$$|K_i(z,w)| = (\det A)^{p-2} \cdot \prod_{j=1}^{n} \left| k^{U_j}_{p, \gamma_j, \alpha_{i,j}, d}(z_j, w_j) \right| .$$

Thanks to Proposition 5.16 for each $1 \leq j \leq n$ there exist functions ϕ_j, ψ_j and constants C^j_1, C^j_2 such that

$$\begin{aligned} &\int_{U_j} \left| k^{U_j}_{p,\gamma_j,\alpha_{i,j},d}(z,w) \right| \psi_j(w)^q \mu_{\gamma_j}(w) \, dV(w) \leq C_1 \phi_j(z)^q, \\ &\int_{U_j} \phi_j(z)^p \left| k^{U_j}_{p,\gamma_j,\alpha_{i,j},d}(z,w) \right| \mu_{\gamma_j}(z) \, dV(z) \leq C_2 \psi_j(w)^p. \end{aligned}$$

The result follows by Proposition 5.11. $\square$

8.4. **Dual spaces on monomial polyhedra.** The general duality theory developed in Section 4.4 together with the L^p-boundedness of the AMBO $(\boldsymbol{P}^{\mathscr{U}}_{p,1})^+$ established in previous section has immediate implications in characterizing the dual space of $A^p(\mathscr{U})$.

On monomial polyhedra, the general duality statement given in Proposition 4.36 simplifies due to the following observation:

Proposition 8.23. *Let $\mathscr{U} \subset \mathbb{C}^n$ be a monomial polyhedron of the form* (8.3). *Then for each $m > 0$, the Reinhardt power $\mathscr{U}^{(m)} = \mathscr{U}$.*

Proof. Write $\mathscr{U} = \mathscr{U}_B$, where the rows of B are written as $b^j = (b^j_1, \dots, b^j_n) \in \mathbb{Z}^{1\times n}$. From the definition of the Reinhardt power of a domain given in (3.9), we see

$$\begin{aligned}\mathscr{U}^{(m)} &= \{z \in \mathbb{C}^n : (|z_1|^{\frac{1}{m}}, \dots, |z_n|^{\frac{1}{m}}) \in \mathscr{U}\} \\ &= \{z \in \mathbb{C}^n : |e_{b^j}\big(|z_1|^{\frac{1}{m}}, \dots, |z_n|^{\frac{1}{m}}\big)| < 1,\ 1 \le j \le n\} \\ &= \big\{z \in \mathbb{C}^n : |e_{b^j}(z)|^{\frac{1}{m}} < 1,\ 1 \le j \le n\big\} = \big\{z \in \mathbb{C}^n : |e_{b^j}(z)| < 1,\ 1 \le j \le n\big\} = \mathscr{U}.\end{aligned}$$

□

Theorem 8.1 shows that for $1 < p < \infty$, the AMBO $(\boldsymbol{P}^{\mathscr{U}}_{p,1})^+$ is a bounded operator on $L^p(\mathscr{U})$. This fact can be combined with Proposition 8.23 and Proposition 4.36 to show the following

Theorem 8.24. *The duality pairing of $L^p(\mathscr{U}) \times L^q(\mathscr{U}, \eta_q) \to \mathbb{C}$ given in* (4.30) *by $(f, g) \mapsto \{f, g\}_{p,1}$ restricts to a duality pairing on the holomorphic subspaces*

$$A^p(\mathscr{U}) \times A^q(\mathscr{U}, \eta_q) \to \mathbb{C}.$$

This gives the identification of the dual space $A^p(\mathscr{U})'$ with $A^q(\mathscr{U}, \eta_q)$.

Proof. The boundedness of $(\boldsymbol{P}^{\mathscr{U}}_{p,1})^+$ in $L^p(\mathscr{U})$ allows for the use of Proposition 4.36. In this setting $\mathscr{U}^{(p-1)} = \mathscr{U}$ by Proposition 8.23, which yields the result. □

9. Comparison with the Bergman projection

In this section we compare the behavior of the Bergman projection on L^p, $p \neq 2$, with that of the MBP on certain domains where the Bergman projection is not well-behaved. Let $\Omega \subset \mathbb{C}^n$ be a bounded Reinhardt domain such that the origin lies on its boundary, the simplest example being the punctured disc $\mathbb{D}^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$. The anomalous nature of the singularity is already seen in the classical *Riemann removable singularity theorem*: a holomorphic function on $\mathbb{D}^*$ which is bounded near the boundary point 0 of this domain extends to a function holomorphic on $\mathbb{D}$. A higher dimensional version of this phenomenon was noticed by Sibony in [Sib75] in the Hartogs triangle and generalized in [Cha19]: there is a larger domain $\widetilde{\Omega}$ such that each holomorphic function on Ω which extends $\mathcal{C}^\infty$-smoothly to the origin in fact extends holomorphically to $\widetilde{\Omega}$.

In understanding the L^p function theory of a bounded Reinhardt domain $\Omega \subset \mathbb{C}^n$ with boundary passing through the origin, it is useful to consider the behavior of the sets of p-allowable indices introduced in Section 1.4

$$\mathcal{S}_p(\Omega) = \{\alpha \in \mathbb{Z}^n : e_\alpha \in L^p(\Omega)\},$$

as p traverses the interval $(1, \infty)$. It is clear that the sets can only *shrink* as p increases, as fewer monomials become integrable due to increase in the exponent p in the integral $\int_\Omega |e_\alpha|^p\, dV$. However, the set $\mathcal{S}_p(\Omega)$ is always nonempty, since $\mathbb{N}^n \subset \mathcal{S}_p(\Omega)$, Ω being bounded. For example on the punctured disc, if $p < 2$, then $\mathcal{S}_p(\mathbb{D}^*) = \{\alpha \in \mathbb{Z} : \alpha \ge -1\}$, and if $p \ge 2$, then $\mathcal{S}_p(\mathbb{D}^*) = \{\alpha \in \mathbb{Z} : \alpha \ge 0\}$. The exponent $p = 2$ where the set of indices shrinks is a so-called *threshold*. On a monomial polyhedron $\mathscr{U}$, it was shown in [BCEM22] that there is a positive integer $\kappa(\mathscr{U})$ from which the two thresholds closest to $p = 2$ are easily determined. Define

$$q^* = \frac{2\kappa(\mathscr{U})}{\kappa(\mathscr{U}) + 1}, \qquad p^* = \frac{2\kappa(\mathscr{U})}{\kappa(\mathscr{U}) - 1}.$$

Notice that $q^* < 2 < p^*$ and that p^* and q^* are Hölder conjugates. The smallest threshold greater than 2 is p^*, and for each p with $2 \le p < p^*$, we have $\mathcal{S}_p(\mathscr{U}) = \mathcal{S}_2(\mathscr{U})$. The largest threshold less than 2 is q^* and for each $q^* \le p < 2$, $\mathcal{S}_{q^*}(\mathscr{U}) = \mathcal{S}_p(\mathscr{U})$. It is no accident that the range of values of p for which the Bergman projection is bounded in $L^p(\mathscr{U})$ is $q^* < p < p^*$, as seen in (1.11) above ([BCEM22, Theorem 1.2]).

9.1. **The L^p-irregularity of the Bergman projection.** Outside the interval (q^*, p^*), the L^p-boundedness of the Bergman projection on the monomial polyhedron $\mathscr{U}$ fails in different ways depending on whether $p \geq p^*$ or $p \leq q^*$. Since $\mathscr{U}$ is bounded, we have $L^p(\mathscr{U}) \subset L^2(\mathscr{U})$ if $p \geq p^* > 2$, so the integral operator defining the Bergman projection in (1.1) is defined for each $f \in L^p(\mathscr{U})$. The failure of boundedness of the Bergman projection corresponds to the fact that there are functions $f \in L^p(\mathscr{U})$ for which the projection $\boldsymbol{B}^{\mathscr{U}} f$ is not in $A^p(\mathscr{U})$. It is easy to give an explicit example when $\mathscr{U} = \mathbb{H}$, the Hartogs triangle. Suppose $p \geq \frac{2\kappa(\mathbb{H})}{\kappa(\mathbb{H})-1} = 4$ and let $f(z) = \overline{z}_2$, which is bounded and therefore in $L^p(\mathbb{H})$. A computation shows that there is a constant C such that $\boldsymbol{B}^{\mathbb{H}} f(z) = C{z_2}^{-1} \notin L^p(\mathbb{H})$. This idea can be generalized to an arbitrary monomial polyhedron $\mathscr{U}$ to show that if $p \geq p^*$, there is a function in $L^p(\mathscr{U})$ which projects to a monomial which is in $L^2(\mathscr{U})$ but not in $L^p(\mathscr{U})$. In [CZ16] the range of the map $\boldsymbol{B}^{\mathbb{H}} : L^p(\mathbb{H}) \to L^2(\mathbb{H})$ for $p \geq 4$ was identified as a weighted L^p-Bergman space strictly larger than $L^p(\mathbb{H})$, and a similar result holds on any monomial polyhedron. Recent work of Huo and Wick [HW20] also shows that $\boldsymbol{B}^{\mathbb{H}}$ is of weak-type (4,4). For $p \leq q^*$, the situation is worse:

Proposition 9.1. *If $1 < p \leq \frac{2\kappa(\mathscr{U})}{\kappa(\mathscr{U})+1}$ and $z \in \mathscr{U}$, there is a function $f \in L^p(\mathscr{U})$ such that*

$$\int_{\mathscr{U}} B^{\mathscr{U}}(z,w) f(w)\, dV(w)$$

diverges. Consequently there is no way to extend the Bergman projection to $L^p(\mathscr{U})$ using its integral representation.

Proof. Let q denote the Hölder conjugate of p so that $q \geq \frac{2\kappa(\mathscr{U})}{\kappa(\mathscr{U})-1}$. The holomorphic function on the Reinhardt domain $\mathscr{U}$ given by $g(\zeta) = B(\zeta, z)$ has Laurent expansion

$$g(\zeta) = \sum_{\alpha \in \mathcal{S}_2(\mathscr{U})} \frac{\overline{z}^\alpha}{\|e_\alpha\|_2} \zeta^\alpha.$$

Since $q \geq \frac{2\kappa(\mathscr{U})}{\kappa(\mathscr{U})-1} = p^*$, and the set of integrable monomials shrinks at p^*, it follows that there is a monomial $e_\alpha \in A^2(\mathscr{U}) \setminus A^q(\mathscr{U})$. Since this non-$A^q$ monomial appears in the above Laurent series with a nonzero coefficient, and by Theorem 2.16, the Laurent expansion of a function in A^q can only have monomials which are in A^q, it follows that $g \notin A^q(\mathscr{U})$. By symmetry therefore, $B(z, \cdot) \notin L^q(\mathscr{U})$. It now follows that there is a function $f \in L^p(\mathscr{U})$ such that the integral above does not converge. □

When $\mathscr{U} = \mathbb{H}$, one can show by explicit computation that if $1 < p < \frac{4}{3} = \frac{2\kappa(\mathbb{H})}{\kappa(\mathbb{H})+1}$, we can take $f(w) = w_2^{-3}$ in the above result for each $z \in \mathbb{H}$. It was shown in [HW20] that $\boldsymbol{B}^{\mathbb{H}}$ fails to be weak-type $(\frac{4}{3}, \frac{4}{3})$, but the above proposition shows that $\boldsymbol{B}^{\mathbb{H}}$ in fact does not even exist as an everywhere defined operator on $L^{\frac{4}{3}}(\mathbb{H})$.

In contrast with the above, Proposition 8.9 guarantees that for $1 < p < \infty$ and $\mathscr{U}$ a monomial polyhedron, that the MBP $\boldsymbol{P}^{\mathscr{U}}_{p,1}$ is a bounded operator from $L^p(\mathscr{U})$ onto $A^p(\mathscr{U})$, and Theorem 4.1 says that for $z \in \mathscr{U}$, the function $K^{\mathscr{U}}_{p,1}(z, \cdot) \in L^q(\mathscr{U})$, where $\frac{1}{p} + \frac{1}{q} = 1$.

9.2. **Failure of reproducing property, $1 < p < 2$.** The Bergman projection extended to L^p need not reproduce A^p, as one sees in the case of the punctured disc $\mathbb{D}^*$. Here, since $A^2(\mathbb{D}^*)$ and $A^2(\mathbb{D})$ are identical, the Bergman kernels have the same formula, and the Bergman projection on $\mathbb{D}^*$ extends to a bounded operator on $L^p(\mathbb{D}^*)$ for every $1 < p < \infty$, but fails to be surjective onto $A^p(\mathbb{D}^*)$ for certain p. This happens because the range of the Bergman projection can be naturally identified with $A^p(\mathbb{D})$, and when $1 < p < 2$, the space $A^p(\mathbb{D})$ is a strict subspace of $A^p(\mathbb{D}^*)$ (for example the function $g(z) = z^{-1}$ belongs to $A^p(\mathbb{D}^*) \setminus A^2(\mathbb{D}^*)$). In particular, the space $A^p(\mathbb{D}^*)$ is not reproduced by the extended Bergman projection, i.e., $\boldsymbol{B}^{\mathbb{D}^*}$ is not the identity on $A^p(\mathbb{D}^*)$. In fact, the one-dimensional span of the function $g(z) = z^{-1}$ is the nullspace of the operator $\boldsymbol{B}^{\mathbb{D}^*}$ restricted to $A^p(\mathbb{D}^*)$.

On the Hartogs triangle, it was seen in Section 9.1 that the Bergman projection is bounded on $L^p(\mathbb{H})$ for $\frac{4}{3} < p < 4$. However, $\boldsymbol{B}^{\mathbb{H}}$ fails to reproduce $A^p(\mathbb{H})$ for $\frac{4}{3} < p < 2$. Let $\mathcal{N} \subset A^p(\mathbb{H})$ be the closed subspace spanned by the monomials in $A^p(\mathbb{H}) \setminus A^2(\mathbb{H})$. One sees from a computation that the monomials in $A^p(\mathbb{H}) \setminus A^2(\mathbb{H})$ are e_α with $\alpha_1 \geq 0$ and $\alpha_1 + \alpha_2 = -2$. Then one can verify using orthogonality of L^p and L^q monomials that the nullspace of $\boldsymbol{B}^{\mathbb{H}}$ restricted to $A^p(\mathbb{H})$ is $\mathcal{N}$, and thus again the reproducing property fails.

In contrast, the MBP of $A^p(\mathscr{U})$ accounts for *all* monomials appearing in the Schauder basis $\{e_\beta : \beta \in \mathcal{S}_p(\mathscr{U})\}$, and Proposition 8.9 shows that for $1 < p < \infty$, $\boldsymbol{P}^{\mathscr{U}}_{p,1}$ is a bounded *surjective* projection of $L^p(\mathscr{U})$ onto $A^p(\mathscr{U})$. In particular, the MBP reproduces $A^p(\mathscr{U})$.

9.3. **The Bergman projection and holomorphic dual spaces.** The following is a reformulation of [CEM19, Theorem 2.15]:

Theorem 9.2. *Suppose that the following two conditions hold on a domain $U \subset \mathbb{C}^n$.*

(1) The absolute Bergman operator $(\boldsymbol{B}^U)^+ : L^p(U) \to L^p(U)$ is bounded.

(2) The Bergman projection acts as the identity operator on both $A^p(U)$ and $A^q(U)$.

Then the sesquilinear Hölder pairing restricts to a duality pairing of $A^p(U)$ with $A^q(U)$:

$$\langle f, g\rangle = \int_U f\overline{g}\, dV, \qquad f \in A^p(U), \quad g \in A^q(U), \tag{9.3}$$

providing the dual space identification $A^p(U)' \simeq A^q(U)$.

For instance, on smoothly bounded strongly pseudoconvex domains, properties (1) and (2) both hold, see [PS77] and [Cat80], so the conclusion holds.

When one of the properties (1) or (2) fails, the conclusion can fail. On the punctured disc $\mathbb{D}^* \subset \mathbb{C}$, under the pairing (9.3) $A^p(\mathbb{D}^*)'$ can only be identified with $A^q(\mathbb{D}^*)$ if $p = q = 2$. If $p > 2$, there are non-zero elements of $A^q(\mathbb{D}^*)$ that represent the zero-functional on $A^p(\mathbb{D}^*)$. In particular, the monomial $e_{-1} \in L^q(\mathbb{D}^*)$, and $\langle f, e_{-1}\rangle = 0$, for every $f \in A^p(\mathbb{D}^*)$. If $p < 2$, there are functionals $\phi \in A^p(\mathbb{D}^*)'$ which cannot be realized by a function in $A^q(\mathbb{D}^*)$ via the pairing (9.3). Consider the coefficient functional $a_{-1} : A^p(\mathbb{D}^*) \to \mathbb{C}$ mapping a function to its coefficient of $e_{-1}(z) = z^{-1}$. This functional can not be represented by a function in $A^q(\mathbb{D}^*)$ under the pairing (9.3), because $\langle e_{-1}, f\rangle = 0$ for all $f \in A^q(\mathbb{D}^*)$.

On the Hartogs triangle, let $2 < p < 4$, and q be the Hölder conjugate of p. The nullspace $\mathcal{N} \subset A^q(\mathbb{H})$ of the restriction of $\boldsymbol{B}^{\mathbb{H}}$ to $A^q(\mathbb{H})$ (as discussed in Section 9.2) has the property that if $g \in \mathcal{N}$ then $\langle f, g\rangle = 0$ for every $f \in A^p(\mathbb{H})$. Therefore, g represents the zero functional under the pairing (9.3). Now let $\frac{4}{3} < p < 2$, and q be the Hölder conjugate of p. Let $\alpha \in \mathbb{Z}^2$ with $\alpha_1 \geq 0$ and $\alpha_1 + \alpha_2 = -2$. Then the coefficient functional $a_\alpha : A^p(\mathbb{H}) \to \mathbb{C}$ is a bounded linear functional on $A^p(\mathbb{H})$ by Theorem 2.16. But since $\langle e_\alpha, f\rangle = 0$ for $f \in A^q(\mathbb{H})$, we are unable to identify a_α with a function in $A^q(\mathbb{H})$.

In contrast with the above, the duality theory set forth in Section 4.4 lets us characterize duals of Bergman spaces of Reinhardt domains via the pairing (4.30) whenever the MBP is absolutely bounded. In the case of the punctured disc, we have seen in Corollary 5.23 that

$$A^p(\mathbb{D}^*)' \simeq A^q(\mathbb{D}^*, \eta_q), \qquad \eta_q(\zeta) = (q-1)|\zeta|^{2q-4}.$$

Theorem 8.24 shows that on monomial polyhedra, the pairing of $A^p(\mathscr{U})$ and $A^q(\mathscr{U},\eta_q)$ by $(f,g)\mapsto \{f,g\}_{p,1}$ is a duality pairing. In the special case when $\mathscr{U}$ is the Hartogs triangle,

$$A^p(\mathbb{H})' \simeq A^q(\mathbb{H},\eta_q), \qquad \eta_q(\zeta) = (q-1)^2|\zeta_1\zeta_2|^{2q-4}.$$

References

[Bar84] David E. Barrett. Irregularity of the Bergman projection on a smooth bounded domain in $\mathbf{C}^2$. *Ann. of Math. (2)*, 119(2):431–436, 1984.

[BC82] Steven Bell and David Catlin. Boundary regularity of proper holomorphic mappings. *Duke Math. J.*, 49(2):385–396, 1982.

[BCEM22] Chase Bender, Debraj Chakrabarti, Luke Edholm, and Meera Mainkar. L^p-regularity of the Bergman projection on quotient domains. *Canad. J. Math.*, 74(3):732–772, 2022.

[Bel81] Steven R. Bell. Proper holomorphic mappings and the Bergman projection. *Duke Math. J.*, 48(1):167–175, 1981.

[Bel82] Steven R. Bell. The Bergman kernel function and proper holomorphic mappings. *Trans. Amer. Math. Soc.*, 270(2):685–691, 1982.

[Ber70] Stefan Bergman. *The kernel function and conformal mapping.* Mathematical Surveys, No. V. American Mathematical Society, Providence, R.I., revised edition, 1970.

[Cat80] David Catlin. Boundary behavior of holomorphic functions on pseudoconvex domains. *J. Differential Geom.*, 15(4):605–625 (1981), 1980.

[CD22] D. Chakrabarti and A. Dawn. Power series as fourier series. *Rocky Mountain Journal of Mathematics*, 52(5):1539–1574, 2022.

[CEM19] D. Chakrabarti, L. D. Edholm, and J. D. McNeal. Duality and approximation of Bergman spaces. *Adv. Math.*, 341:616–656, 2019.

[Cha19] Debraj Chakrabarti. On an observation of Sibony. *Proc. Amer. Math. Soc.*, 147(8):3451–3454, 2019.

[Che17] Liwei Chen. The L^p boundedness of the Bergman projection for a class of bounded Hartogs domains. *J. Math. Anal. Appl.*, 448(1):598–610, 2017.

[CZ16] Debraj Chakrabarti and Yunus E. Zeytuncu. L^p mapping properties of the Bergman projection on the Hartogs triangle. *Proc. Amer. Math. Soc.*, 144(4):1643–1653, 2016.

[DF82] Klas Diederich and John Erik Fornaess. Boundary regularity of proper holomorphic mappings. *Invent. Math.*, 67(3):363–384, 1982.

[DF04] David S. Dummit and Richard M. Foote. *Abstract algebra.* John Wiley & Sons, Inc., Hoboken, NJ, third edition, 2004.

[DM23] G. Dall'Ara and A. Monguzzi. Nonabelian Ramified Coverings and L^p-boundedness of Bergman Projections in $\mathbb{C}^2$. *J. Geom. Anal.*, 33(2):52, 2023.

[Edh16] Luke D. Edholm. Bergman theory of certain generalized Hartogs triangles. *Pacific J. Math.*, 284(2):327–342, 2016.

[EM16] L. D. Edholm and J. D. McNeal. The Bergman projection on fat Hartogs triangles: L^p boundedness. *Proc. Amer. Math. Soc.*, 144(5):2185–2196, 2016.

[EM17] L. D. Edholm and J. D. McNeal. Bergman subspaces and subkernels: degenerate L^p mapping and zeroes. *J. Geom. Anal.*, 27(4):2658–2683, 2017.

[EM20] L. D. Edholm and J. D. McNeal. Sobolev mapping of some holomorphic projections. *J. Geom. Anal.*, 30(2):1293–1311, 2020.

[FK72] G. B. Folland and J. J. Kohn. *The Neumann problem for the Cauchy-Riemann complex.* Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1972. Annals of Mathematics Studies, No. 75.

[FKP99] Luigi Fontana, Steven G. Krantz, and Marco M. Peloso. The $\overline{\partial}$-Neumann problem in the Sobolev topology. *Indiana Univ. Math. J.*, 48(1):275–293, 1999.

[FKP01] Luigi Fontana, Steven G. Krantz, and Marco M. Peloso. Estimates for the $\overline{\partial}$-Neumann problem in the Sobolev topology on $Z(q)$ domains. *Houston J. Math.*, 27(1):123–175, 2001.

[FR74] Frank Forelli and Walter Rudin. Projections on spaces of holomorphic functions in balls. *Ind. Univ. Math. J.*, 24:593–602, 1974.

[HW20] Zhenghui Huo and Brett D. Wick. Weak-type estimates for the Bergman projection on the polydisc and the Hartogs triangle. *Bull. Lond. Math. Soc.*, 52(5):891–906, 2020.

[JP08] Marek Jarnicki and Peter Pflug. *First steps in several complex variables: Reinhardt domains.* EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, 2008.

[Kob59] Shoshichi Kobayashi. Geometry of bounded domains. *Trans. Amer. Math. Soc.*, 92:267–290, 1959.

[Kra99] Steven G. Krantz. *A panorama of harmonic analysis*, volume 27 of *Carus Mathematical Monographs*. Mathematical Association of America, Washington, DC, 1999.

[Kra13] Steven G. Krantz. *Geometric analysis of the Bergman kernel and metric*, volume 268 of *Graduate Texts in Mathematics*. Springer, New York, 2013.

[LT77] Joram Lindenstrauss and Lior Tzafriri. *Classical Banach spaces. I.* Springer-Verlag, Berlin-New York, 1977. Sequence spaces, Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 92.

[Mas91] William S. Massey. *A basic course in algebraic topology*, volume 127 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991.

[McN94] Jeffery D. McNeal. Estimates on the Bergman kernels of convex domains. *Adv. Math.*, 109(1):108–139, 1994.

[Mon21] Alessandro Monguzzi. Holomorphic function spaces on the Hartogs triangle. *Math. Nachr.*, 294(11):2209–2231, 2021.

[MS94] J. D. McNeal and E. M. Stein. Mapping properties of the Bergman projection on convex domains of finite type. *Duke Math. J.*, 73(1):177–199, 1994.

[Muj86] Jorge Mujica. *Complex analysis in Banach spaces*, volume 120 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1986. Holomorphic functions and domains of holomorphy in finite and infinite dimensions, Notas de Matemática [Mathematical Notes], 107.

[Nic07] Liviu I. Nicolaescu. *Lectures on the geometry of manifolds*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, second edition, 2007.

[NP09] Alexander Nagel and Malabika Pramanik. Maximal averages over linear and monomial polyhedra. *Duke Math. J.*, 149(2):209–277, 2009.

[NP21] Alexander Nagel and Malabika Pramanik. Bergman spaces under maps of monomial type. *J. Geom. Anal.*, 31(5):4531–4560, 2021.

[NRSW89] A. Nagel, J.-P. Rosay, E. M. Stein, and S. Wainger. Estimates for the Bergman and Szegő kernels in $\mathbb{C}^2$. *Ann. of Math. (2)*, 129(1):113–149, 1989.

[PS77] D. H. Phong and E. M. Stein. Estimates for the Bergman and Szegö projections on strongly pseudo-convex domains. *Duke Math. J.*, 44(3):695–704, 1977.

[Ran86] R. Michael Range. *Holomorphic functions and integral representations in several complex variables*, volume 108 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1986.

[Sib75] Nessim Sibony. Prolongement des fonctions holomorphes bornées et métrique de Carathéodory. *Invent. Math.*, 29(3):205–230, 1975.

[Tay39] A. E. Taylor. The extension of linear functionals. *Duke Math. J.*, 5:538–547, 1939.

[Zey13] Yunus E. Zeytuncu. L^p regularity of weighted Bergman projections. *Trans. Amer. Math. Soc.*, 365(6):2959–2976, 2013.

[Zey20] Yunus E. Zeytuncu. A survey of the L^p regularity of the Bergman projection. *Complex Anal. Synerg.*, 6(2):Paper No. 19, 7, 2020.

[Zha21a] Shuo Zhang. L^p boundedness for the Bergman projections over n-dimensional generalized Hartogs triangles. *Complex Var. Elliptic Equ.*, 66(9):1591–1608, 2021.

[Zha21b] Shuo Zhang. Mapping properties of the Bergman projections on elementary Reinhardt domains. *Math. Slovaca*, 71(4):831–844, 2021.

[Zhu91] Kehe Zhu. Duality of Bloch spaces and norm convergence of Taylor series. *Michigan Math. J.*, 38(1):89–101, 1991.

[Zhu07] Kehe Zhu. *Operator theory in function spaces*, volume 138 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2007.

[ZJ64] V.P. Zaharjuta and V.I. Judovic. The general form of a linear functional on H'_p. *Uspekhi Mat. Nauk.*, 19(2):139–142, 1964.

[Zwo00] Włodzimierz Zwonek. Completeness, Reinhardt domains and the method of complex geodesics in the theory of invariant functions. *Dissertationes Math. (Rozprawy Mat.)*, 388:103, 2000.

Department of Mathematics, Central Michigan University, Mt. Pleasant, MI 48858, USA
Email address: chakr2d@cmich.edu

Department of Mathematics, Universität Wien, Vienna, Austria
Email address: luke.david.edholm@univie.ac.at