L^p MAPPING PROPERTIES OF THE BERGMAN PROJECTION ON THE HARTOGS TRIANGLE

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ABSTRACT. We prove optimal estimates for the mapping properties of the Bergman projection on the Hartogs triangle in weighted L^p spaces when $p > \frac{4}{3}$, where the weight is a power of the distance to the singular boundary point. For $1 < p \leq \frac{4}{3}$ we show that no such weighted estimates are possible.

1. INTRODUCTION

1.1. **Results.** In this note we describe the L^p regularity of the Bergman projection on the *Hartogs triangle*, the pseudoconvex domain in $\mathbb{C}^2$ defined as

$$\mathbb{H} = \left\{ (z_1, z_2) \in \mathbb{C}^2 \mid |z_2| < |z_1| < 1 \right\}, \tag{1}$$

for $1 < p < \infty$. To state our results we use *weighted Bergman spaces* $A^p(\mathbb{H}, \omega)$, where $\omega > 0$ is a continuous function on $\mathbb{H}$, and a holomorphic function $f \in \mathcal{O}(\mathbb{H})$ belongs to $A^p(\mathbb{H}, \omega)$ if $\|f\|_{L^p(\mathbb{H},\omega)}^p = \int_{\mathbb{H}} |f|^p \, \omega dV < \infty$, where dV is Lebesgue measure of $\mathbb{C}^2$. Let δ_1 be defined on $\mathbb{C}^2$ by

$$\delta_1(z_1, z_2) = |z_1| \, . \tag{2}$$

On $\mathbb{H}$, we have $\frac{1}{\sqrt{2}} |z| \leq \delta_1(z) \leq |z|$, i.e. δ_1 is comparable to the distance to the singular point 0. Let $\mathbf{B}_{\mathbb{H}} : L^2(\mathbb{H}) \to A^2(\mathbb{H})$ denote the Bergman projection. We begin with the case $p \geq 2$:

Theorem 1.1. *If $p \geq 2$, the map $\mathbf{B}_{\mathbb{H}}$ is bounded and surjective from $L^p(\mathbb{H})$ to $A^p(\mathbb{H}, \delta_1^{p-2})$.*

Note that the surjectivity of $\mathbf{B}_{\mathbb{H}}$ means that we have the best possible estimates. From Theorem 1.1 we can recover the following folk result:

Corollary 1.1. *(a) If $\frac{4}{3} < p < 4$, then $\mathbf{B}_{\mathbb{H}}$ is bounded (and surjective) from $L^p(\mathbb{H})$ to $A^p(\mathbb{H})$.*
(b) If $p \geq 4$, then $\mathbf{B}_{\mathbb{H}}$ does not map $L^p(\mathbb{H})$ into $A^p(\mathbb{H})$.

See [Che13] for a generalization of part (a) to a class of domains closely related to $\mathbb{H}$. Note that for $2 \leq p < 4$, both Theorem 1.1 and Corollary 1.1 apply. This is because in this range the space $A^p(\mathbb{H}, \delta_1^{p-2})$ coincides with $A^p(\mathbb{H})$ (cf. Proposition 5.1 below).

We now consider what happens under Bergman projection if $1 < p \leq \frac{4}{3}$. For such p, it turns out that there is *absolutely no way* to control the Bergman projection of an L^p function on the Hartogs triangle using a weight depending on the distance to the singularity:

Theorem 1.2. *Let $1 < p \leq \frac{4}{3}$, and let $\lambda > 0$ be a continuous function on $(0, 1]$. Then $\mathbf{B}_{\mathbb{H}}$ does not map $L^p(\mathbb{H})$ into $A^p(\mathbb{H}, \lambda \circ \delta_1)$.*

This pathological phenomenon does not seem to have been noticed before.

1.2. **The Bergman projection.** We recall some basic definitions and facts regarding the Bergman projection operator. Let $U \subset \mathbb{C}^n$ be a bounded domain. The *Bergman space* $A^2(U)$ is defined to be the intersection $L^2(U) \cap \mathcal{O}(U)$ of the space $L^2(U)$ of square integrable functions on U (with respect to the Lebesgue measure of $\mathbb{C}^n$) with the space $\mathcal{O}(U)$ of holomorphic functions on U. By the Bergman inequality, $A^2(U)$ is a closed subspace of $L^2(U)$. The orthogonal projection operator $\mathbf{B}_U : L^2(U) \to A^2(U)$ is the *Bergman projection* associated with the domain U. It follows from the Riesz representation theorem that the Bergman projection is an integral operator with the kernel $\mathbb{B}_U(z,w)$ on $U \times U$ (called the *Bergman kernel*), i.e. $\mathbf{B}_U f(z) = \int_U \mathbb{B}_U(z,w)f(w)dV(w)$ for all $f \in L^2(U)$. We refer to [Kra01] for the general theory.

It is natural to consider the mapping properties of $\mathbf{B}_U$ on other spaces of functions on U, for example L^p spaces or Sobolev spaces. A survey of mapping properties on Sobolev spaces on smoothly bounded pseudoconvex domains can be found in [BS99]. For results on the regularity of the Bergman projection in L^p space on bounded pseudoconvex domains we refer to the following articles and the references therein [LS12, PS77, Bar84, MS94, KP08, BŞ12, Zey13]. In [EL08], estimates were obtained for the Bergman projection on certain non-smooth domains in L^p spaces with weights which vanish at the singularities of the boundary. This is similar to our Theorem 1.1, but the Hartogs triangle is not among the domains to which the results of [EL08] apply.

1.3. **The Hartogs triangle.** The Hartogs triangle defined in (1) has remarkable geometric and function-theoretic properties, and is a classical source of counterexamples in complex analysis. The boundary $b\mathbb{H}$ of the domain $\mathbb{H}$ has a serious singularity at the point 0, where $b\mathbb{H}$ cannot even be represented as a graph of a continuous function. The closure $\overline{\mathbb{H}}$ does not have a Stein neighborhood basis. The $\overline{\partial}$-problem on $\mathbb{H}$ is not globally regular, i.e., there is a $\overline{\partial}$-closed $(0,1)$-form $g \in \mathcal{C}^\infty_{0,1}(\overline{\mathbb{H}})$ which is smooth up to the boundary on $\mathbb{H}$, such that *no solution u* of the equation $\overline{\partial}u = g$ lies in $\mathcal{C}^\infty(\overline{\mathbb{H}})$ (see [CC91]).

A standard way of understanding function theory on domains with singular boundary is the use of weights which vanish or blow up at the singular points of the boundary. Consequently, one can obtain estimates which take into account the behavior of functions and forms near the singular points. For the Hartogs triangle, this was done in [CS13], where estimates for the canonical solution of the $\overline{\partial}$-problem were obtained in weighted Sobolev spaces. While the Hartogs triangle has a non-Lipschitz boundary, as a complex manifold it has a very simple structure: it is biholomorphic to the product $\mathbb{D}^* \times \mathbb{D}$ of the punctured unit disc $\mathbb{D}^* = \{\, z \in \mathbb{C} \mid 0 < |z| < 1 \,\}$ and the unit disc $\mathbb{D}$. Consequently, one can pull back problems on $\mathbb{H}$ to problems on $\mathbb{D}^* \times \mathbb{D}$, and one gets weights coming from the Jacobian factor. This technique was used in [MM92, CS13] to study function theory on $\mathbb{H}$. Here we use the same method to study the mapping properties of $\mathbf{B}_{\mathbb{H}}$ in L^p spaces.

This paper is organized as follows. In the following section, we give a proof of Theorem 1.2, using a duality argument. After that, in Section 3 we consider the Bergman projection on the product $\mathbb{D}^* \times \mathbb{D}$ with respect to a radial weight on the first factor. In Proposition 3.1, we obtain estimates on $\mathbb{D}^* \times \mathbb{D}$ using operator-theoretic methods relating to norm-convergence in L^p spaces of Taylor series. In Section 4 we prove Theorem 1.1 using biholomorphic mapping of $\mathbb{H}$ with $\mathbb{D}^* \times \mathbb{D}$. Finally, in the last section, we deduce Corollary 1.1 from Theorem 1.1 using a description of weighted Bergman spaces on $\mathbb{H}$ (see Proposition 5.1).

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2. PROOF OF THEOREM 1.2

For a continuous function $\omega > 0$ on a domain U, the space $L^p(U, \omega)$ consists of functions f on U for which
$$\|f\|^p_{L^p(U,\omega)} = \int_U |f|^p \, \omega dV < \infty,$$
where dV is Lebesgue measure. We use the standard duality of L^p spaces to prove the following:

Lemma 2.1. *Let $U \subset \mathbb{C}^n$ be a bounded domain, $p > 1$, and $\omega > 0$ a weight function on U. Then if $\mathbf{B}_U$ is bounded from $L^p(U)$ to $A^p(U, \omega)$, then $\mathbf{B}_U$ is also bounded from $L^q(U, \omega^{1-q})$ to $A^q(U)$, where q is the conjugate exponent to p, i.e. $\frac{1}{p} + \frac{1}{q} = 1$.*

Proof. For $f \in L^q(U, \omega^{(1-q)}) \cap L^2(U)$ we estimate $\|\mathbf{B}_U f\|_{L^q(U)}$ by using duality of L^p spaces on U. Here and later we denote $\langle u, v \rangle = \int_{\mathbb{H}} u\bar{v}dV$, which if $u, v \in L^2(\mathbb{H})$ is the standard inner product of the Hilbert space $L^2(\mathbb{H})$. Note that on the bounded domain U, we have $L^p(U) \subset L^2(U)$ (for $p \geq 2$) by Hölder's inequality. We let $g \in L^p(U)$ range over the unit sphere $\|g\|_{L^p(U)} = 1$, so that we have

$$
\begin{aligned}
\|\mathbf{B}_U f\|_{L^q(U)} &= \sup_{\|g\|_{L^p(U)}=1} |\langle \mathbf{B}_U f, g \rangle| \\
&= \sup_{\|g\|_{L^p(U)}=1} |\langle f, \mathbf{B}_U g \rangle|, \text{ since } \mathbf{B}_U \text{ is self-adjoint} \\
&= \sup_{\|g\|_{L^p(U)}=1} \left| \left\langle f \cdot \omega^{-\frac{1}{p}}, (\mathbf{B}_U g) \cdot \omega^{\frac{1}{p}} \right\rangle \right| \\
&\leq \sup_{\|g\|_{L^p(U)}=1} \left(\left\| f \cdot \omega^{-\frac{1}{p}} \right\|_{L^q(U)} \cdot \left\| (\mathbf{B}_U g) \cdot \omega^{\frac{1}{p}} \right\|_{L^p(U)} \right), \text{ by Hölder's inequality} \\
&= \left(\sup_{\|g\|_{L^p(U)}=1} \|\mathbf{B}_U g\|_{L^p(U,\omega)} \right) \cdot \|f\|_{L^q\left(U, \omega^{-\frac{q}{p}}\right)}
\end{aligned}
$$

since by assumption the first factor is finite, we have $\left(-\frac{q}{p}\right) = 1 - q$ and $L^q(U, \omega^{(1-q)}) \cap L^2(U)$ is dense in $L^q(U, \omega^{(1-q)})$, it follows that $\mathbf{B}_U$ is bounded from $L^q(U, \omega^{(1-q)})$ to $A^q(U)$. $\qquad\square$

Proof of Theorem 1.2. If $1 < p \leq \frac{4}{3}$, then the conjugate exponent $q \geq 4$. Therefore, by Lemma 2.1, to prove the result we only need to show that $\mathbf{B}_{\mathbb{H}}$ does not map $L^q(\mathbb{H}, (\lambda \circ \delta_1)^{1-q})$ into $A^q(\mathbb{H})$. It therefore suffices to present a function $f \in L^2(\mathbb{H}) \cap L^q(\mathbb{H}, (\lambda \circ \delta_1)^{1-q})$ such that $\mathbf{B}_{\mathbb{H}} f \notin A^q(\mathbb{H})$.

Let $\chi \geq 0$ be a continuous function on $[0, 1]$ such that $\chi \equiv 0$ near 0, and $\chi \equiv 1$ near 1. Let f be the function on $\overline{\mathbb{H}}$ given by $f(z_1, z_2) = (\chi \circ \delta_1) \cdot \overline{z_1}$, which is continuous on $\overline{\mathbb{H}}$ and vanishes near the singularity 0, so that $f \in L^2(\mathbb{H}) \cap L^q(\mathbb{H}, (\lambda \circ \delta_1)^{1-q})$ for any choice of λ. We claim that there is a constant $C > 0$ such that $\mathbf{B}_{\mathbb{H}}(f) = Cz_1^{-1}$. Note that the system of monomials $\{z_1^m z_2^n\}$ for $m \geq -(n+1)$ and $n \geq 0$ forms a complete orthogonal set in $A^2(\mathbb{H})$, so that it suffices to show

that unless $m = -1$ and $n = 0$, the function f is orthogonal to each $z_1^m z_2^n$. We have, using polar coordinates:

$$
\begin{aligned}
\langle f, z_1^m z_2^n \rangle &= \int_{\mathbb{H}} \chi(|z_1|) \overline{z_1} \cdot \overline{z_1}^m \overline{z_2}^n dV \\
&= \int_{\mathbb{H}} \chi(r_1) r_1 e^{-i\theta_1} r_1^m e^{-im\theta_1} r_2^n e^{-in\theta_2} r_1 r_2 dr_1 dr_2 d\theta_1 d\theta_2 \\
&= \left(\int_0^{2\pi} e^{-i(m+1)\theta_1} d\theta_1 \right) \cdot \left(\int_0^{2\pi} e^{-in\theta_2} d\theta_2 \right) \cdot \int_\Delta \chi(r_1) r_1^{m+2} r_2^{n+1} dr_1 dr_2,
\end{aligned}
$$

where $\Delta \subset \mathbb{R}^2$ is the subset defined by $\{(r_1, r_2) : 0 \le r_2 < r_1 < 1\}$. This integral vanishes unless $m = -1$, $n = 0$.

To complete the proof, we verify that $\frac{1}{z_1}$ does not belong to $A^q(\mathbb{H})$, if $q \ge 4$. We have,

$$
\begin{aligned}
\int_{\mathbb{H}} \left| \frac{1}{z_1} \right|^q dV &= \int_{\mathbb{H}} \frac{1}{r_1^q} r_1 r_2 dr_1 dr_2 d\theta_1 d\theta_2 \\
&= 4\pi^2 \int_{r_1=0}^1 r_1^{1-q} \left(\int_{r_2=0}^{r_1} r_2 dr_2 \right) dr_1 \\
&= 2\pi^2 \int_0^1 r_1^{3-q} dr_1,
\end{aligned}
\tag{3}
$$

which diverges if $q \ge 4$. $\qquad\square$

3. Weighted L^p-regularity of the Bergman projection on $\mathbb{D}^* \times \mathbb{D}$.

We prove the following regularity result for the Bergman projection on the bidisc. As in (2), we set $\delta_1(z) = |z_1|$.

Proposition 3.1. *For $p \ge 2$, the Bergman projection $\mathbf{B}_{\mathbb{D}^* \times \mathbb{D}}$ is bounded and surjective from $L^p(\mathbb{D}^* \times \mathbb{D}, \delta_1^{2-p})$ onto $A^p(\mathbb{D}^* \times \mathbb{D})$.*

Note that when $p \ge 2$, $A^p(\mathbb{D}^* \times \mathbb{D})$ is identical to the space $A^p(\mathbb{D}^2)$, since each holomorphic function in $L^p(\mathbb{D}^* \times \mathbb{D})$ extends holomorphically to $\mathbb{D}^2$ (see the proof of Lemma 5.1 below).

For a non-negative integer N, we define a map $S_N : \mathcal{O}(\mathbb{D}^2) \to \mathcal{O}(\mathbb{D}^2)$ in the following way. For $f \in \mathcal{O}(\mathbb{D}^2)$, with Taylor representation

$$
f(w_1, w_2) = \sum_{\mu, \nu = 0}^\infty a_{\mu, \nu} w_1^\mu w_2^\nu,
\tag{4}
$$

we define the function $S_N f$ by setting

$$
S_N f(w_1, w_2) = \sum_{\mu=0}^N \sum_{\nu=0}^\infty a_{\mu, \nu} w_1^\mu w_2^\nu,
\tag{5}
$$

i.e., S_N is the N-th partial sum in the w_1 variable. We have the following analog of a result of Zhu ([Zhu91, Corollary 4]):

Lemma 3.1. *For $p \ge 1$ and for any $f \in A^p(\mathbb{D}^2)$, as N goes to infinity, $S_N f \to f$ in the L^p-norm.*

Proof. Denote by $\mathbb{T}$ the unit circle $\{|z| = 1\}$ in the plane and by $\mathbb{T}^2$ the two-torus $\{(z_1, z_2) \in \mathbb{C}^2 | |z_1| = |z_2| = 1\}$. Suppose that $g \in H^p(\mathbb{T}^2)$, the Hardy space on $\mathbb{T}^2$. This means that g admits a holomorphic extension to $\mathbb{D}^2$ and has boundary values in $L^p(\mathbb{T}^2)$. For almost all $z_2 \in \mathbb{T}$, the function $z_1 \mapsto g(z_1, z_2)$ is in $L^p(\mathbb{T})$, and note that the N-th partial sum of the Fourier series representation of $g(\cdot, z_2)$ is precisely $S_N g(\cdot, z_2)$. From the classical Riesz theory of convergence of Fourier series in $L^p(\mathbb{T})$ (see e.g. [Gra08, Section 3.5]), we conclude that there is a constant C_0, independent of the function g such that we have for almost all $z_2 \in \mathbb{T}$ that

$$\int_0^{2\pi} \left| S_N g(e^{i\theta}, z_2) \right|^p d\theta \leq C_0 \int_0^{2\pi} \left| g(e^{i\theta}, z_2) \right|^p d\theta. \tag{6}$$

Following a standard functional-analytic argument (cf. [Zhu91, Proposition 1]), to prove Lemma 3.1, it is sufficient to show that there is a constant $C > 0$ such that $\|S_N\| \leq C$ for all non-negative integers N, where we think of S_N as an operator from $A^p(\mathbb{D}^2)$ to itself, and $\|\cdot\|$ is the operator norm. Now

$$\begin{aligned}
\|S_N f\|^p_{L^p(\mathbb{D}^2)} &= \int_{\mathbb{D}} \int_{\mathbb{D}} |S_N f(w_1, w_2)|^p \, dV(w_1) dV(w_2) \\
&= \int_0^1 \int_0^1 r_1 r_2 \left(\int_0^{2\pi} \int_0^{2\pi} \left| S_N f(r_1 e^{i\theta_1}, r_2 e^{i\theta_2}) \right|^p d\theta_1 d\theta_2 \right) dr_1 dr_2 \\
&\leq C_0 \int_0^1 \int_0^1 r_1 r_2 \left(\int_0^{2\pi} \int_0^{2\pi} \left| f(r_1 e^{i\theta_1}, r_2 e^{i\theta_2}) \right|^p d\theta_1 d\theta_2 \right) dr_1 dr_2 \\
&= C_0 \|f\|^p_{L^p(\mathbb{D}^2)},
\end{aligned}$$

where in the last-but-one line, we have used (6). $\qquad \square$

We now give a sufficient condition for an operator on $A^p(\mathbb{D}^2)$ defined by a certain multiplier sequence on the Taylor coefficients to be bounded. Let $\{t_\mu\}_{\mu=0}^\infty$ be a sequence of complex numbers such that

$$\sum_{\mu=0}^\infty |t_{\mu+1} - t_\mu| < \infty. \tag{7}$$

Writing $t_N = \sum_{\mu=0}^{N-1} (t_{\mu+1} - t_\mu) + t_0$, and using (7) we see that the limit $\lim_{N \to \infty} t_N$ exists, and in particular, the sequence $\{t_\mu\}_{\mu=0}^\infty$ is bounded. It follows that for a holomorphic f on $\mathbb{D}^2$, represented as in (4), the new function

$$T f(w_1, w_2) = \sum_{\mu,\nu=0}^\infty t_\mu a_{\mu,\nu} w_1^\mu w_2^\nu \tag{8}$$

is also a holomorphic function on $\mathbb{D}^2$.

Lemma 3.2. *The map T is bounded from $A^p(\mathbb{D}^2)$ to itself.*

Proof. Let $f \in A^p(\mathbb{D}^2)$. We will first show that the partial sums $S_N T f$ of $T f$ (defined using the operator S_N of (5)) converge in $A^p(\mathbb{D}^2)$. Indeed, using summation by parts, we can write

$$S_N T f = t_N \cdot S_N f - \sum_{k=0}^{N-1} (t_{k+1} - t_k) S_k f.$$

Note that each of the factors t_N and $S_N f$ in the first term converges to a limit as $N \to \infty$ so that the first term converges to a limit in $L^p(\mathbb{D}^2)$ as $N \to \infty$. Let $C > 0$ be such that $\|S_N f\|_{L^p(\mathbb{D}^2)} \leq C$ for all N. Such C exists since $S_N f \to f$ in $L^p(\mathbb{D}^2)$. Therefore, we see that

$$\sum_{k=0}^{\infty} \|(t_{k+1} - t_k) S_k f\|_{L^p(\mathbb{D}^2)} \leq C \sum_{k=0}^{\infty} |t_{k+1} - t_k|$$
$$< \infty.$$

This shows the second term is the $(N-1)$-th partial sum of an absolutely convergent series in a Banach space, and therefore has a limit as $N \to \infty$.

Now let $g = \lim_{N \to \infty} S_N T f$, where the limit is in the topology of $A^p(\mathbb{D}^2)$. Let $M \geq 0$ be an integer. By the continuity of S_M on $A^p(\mathbb{D}^2)$ established in the proof of Lemma 3.1, it follows that $S_M g = S_M (\lim_{N \to \infty} S_N T f) = \lim_{N \to \infty} S_M S_N T f = S_M T f$. Since this is true for each M, we have $Tf = g$. Since $g \in A^p(\mathbb{D}^2)$, the operator T maps $A^p(\mathbb{D}^2)$ boundedly to itself. $\qquad \square$

Proof of Proposition 3.1. We first prove that $\mathbf{B}_{\mathbb{D}^* \times \mathbb{D}}$ is bounded from $L^p(\mathbb{D}^* \times \mathbb{D}, \delta_1^{2-p})$ to $A^p(\mathbb{D}^2)$ (note that $A^p(\mathbb{D}^2)$ is identical to $A^p(\mathbb{D}^* \times \mathbb{D})$.) Recall that the Bergman projection on the unit disc is bounded from $L^p(\mathbb{D})$ to $A^p(\mathbb{D})$ (see e.g. [Zhu07, FR75]). Using the fact that the Bergman kernel of the product $\mathbb{D}^2$ is the tensor product of the Bergman kernels of the factors, by a simple application of Fubini's theorem it follows that the Bergman projection on the bidisc $\mathbb{D}^2$ is bounded from $L^p(\mathbb{D}^2)$ to $A^p(\mathbb{D}^2)$. On $\mathbb{D}^* \times \mathbb{D}$ we have $0 < \delta_1 < 1$, and therefore for $p \geq 2$, we obtain,

$$\|f\|_{L^p(\mathbb{D}^2)}^p = \int_{\mathbb{D}^2} |f|^p \, dV \leq \int_{\mathbb{D}^2} |f|^p \, \delta_1^{2-p} dV = \|f\|_{L^p(\mathbb{D}^2, \delta_1^{2-p})}^p \,,$$

so that $L^p(\mathbb{D}^* \times \mathbb{D}, \delta_1^{2-p}) = L^p(\mathbb{D}^2, \delta_1^{2-p})$ is continuously embedded in $L^p(\mathbb{D}^2)$. But as noted above, the Bergman projection on $\mathbb{D}^2$ is bounded from $L^p(\mathbb{D}^2)$ to $A^p(\mathbb{D}^2)$, so that by composing the two maps, we see that $\mathbf{B}_{\mathbb{D}^* \times \mathbb{D}}$ is bounded from $L^p(\mathbb{D}^* \times \mathbb{D}, \delta^{2-p})$ to $A^p(\mathbb{D}^2) = A^p(\mathbb{D}^* \times \mathbb{D})$.

To show that $\mathbf{B}_{\mathbb{D}^* \times \mathbb{D}}$ surjective onto $A^p(\mathbb{D}^* \times \mathbb{D})$, we construct an operator $U : A^p(\mathbb{D}^* \times \mathbb{D}) \to L^p(\mathbb{D}^* \times \mathbb{D}, \delta^{2-p})$ such that $\mathbf{B}_{\mathbb{D}^* \times \mathbb{D}} \circ U$ is the identity on $A^p(\mathbb{D}^* \times \mathbb{D})$, i.e, for $f \in A^p(\mathbb{D}^* \times \mathbb{D})$, the function Uf is projected by $\mathbf{B}_{\mathbb{D}^* \times \mathbb{D}}$ to f, which shows the surjectivity of $\mathbf{B}_{\mathbb{D}^* \times \mathbb{D}}$.

To construct U, in the equation (8) we let $t_\mu = 1 + (\mu + 1)^{-1}$, which being monotone and bounded certainly satisfies (7). Further, a computation shows that

$$\frac{\left\langle |w_1|^2 \, w_1^\mu, w_1^m \right\rangle_{\mathbb{D}}}{\langle w_2^m, w_2^m \rangle_{\mathbb{D}}} = \frac{1}{t_\mu} \delta_{\mu m}. \tag{9}$$

Defining T as in (8), we let $Uf = \delta_1^2 \cdot Tf$. Then U maps $A^p(\mathbb{D}^2)$ to $L^p(\mathbb{D}^* \times \mathbb{D}, \delta_1^{2-p})$. Indeed:

$$\|Uf\|_{L^p(\mathbb{D}^* \times \mathbb{D}, \delta_1^{2-p})}^p = \int_{\mathbb{D}^* \times \mathbb{D}} \left| \delta_1^2 Tf \right|^p \delta_1^{2-p} dV$$
$$= \int_{\mathbb{D}^* \times \mathbb{D}} \delta_1^{2+p} |Tf|^p \, dV$$
$$\leq \int_{\mathbb{D}^* \times \mathbb{D}} |Tf|^p \, dV$$
$$\leq C \|f\|_{L^p(\mathbb{D}^2)}^p < \infty,$$

where in the last line we have used Lemma 3.2. Further, since $\{w_1^m w_2^n\}_{m,n=0}^\infty$ is an orthogonal set in $L^2(\mathbb{D}^2)$, we see that

$$
\mathbf{B}_{\mathbb{D}^* \times \mathbb{D}}(Uf)(w_1, w_2) = \sum_{m,n=0}^\infty \left\langle |w_1|^2 \sum_{\mu,\nu=0}^\infty t_\mu a_{\mu,\nu} w_1^\mu w_2^\nu, w_1^m w_2^n \right\rangle_{\mathbb{D}^* \times \mathbb{D}} \cdot \frac{w_1^m w_2^n}{\langle w_1^m, w_1^m \rangle_{\mathbb{D}} \cdot \langle w_2^n, w_2^n \rangle_{\mathbb{D}}}
$$

$$
= \sum_{m,n=0}^\infty \sum_{\mu,\nu=0}^\infty t_\mu a_{\mu,\nu} \left\langle |w_1|^2 w_1^\mu w_2^\nu, w_1^m w_2^n \right\rangle_{\mathbb{D}^* \times \mathbb{D}} \cdot \frac{w_1^m w_2^n}{\langle w_1^m, w_1^m \rangle_{\mathbb{D}} \cdot \langle w_2^n, w_2^n \rangle_{\mathbb{D}}}
$$

$$
= \sum_{m,n=0}^\infty \sum_{\mu,\nu=0}^\infty t_\mu a_{\mu,\nu} \frac{\left\langle |w_1|^2 w_1^\mu, w_1^m \right\rangle_{\mathbb{D}}}{\langle w_1^m, w_1^m \rangle_{\mathbb{D}}} \cdot \frac{\langle w_2^\nu, w_2^n \rangle_{\mathbb{D}}}{\langle w_2^n, w_2^n \rangle_{\mathbb{D}}} w_1^m w_2^n
$$

$$
= \sum_{m,n=0}^\infty \sum_{\mu,\nu=0}^\infty t_\mu a_{\mu,\nu} \cdot \frac{1}{t_\mu} \delta_{\mu m} \delta_{\nu n} w_1^m w_2^n, \quad \text{using (9)}
$$

$$
= \sum_{m,n=0}^\infty a_{m,n} w_1^m w_2^n = f(w_1, w_2).
$$

$\square$

4. Proof of Theorem 1.1.

4.1. **Change of Variables.** Let U and V be two bounded domains in $\mathbb{C}^n$, and let $\Phi : U \to V$ be a biholomorphic map. We set

$$
u = \left| \det \Phi' \right| \text{ and } v = \left| (\det \Phi') \circ \Phi^{-1} \right|.
$$

We prove the following:

Lemma 4.1. Let α be a real number. The following are equivalent:
   (1) $\mathbf{B}_U$ is bounded and surjective from $L^p(U, u^{2-p})$ to $A^p(U, u^{2-p+\alpha})$.
   (2) $\mathbf{B}_V$ is bounded and surjective from $L^p(V)$ to $A^p(V, v^\alpha)$.

For a function h on V define

$$
T_\Phi h = \det(\Phi') \cdot (h \circ \Phi). \tag{10}
$$

Let α be a real number. We first note the following

Lemma 4.2. The map

$$
T_\Phi : L^p(V, v^\alpha) \to L^p(U, u^{2-p+\alpha}) \tag{11}
$$

is an isometric isomorphism of Banach spaces.

Proof. Indeed,

$$
\|T_\Phi f\|_{L^p(U, u^{2-p+\alpha})}^p = \int_U \left| (\det \Phi') \cdot (f \circ \Phi) \right|^p \left| \det \Phi' \right|^{2-p+\alpha} dV
$$

$$
= \int_U |f \circ \Phi|^p \left| \det \Phi' \right|^\alpha \left| \det \Phi' \right|^2 dV
$$

$$
= \int_V |f|^p \left| \det \Phi' \circ \Phi^{-1} \right|^\alpha dV,
$$

where in the last line we use the change of variables formula. $\square$

Proof of Lemma 4.1. Recall the Bell transformation formula for the Bergman projection (cf. [Bel81]), which can be written as

$$\mathbf{B}_U \circ T_\Phi = T_\Phi \circ \mathbf{B}_V, \tag{12}$$

So that we have $\mathbf{B}_V = T_\Phi^{-1} \circ \mathbf{B}_U \circ T_\Phi$. Start with $1 \Longrightarrow 2$. We know that T_Φ is an isometry from $L^p(V)$ to $L^p(U, u^{2-p})$, by hypothesis, $\mathbf{B}_U$ is bounded from $L^p(U, u^{2-p})$ to $A^p(U, u^{2-p+\alpha})$, and we also know that T_Φ^{-1} is an isometry from $A^p(U, u^{2-p+\alpha})$ to $A^p(V, v^\alpha)$, so that we have (2). The part $2 \Longrightarrow 1$ can be done exactly the same way. $\square$

4.2. **Proof of Theorem 1.1.** We apply Lemma 4.1. Let $\Phi : \mathbb{D}^* \times \mathbb{D} \to \mathbb{H}$ be given by

$$\Phi(w_1, w_2) = (w_1, w_1 w_2), \tag{13}$$

so that in Lemma 4.1, we have

$$u(w_1, w_2) = |w_1| \text{ and } v(z_1, z_2) = |z_1|,$$

and we take $\alpha = p - 2$. Lemma 4.1 shows that the following statements are equivalent:

(1) $\mathbf{B}_\mathbb{H}$ is bounded and surjective from $L^p(\mathbb{H})$ to $A^p(\mathbb{H}, \delta_1^{p-2})$.
(2) $\mathbf{B}_{\mathbb{D}^* \times \mathbb{D}}$ is bounded and surjective from $L^p(\mathbb{D}^* \times \mathbb{D}, \delta_1^{2-p})$ to $A^p(\mathbb{D}^* \times \mathbb{D})$.

Since the second statement was proved in Proposition 3.1, the result follows for $p \geq 2$.

5. Proof of Corollary 1.1

5.1. **Weighted Bergman spaces on the punctured disc.** On the punctured unit disc $\mathbb{D}^* = \{ z \in \mathbb{C} \mid 0 < |z| < 1 \}$, denote by δ the weight function $\delta(z) = |z|$.

Lemma 5.1. *Let $p \geq 2$ and $-2 < \alpha \leq 0$. Then we have*

$$A^p(\mathbb{D}^*, \delta^\alpha) = A^p(\mathbb{D}), \tag{14}$$

and the two norms $\|\cdot\|_{L^p(\mathbb{D})}$ and $\|\cdot\|_{L^p(\mathbb{D}, \delta^\alpha)}$ on this space are equivalent.

Proof. We first claim that each function in $A^p(\mathbb{D}^*, \delta^\alpha)$ has a *removable* singularity at 0. Since $-2 < \alpha \leq 0$, we have

$$\int_\mathbb{D} |w|^\alpha \, dV(w) = 2\pi \int_0^1 r^{1+\alpha} dr < \infty, \tag{15}$$

so that $\delta^\alpha \in L^1(\mathbb{D})$. It follows by Hölder's inequality that $A^p(\mathbb{D}^*) \subset A^2(\mathbb{D}^*)$, so that to establish the claim it is sufficient to show that each function in $A^2(\mathbb{D}^*)$ extends holomorphically to 0. Now if $f \in \mathcal{O}(\mathbb{D}^*)$, writing $f(w) = \sum_{\nu=-\infty}^\infty a_\nu w^\nu$, we have $\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = 2\pi \sum_{\nu=-\infty}^\infty |a_\nu|^2 r^{2\nu}$. If $f \in A^2(\mathbb{D}^*)$ we have

$$\|f\|_{A^2(\mathbb{D}^*)}^2 = \int_0^1 \left(\int_0^{2\pi} \left| f\left(re^{i\theta}\right) \right|^2 d\theta \right) r dr$$

$$= \sum_{\nu=-\infty}^\infty |a_\nu|^2 \int_0^1 r^{2\nu+1} dr,$$

by the monotone convergence theorem. Since this is finite, we must have $a_\nu = 0$ unless $2\nu+1 > -1$, i.e., $\nu \geq 0$, which verifies the claim that the singularity of f at 0 is removable.

Since every element of $A^p(\mathbb{D}^*)$ has a removable singularity at 0, it follows that $A^p(\mathbb{D}^*) = A^p(\mathbb{D})$. Now since $\alpha \leq 0$, we have $\delta^\alpha \geq 1$ on $\mathbb{D}$, so we have a continuous embedding

$$A^p(\mathbb{D}^*, \delta^\alpha) \hookrightarrow A^p(\mathbb{D}^*) = A^p(\mathbb{D}). \tag{16}$$

We claim that the injective map in (16) is in fact surjective. Let $f \in A^p(\mathbb{D})$. We have

$$\|f\|^p_{L^p(\mathbb{D}^*, \delta^\alpha)} = \int_{\mathbb{D}} |f(w)|^p |w|^\alpha \, dV(w)$$

$$= \int_{|w|<\frac{1}{2}} |f(w)|^p |w|^\alpha \, dV(w) + \int_{|w|\geq\frac{1}{2}} |f(w)|^p |w|^\alpha \, dV(w)$$

$$\leq C \int |w|^\alpha \, dV(w) + \frac{1}{2^\alpha} \int_{|w|\geq\frac{1}{2}} |f(w)|^p \, dV(w),$$

The first of these integrals is finite (see (15)) and so is the second one since by hypothesis $f \in L^p(\mathbb{D})$. Therefore the map in (16) is a continuous bijection and the result follows by the open mapping theorem. $\qquad \square$

We deduce the following from Lemma 5.1:

Proposition 5.1. *If $2 \leq p < 4$ then*

$$A^p(\mathbb{H}, \delta_1^{p-2}) = A^p(\mathbb{H}),$$

and the two norms $\|\cdot\|_{L^p(\mathbb{H})}$ and $\|\cdot\|_{L^p(\mathbb{H}, \delta_1^\alpha)}$ on this space are equivalent.

Proof. Let δ_1 be as in (2). Then by an application of Fubini's theorem to the bidisc and Lemma 5.1, we see that $A^p(\mathbb{D}^* \times \mathbb{D}, \delta_1^\alpha) = A^p(\mathbb{D}^2)$, with equivalence of norms, provided $p \geq 2$ and $-2 < \alpha \leq 0$. In particular, if $2 \leq p < 4$, then $-2 < 2 - p \leq 0$, so that we have

$$A^p(\mathbb{D}^* \times \mathbb{D}, \delta_1^{2-p}) = A^p(\mathbb{D}^2). \tag{17}$$

Let the map $\Phi : \mathbb{D}^* \times \mathbb{D} \to \mathbb{H}$ be given by (13), and let T_Φ be as in (10). Noting that T_ϕ maps holomorphic functions to holomorphic functions, we see from Lemma 4.2 that for each real α, $T_\Phi : A^p(\mathbb{H}, \delta_1^\alpha) \to A^p(\mathbb{D}^* \times \mathbb{D}, \delta_1^{2-p+\alpha})$ is an isometric isomorphism of Banach spaces. Letting $\alpha = 0$, we see that $T_\Phi : A^p(\mathbb{H}) \to A^p(\mathbb{D}^* \times \mathbb{D}, \delta_1^{2-p})$ is an isometric isomorphism, and letting $\alpha = p - 2$, we see that $T_\Phi : A^p(\mathbb{H}, \delta_1^{p-2}) \to A^p(\mathbb{D}^* \times \mathbb{D})$ is also an isometric isomorphism. But we already saw that $A^p(\mathbb{D}^* \times \mathbb{D}, \delta_1^{2-p})$ and $A^p(\mathbb{D}^* \times \mathbb{D})$ are the same space of functions on $\mathbb{D}^* \times \mathbb{D}$, and the two Banach-space norms are equivalent. The result follows. $\qquad \square$

5.2. **Proof of Corollary 1.1.** For part (a), we combine Theorem 1.1 with Proposition 5.1. By the former, if $2 \leq p < 4$, the operator $\mathbf{B}_\mathbb{H}$ is bounded from $L^p(\mathbb{H})$ to $A^p(\mathbb{H}, \delta_1^{p-2})$ and by the latter, the space $A^p(\mathbb{H}, \delta_1^{p-2})$ is the same as $A^p(\mathbb{H})$ as a topological vector space, therefore the result follows for $2 \leq p < 4$. Applying Lemma 2.1 with $\omega \equiv 1$, we obtain that $\mathbf{B}_\mathbb{H}$ is bounded (and hence surjective) from $L^p(\mathbb{H})$ to $A^p(\mathbb{H})$ when $\frac{4}{3} < p < 2$.

We showed in Section 3 above that $\mathbf{B}_\mathbb{H}$ is bounded and *surjective* from $L^p(\mathbb{H})$ onto $A^p(\mathbb{H}, \delta_1^{p-2})$. Consequently to prove part (b) of Corollary 1.1 it suffices to show that, for $p \geq 4$, there is a function $f \in A^p(\mathbb{H}, \delta_1^{p-2})$ which is not in $A^p(\mathbb{H})$. Such a function is $f(z_1, z_2) = \frac{1}{z_1}$. The computation leading to (2) shows that $f \notin A^p(\mathbb{H})$ if $p \geq 4$. On the other hand, by a direct computation;

$$\|f\|^p_{L^p(\mathbb{H}, \delta_1^{p-2})} = \int_{\mathbb{H}} \frac{1}{|z_1|^p} |z_1|^{p-2} \, dV$$

$$= 4\pi^2 \int_{r_1=0}^1 \int_{r_2=0}^{r_1} \frac{1}{r_1^2} r_1 r_2 \, dr_1 dr_2$$

$$= 2\pi^2,$$

so that the result follows.

References

[Bar84] David E. Barrett. Irregularity of the Bergman projection on a smooth bounded domain in $\mathbf{C}^2$. *Ann. of Math. (2)*, 119(2):431–436, 1984.

[Bel81] Steven R. Bell. Proper holomorphic mappings and the Bergman projection. *Duke Math. J.*, 48(1):167–175, 1981.

[BS99] Harold P. Boas and Emil J. Straube. Global regularity of the $\overline{\partial}$-Neumann problem: a survey of the L^2-Sobolev theory. In *Several complex variables (Berkeley, CA, 1995–1996)*, volume 37 of *Math. Sci. Res. Inst. Publ.*, pages 79–111. Cambridge Univ. Press, Cambridge, 1999.

[BŞ12] David Barrett and Sönmez Şahutoğlu. Irregularity of the Bergman projection on worm domains in $\mathbb{C}^n$. *Michigan Math. J.*, 61(1):187–198, 2012.

[CC91] J. Chaumat and A.-M. Chollet. Régularité höldérienne de l'opérateur $\overline{\partial}$ sur le triangle de Hartogs. *Ann. Inst. Fourier (Grenoble)*, 41(4):867–882, 1991.

[Che13] Liwei Chen. The L^p boundedness of the Bergman projection for a class of bounded Hartogs domains. *arXiv:1304.7898*, 2013.

[CS13] Debraj Chakrabarti and Mei-Chi Shaw. Sobolev regularity of the $\overline{\partial}$-equation on the Hartogs triangle. *Math. Ann.*, 356(1):241–258, 2013.

[EL08] Dariush Ehsani and Ingo Lieb. L^p-estimates for the Bergman projection on strictly pseudoconvex non-smooth domains. *Math. Nachr.*, 281(7):916–929, 2008.

[FR75] Frank Forelli and Walter Rudin. Projections on spaces of holomorphic functions in balls. *Indiana Univ. Math. J.*, 24:593–602, 1974/75.

[Gra08] Loukas Grafakos. *Classical Fourier analysis*, volume 249 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2008.

[KP08] Steven G. Krantz and Marco M. Peloso. Analysis and geometry on worm domains. *J. Geom. Anal.*, 18(2):478–510, 2008.

[Kra01] Steven G. Krantz. *Function theory of several complex variables*. AMS Chelsea Publishing, Providence, RI, 2001. Reprint of the 1992 edition.

[LS12] Loredana Lanzani and Elias M. Stein. The Bergman projection in L^p for domains with minimal smoothness. *Illinois J. Math.*, 56(1):127–154 (2013), 2012.

[MM92] Lan Ma and Joachim Michel. $\mathcal{C}^{k+a}$-estimates for the $\overline{\partial}$-equation on the Hartogs triangle. *Math. Ann.*, 294(4):661–675, 1992.

[MS94] J. D. McNeal and E. M. Stein. Mapping properties of the Bergman projection on convex domains of finite type. *Duke Math. J.*, 73(1):177–199, 1994.

[PS77] D. H. Phong and E. M. Stein. Estimates for the Bergman and Szegö projections on strongly pseudo-convex domains. *Duke Math. J.*, 44(3):695–704, 1977.

[Zey13] Yunus E. Zeytuncu. L^p regularity of weighted Bergman projections. *Trans. Amer. Math. Soc.*, 365(6):2959–2976, 2013.

[Zhu91] Ke He Zhu. Duality of Bloch spaces and norm convergence of Taylor series. *Michigan Math. J.*, 38(1):89–101, 1991.

[Zhu07] Kehe Zhu. *Operator theory in function spaces*, volume 138 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2007.

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