\textbf{Lp-regularity of the Bergman projection on quotient domains}

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Abstract. We relate the Lp-mapping properties of the Bergman projections on two domains in \( \mathbb{C}^n \), one of which is the quotient of the other under the action of a finite group of biholomorphic automorphisms. We use this relation to deduce the sharp ranges of \( L^p \)-boundedness of the Bergman projection on certain \( n \)-dimensional model domains generalizing the Hartogs triangle.

1. Introduction

1.1. \( L^p \)-regularity on singular Reinhardt domains. A recent series of intriguing results has drawn considerable attention to the boundedness of the Bergman projection in the \( L^p \)-norm on a class of highly singular domains. Among the most remarkable results in this vein is one that concerns the so-called \textit{generalized Hartogs triangles}, defined for coprime positive integers \( m, n \) as

\[ H_{m/n} = \{ (z_1, z_2) \in \mathbb{C}^2 : |z_1|^{m/n} < |z_2| < 1 \} \]  

(1.1)

The result is that the Bergman projection is bounded in \( L^p(H_{m/n}) \) if and only if

\[ \frac{2(m + n)}{m + n + 1} < p < \frac{2(m + n)}{m + n - 1}. \]  

(1.2)

It is striking that the range (1.2) should depend on the “arithmetic complexity” \( m + n \) of the domain \( (1.1) \), rather than the “fatness exponent” \( \frac{m}{n} \) which determines the shape of the domain as a subset of \( \mathbb{C}^2 \). The proof of this result – see [EM17] – consists of an explicit computation of the Bergman kernel, followed by an application of Schur’s test to determine the range of \( L^p \)-boundedness. See [CZ16, Edh16, EM16, Huo18, HW19, CKY20, EM20, CJY20] for other results in this circle of ideas.

The range of \( L^p \)-boundedness of the Bergman projection on a domain is a function theoretic property determined by the domain’s Hermitian geometry, but the full extent of this relationship is yet to be understood. In particular, it is of great interest to obtain a clear description of the way in which the origin singularity of \( (1.1) \) gives rise to the range \( (1.2) \). This article brings to bear a new perspective on this problem: one in which \( H_{m/n} \) is realized as a quotient of the domain \( \Omega_1 = \mathbb{D} \times \mathbb{D}^* \) (the product of the unit disc with a punctured disc) under the action of a group of biholomorphic automorphisms. This point of view leads to a transformation law relating the \( L^p \)-Bergman spaces and Bergman projections of two domains related in such a way – one that is closely connected to the well-known Bell’s transformation law relating the Bergman kernels of the two domains under proper holomorphic mappings. The \( L^p \)-mapping properties of the Bergman projection on \( H_{m/n} \) and kindred domains in higher dimensions may thus be obtained as consequences of a general theorem together with an understanding of the branching behavior of a proper map of \textit{quotient type} \( \Phi \) from a domain \( \Omega_1 \) on which we have a full understanding of the Bergman projection in \( L^p \)-norms to the domain \( \Omega_2 \) in which we are interested; see Definition [2.1] The precise connection between the \( L^p \)-boundedness of the Bergman projection on two domains related by such a map is explained in Theorem [3.12] below. The utility of this theorem will then be illustrated for two very

2010 Mathematics Subject Classification. 32A36.

Debraj Chakrabarti was partially supported by National Science Foundation grant DMS-1600371.
different families of domains generalizing the Hartogs triangle of \((1,1)\) to higher dimensions. The number-theoretic aspects of the \(L^p\)-boundedness interval demonstrated in \((1.2)\) are then seen as arising from the branching behavior of the map \(\Phi\).

Let \(k = (k_1, \ldots, k_n)\) be an \(n\)-tuple of positive integers. The first class of domains of the form

\[
\mathcal{H}_k = \left\{ z \in \mathbb{D}^n : |z_1|^{k_1} < \prod_{j=2}^{n} |z_j|^{k_j} \right\},
\]

(1.3)

where \(\mathbb{D}^n\) is the \(n\)-dimensional polydisc. These pseudoconvex Reinhardt domains generalizing the Hartogs triangle \(\mathcal{H}_{(1,1)} = \{ |z_1| < |z_2| < 1 \} \subset \mathbb{C}^2\) were introduced in [CKMM19] where they were called \textit{elementary Reinhardt domains of signature 1}.

The second class of domains considered also involves \(n\)-tuples \(k = (k_1, \ldots, k_n)\) of coprime positive integers. Define

\[
\mathcal{H}_k = \left\{ z \in \mathbb{D}^n : |z_1|^{k_1} < |z_2|^{k_2} \cdots |z_n|^{k_n} < 1 \right\}.
\]

(1.4)

In [Par18], the Bergman kernel of \(\mathcal{H}_k\) was explicitly computed for \(n = 3\), and in [Che17] the special case \(k = (1,1,\ldots, 1)\) was considered and it was shown that the Bergman projection is \(L^p\)-bounded on \(\mathcal{H}_{(1,1,\ldots, 1)}\) if and only if \(\frac{2n}{n+1} < p < \frac{2n}{n-1}\).

For both \(\mathcal{H}_k\) and \(\mathcal{H}_k\), it is possible to use the method of [EM17] to deduce the range of \(L^p\)-boundedness of the Bergman projection, using the Schur’s test and an explicit representation of the Bergman kernel. This has been carried out in the preprints [Zha20, Zha19]. Our main interest here is to demonstrate the geometric origin of the range of \(L^p\)-boundedness in these domains. We now state the precise results on Bergman \(L^p\)-boundedness for these classes of domains:

**Theorem 1.5.** Let \(k = (k_1, \ldots, k_n)\) be an \(n\)-tuple of positive integers. The Bergman projection on \(\mathcal{H}_k\) is bounded from \(L^p(\mathcal{H}_k)\) to \(A^p(\mathcal{H}_k)\) if and only if

\[
\max_{2 \leq j \leq n} \frac{2(k_1 + k_j)}{k_1 + k_j + \gcd(k_1, k_j)} < p < \min_{2 \leq j \leq n} \frac{2(k_1 + k_j)}{k_1 + k_j - \gcd(k_1, k_j)}.
\]

(1.6)

In order to state the result for \(\mathcal{H}_k\) in the most concise way, we introduce the following notation. Given an \(n\)-tuple \(k = (k_1, \ldots, k_n)\) of positive integers, define

\[
\ell_j = \ell_j(k) = \frac{\text{lcm}(k_1, \ldots, k_n)}{k_j} \text{ for } 1 \leq j \leq n.
\]

(1.7)

Then we have

**Theorem 1.8.** Let \(k = (k_1, \ldots, k_n)\) be an \(n\)-tuple of positive integers, and let

\[
\Lambda = \Lambda(k) := \sum_{j=1}^{n} \ell_j,
\]

(1.9)

where the \(\ell_j\) are as in \((1.7)\). The Bergman projection on \(\mathcal{H}_k\) is bounded in the \(L^p\)-norm if and only if

\[
\frac{2\Lambda}{\Lambda + 1} < p < \frac{2\Lambda}{\Lambda - 1}.
\]

(1.10)

More generally, this method applies to images of the polydisc under monomial maps, a class of domains whose Bergman kernels were computed in [NP20]. It should be emphasized that our proofs of Theorems 1.5 and 1.8 make no explicit use to the Bergman kernel formulas on either \(\mathcal{H}_k\) or \(\mathcal{H}_k\), as Theorem 3.12 allows all hard analysis to take place on the covering domain. This method has likely application to a much wider class of domains with boundary singularities.
1.2. Bergman space and projection. We collect here some general information about the Bergman projection, and set up notation for later use.

Let $\Omega$ be a domain (an open connected set) in $\mathbb{C}^n$. The Bergman space $A^2(\Omega)$ – which dates back to the work of S. Bergman in [Ber22] – is the Hilbert space of holomorphic functions on $\Omega$ which are square integrable with respect to the Lebesgue measure; see [Kra13] for a modern treatment. The space $A^2(\Omega)$ is a closed subspace of $L^2(\Omega)$, the usual Hilbert space of measurable functions square integrable with respect to the Lebesgue measure. The Bergman projection is the orthogonal projection

$$B_{\Omega} : L^2(\Omega) \rightarrow A^2(\Omega).$$

The construction of Bergman spaces has a contravariant functorial character. If $\phi : \Omega_1 \rightarrow \Omega_2$ is a holomorphic map of domains, we can associate a continuous linear mapping of Hilbert spaces $\phi^\sharp : L^2(\Omega_2) \rightarrow L^2(\Omega_1)$ defined for each $f \in L^2(\Omega_2)$ by

$$\phi^\sharp(f) = f \circ \phi \cdot \det \phi',$$

where $\phi'(z) : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is the complex derivative the map $\phi$ at $z \in \Omega_2$. It is clear that $\phi^\sharp$ restricts to a map $A^2(\Omega_2) \rightarrow A^2(\Omega_1)$. We will refer to $\phi^\sharp$ as the pullback induced by $\phi$. It is not difficult to see that if $\phi$ is a biholomorphism, then the pullback $\phi^\sharp$ is an isometric isomorphism of Hilbert spaces $L^2(\Omega_2) \cong L^2(\Omega_1)$, and restricts to an isometric isomorphism $A^2(\Omega_2) \cong A^2(\Omega_1)$. This biholomorphic invariance of Bergman spaces can be understood intrinsically by interpreting the Bergman space as a space of top-degree holomorphic forms (cf. Was55, Kob59 or Kra13 pp. 178 ff.), and the map $\phi^\sharp$ as the pullback map of forms induced by the holomorphic map $\phi$. This invariance can be extended to proper holomorphic mappings via Bell’s transformation formula, and lies at the heart of classical applications of Bergman theory to the boundary regularity of holomorphic maps; see Bel81, Bel82, DFS82, BC82.

For $0 < p < \infty$, define $L^p$-Bergman spaces $A^p(\Omega)$ of $p$-th power integrable holomorphic functions on $\Omega$. For $p \geq 1$, these are Banach spaces when equipped with the $L^p$-norm. An extensive theory of these spaces on the unit disc has been developed, in analogy with the theory of Hardy spaces (cf. DS04, HKZ00). Unlike the $L^2$-Bergman space, the general $L^p$-Bergman space is not invariantly determined by the complex structure alone, but also depends on the Hermitian structure of the domain as a subset of $\mathbb{C}^n$. An important question about these spaces has been the boundedness of the Bergman projection in the $L^p$-norm. After initial results were obtained for discs and balls (ZJ64, FR75), the problem was studied on various classes of smoothly bounded pseudoconvex domains using descriptions of kernel asymptotics (e.g. PS77, MS94). On these domains the Bergman projection is bounded in $L^p$ for $1 < p < \infty$. Many examples have been given which show that there are domains on which the Bergman projection fails to be bounded in $L^p$ for certain $p$. See Bar84, BS12, KP08, Hed02, in addition to the domains already mentioned in CZ16, EM16, EM17, Che17, CKY20, Huo18, HW19. Theorem 3.12 provides a quantifiable and precise statement relating the Bergman $L^p$-mapping regularity on domains with geometric boundary singularities induced by the singularities of a proper holomorphic covering maps of quotient type, and the Bergman mapping regularity on certain invariant function spaces associated to the covering space. The germ of the idea of relating the $L^p$-regularity of the Bergman projection with the properties of a “resolving” map from a simpler domain may already be found in CZ16, CKY20. The sharper version of this technique presented in this paper may be thought of as a step towards a unified understanding of the way in which boundary singularities affect the mapping properties of the Bergman projection.

1.3. Acknowledgments. We would like to thank Yuan Yuan for very interesting discussions about this problem and the results of CKY20 during the 2019 Midwest Several Complex Variables Conference at Dearborn, MI, Steven Krantz for comments on the same paper made to the second-named author during a visit to Washington University at St. Louis, MO, in the same year. We would also like to thank Shuo Zhang for pointing out his work in Zha20, Zha19.
2. Preliminaries

2.1. Quotients of domains. We begin with a discussion of certain proper holomorphic maps:

Definition 2.1. Let $\Omega_1, \Omega_2 \subset \mathbb{C}^n$ be domains, let $\Phi : \Omega_1 \to \Omega_2$ be a proper holomorphic mapping and let $\Gamma$ be a group of biholomorphic automorphisms of $\Omega_1$. We will say that $\Phi$ is of quotient type with group $\Gamma$ if for each $z \in \Omega_2$, the action of $\Gamma$ on $\Omega_1$ restricts to a transitive action on the fiber $\Phi^{-1}(z)$.

The name “quotient type” finds justification in the following:

Proposition 2.2. Let $\Phi : \Omega_1 \to \Omega_2$ be a proper holomorphic map of quotient type with group $\Gamma$. Let $\Omega_1^* = \Omega_1 \setminus \{\det \Phi' = 0\}$ be the set of regular points of $\Phi$ and $\Omega_2^* = \Omega_2 \setminus \{\Phi(\{\det \Phi' = 0\})\}$ be the set of regular values of $\Phi$. Then the restriction $\Phi^{\text{reg}} := \Phi|_{\Omega_1^*} : \Omega_1^* \to \Omega_2^*$ is a holomorphic covering map of $|\Gamma|$ sheets, and $\Gamma$ is its group of deck transformations, i.e.,

$$\Gamma = \{\sigma : \Omega_1^* \to \Omega_1^* \text{ is biholomorphic}, \text{ and } \Phi^{\text{reg}} \circ \sigma = \Phi^{\text{reg}}\}.$$ 

Further, $\Gamma$ is a finite group acting properly discontinuously on $\Omega_1^*$, and the quotient $\Omega_1^*/\Gamma$ is biholomorphic to the domain $\Omega_2^*$.

Proof. The restriction $\Phi^{\text{reg}}$ is a local biholomorphism by the (holomorphic) inverse function theorem. Let $w \in \Omega_2^*$ and let $\{z_1, \ldots, z_d\}$ be its preimages in $\Omega_1^*$ (there are finitely many preimages since $\Phi$ is proper). Since $\Phi$ is a local biholomorphism from some neighborhood of each $z_j$ to a neighborhood of $w$, it follows after shrinking that there is a neighborhood $V$ of $w$ and a neighborhood $U_j$ of each $z_j$ ($j = 1, \ldots, d$) such that $\Phi$ maps $U_j$ biholomorphically to $V$, i.e., $\Phi^{\text{reg}}$ is a holomorphic covering map of finite degree. (This is a well-known property of proper holomorphic maps).

The group $\Gamma$ is clearly a subgroup of the group of deck transformations of the covering map $\Phi^{\text{reg}}$. Therefore, the group of deck transformations acts transitively on each fiber of the covering map $\Phi^{\text{reg}}$ and consequently, by well-known results topology, $\Phi^{\text{reg}}$ is a so-called regular covering map (cf. [Mas91, page 135 ff.]). It can consequently be identified with the construction of a quotient by the group of deck-transformations, which acts freely on $\Omega_1$, and therefore simply-transitively on the fibers. Notice that by the hypothesis of transitivity of the group $\Gamma$ on the fibers of $\Phi$, it follows that for each $\sigma \in \Gamma$, we have

$$\Phi^{\text{reg}} \circ \sigma = \Phi^{\text{reg}},$$

so that $\Gamma$ is a subgroup of the group of deck-transformations of $\Phi^{\text{reg}}$. But since the deck-transformations act simply transitively, and $\Gamma$ acts transitively, we see that $\Gamma$ is in fact the full group of deck-transformations. Consequently, topologically, one can identify the covering map $\Phi^{\text{reg}}$ with the quotient map $\Omega_1^* \to \Omega_1^*/\Gamma$, where the group $\Gamma$ acts properly discontinuously on $\Omega_1^*$. It is clear that the quotient $\Omega_1^*/\Gamma$ has a natural complex structure. \qed

Remark 2.3. If $X$ is a complex manifold, and $G$ is a finite group of biholomorphic automorphisms of $X$, then one can show (see [Car57, Car60]) that there is a natural structure of a complex-analytic space (i.e. locally biholomorphic to an analytic set, perhaps singular) on the quotient $X/G$. Therefore, in Proposition 2.2 one can think of $\Omega_2$ as being biholomorphic to $\Omega_1/\Gamma$ in the category of complex-analytic spaces, and $\Phi$ is then identified with the quotient map in the same category. \diamond

2.2. Group invariant Bergman subspaces.

Definition 2.4. Given a group $G$ of biholomorphic automorphisms of a domain $\Omega \subset \mathbb{C}^n$, and a space $\mathcal{F}$ of functions on $\Omega$, we denote by $[\mathcal{F}]^G$ the subspace of $\mathcal{F}$ consisting of functions which are $G$-invariant in the following sense

$$[\mathcal{F}]^G = \{f \in \mathcal{F} : f = \sigma^*(f) \text{ for all } \sigma \in G\},$$
where $\sigma^\sharp$ is the pullback induced by $\sigma$ as in (1.11).

Remark 2.5. Interpreting $\mathfrak{F}$ as a space of holomorphic forms on $\Omega$ by associating $f \in \mathfrak{F}$ with the form $f dz_1 \wedge \cdots \wedge dz_n$, this is simply says that the forms in $[\mathfrak{F}]^G$ are invariant under pullback by elements of $G$.

The following shows that such $G$-invariant subspaces interact well with the Bergman projection:

**Proposition 2.6.** Let $\Omega \subset \mathbb{C}^n$ be a domain, and let $B_{\Omega}$ be its Bergman projection operator.

1. If $\sigma \in \text{Aut}(\Omega)$ is a biholomorphic automorphism, then
   \[ B_{\Omega} \circ \sigma^\sharp = \sigma^\sharp \circ B_{\Omega}. \tag{2.7} \]

2. If $G \subset \text{Aut}(\Omega)$ is a group of biholomorphic automorphisms, then $B_{\Omega}$ restricts to the orthogonal projection operator from $[L^2(\Omega)]^G$ onto $[A^2(\Omega)]^G$.

**Proof.** For (1), note that $\sigma^\sharp$ is a unitary operator on $L^2(\Omega)$ and $B_{\Omega}$ is an orthogonal projection on $L^2(\Omega)$, therefore the unitarily similar operator $Q = \sigma^\sharp \circ B_{\Omega} \circ (\sigma^\sharp)^{-1}$ is also an orthogonal projection. Since $\sigma^\sharp$ (and therefore its inverse) leaves $A^2(\Omega)$ invariant, it follows that the range of $Q$ is $A^2(\Omega)$. Therefore $Q = B_{\Omega}$.

For (2), let $f \in [L^2(\Omega)]^G$. Then using (2.7), we have for $\sigma \in G$
\[ \sigma^\sharp(B_{\Omega}f) = B_{\Omega}(\sigma^\sharp(f)) = B_{\Omega}f, \]
which shows that $B_{\Omega}f \in [A^2(\Omega)]^G$, so that $B_{\Omega}$ maps the $G$-invariant functions $[L^2(\Omega)]^G$ into the $G$-invariant holomorphic functions $[A^2(\Omega)]^G$. Since $B_{\Omega}$ restricts to the identity on $[A^2(\Omega)]^G$, it follows that the range of $B_{\Omega}$ is $[A^2(\Omega)]^G$. Observe that
\[ \ker (B_{\Omega}|[L^2(\Omega)]^G) \subseteq \ker B_{\Omega} = A^2(\Omega)^{\perp} \subseteq ([A^2(\Omega)]^G)^{\perp}, \]
which shows that kernel of the restriction of $B_{\Omega}$ to $[L^2(\Omega)]^G$ is orthogonal to its range, and therefore an orthogonal projection. \( \square \)

3. Transformation of $L^p$-Bergman spaces under quotient maps

3.1. Definitions. In order to state our results, we introduce some terminology:

**Definition 3.1.** We say that a linear map $T$ between Banach spaces $(E_1, \|\cdot\|_1)$ and $(E_2, \|\cdot\|_2)$ is a **homothetic isomorphism** if it is a continuous bijection (and therefore has a continuous inverse) and there is a constant $C > 0$ such that for each $x \in E_1$ we have
\[ \|Tx\|_2 = C \|x\|_1. \]

**Remark 3.2.** Notice that if $T$ is a homothetic isomorphism between Banach spaces, then clearly $\frac{T}{\|T\|}$ is an isometric isomorphism of Banach spaces, so a homothetic isomorphism is simply the product of an isometric isomorphism and a scalar operator. In particular, a homothetic isomorphism between Hilbert spaces preserves angles, and in particular orthogonality of vectors.

The following definition and facts are standard:

**Definition 3.3.** Let $\Omega \subset \mathbb{C}^n$ be a domain, let $\lambda > 0$ be a continuous function (the weight), and let $0 < p < \infty$. Then we define
\[ L^p(\Omega, \lambda) = \left\{ f : \Omega \to \mathbb{C} \text{ measurable} : \int_{\Omega} |f|^p \lambda dV < \infty \right\} \]
and
\[ A^p(\Omega, \lambda) = \left\{ f : \Omega \to \mathbb{C} \text{ holomorphic} : \int_{\Omega} |f|^p \lambda dV < \infty \right\}, \]
where the latter is called a weighted Bergman space. If \( p \geq 1 \), \( L^p(\Omega, \lambda) \) and \( A^p(\Omega, \lambda) \) are Banach spaces with the obvious norm, and \( A^p(\Omega, \lambda) \) is a closed subspace of \( L^p(\Omega, \lambda) \).

### 3.2. Transformation of Bergman spaces

With the above definitions, we are ready to state and prove the following:

**Proposition 3.4.** Let \( \Omega_1, \Omega_2 \) be domains in \( \mathbb{C}^n \), and let \( \Phi : \Omega_1 \to \Omega_2 \) be a proper holomorphic map of quotient type with group \( \Gamma \subset \text{Aut}(\Omega_1) \). Then for \( 1 < p < \infty \), the pullback map \( \Phi^\# \) gives rise to a homothetic isomorphism

\[
\Phi^\# : L^p(\Omega_1, |\det \Phi'|^{-p}) \to \left[ L^p \left( \Omega_1, |\det \Phi'|^{-2p} \right) \right]^{\Gamma}.
\]

This restricts to a homothetic isomorphism

\[
\Phi^\# : A^p(\Omega_2) \to \left[ A^p \left( \Omega_1, |\det \Phi'|^{-p} \right) \right]^{\Gamma}.
\]

**Proof.** Let \( f \) be a function on \( \Omega_2 \), and let \( g = \Phi^\# f \) be its pullback to \( \Omega_1 \), then we have for each \( \sigma \in \Gamma \) that

\[
\sigma^\# (g) = \sigma^\# (\Phi^\# f) = (\Phi \circ \sigma)^\# f = \Phi^\# f = g,
\]

where we have used the contravariance of the pullback \((\Phi \circ \sigma)^\# f = \sigma^\# \Phi^\# f\) and the fact that \( \Phi \circ \sigma = \Phi \) which follows since the action of \( \Gamma \) on \( \Omega_1 \) restricts to actions on each of the fibers. This shows that the range of \( \Phi^\# \) consists of \( \Gamma \)-invariant functions. Special cases of this invariance were already noticed \([\text{MSRZ13, CKY20}]\).

To complete the proof of (3.5) we must show that

(i) \( \|\Phi^\# f\|_{L^p(\Omega_1, |\det \Phi'|^{-p})} = |\Gamma| \cdot \|f\|_{L^p(\Omega_2)} \) for each \( f \in L^p(\Omega_2) \), and

(ii) The image \( \Phi^\# (L^p(\Omega_2)) \) is precisely \( \left[ L^p \left( \Omega_1, |\det \Phi'|^{-2p} \right) \right]^{\Gamma} \).

Let \( \Omega_1^* \) and \( \Omega_2^\* \) be as in Proposition 2.2. Notice that \( \Omega_2 \setminus \Omega_1^\* \) is an analytic set for \( j = 1, 2 \) (for \( j = 2 \) this follows from Remmert’s theorem on the images of proper maps.) Let \( U \) be an open set in \( \Omega_2^\* \) which is evenly covered by \( \Phi \), and let \( V \) be an open set of \( \Omega_1^\* \) which is mapped biholomorphically by \( \Phi \) onto \( U \). Then the inverse image \( \Phi^{-1}(U) \) is the finite disjoint union \( \bigcup_{\sigma \in \Gamma} \sigma V \). Therefore if \( f \in L^p(\Omega_2) \) is supported in \( U \), then \( \Phi^\# f \) is supported in \( \bigcup_{\sigma \in \Gamma} \sigma V \), and we have

\[
\|\Phi^\# f\|_{L^p(\Omega_1, |\det \Phi'|^{-p})} = \left( \int_{\Omega_1} |f \cdot \Phi \cdot \det \Phi'|^p |\det \Phi'|^{-p} \right) dV = \sum_{\sigma \in \Gamma} \int_{\sigma V} |f \cdot \Phi| \cdot \det \Phi' \right|^2 dV
\]

\[
= \sum_{\sigma \in \Gamma} \int_{U} |f| \cdot \det \Phi' \right|^2 dV = |\Gamma| \cdot \|f\|_{L^p(\Omega_2)}^p
\]

where we have used the change of variables formula applied to the biholomorphic map \( \Phi|_{\sigma V} \) along with the fact that the real Jacobian determinant of the map \( \Phi \) is equal to \( |\det \Phi'\|^2 \).

For a general \( f \in L^p(\Omega_2) \), modify the proof as follows. There is clearly a collection of pairwise disjoint open sets \( \{U_j\}_{j \in J} \) in \( \Omega_2 \) such that each \( U_j \) is evenly covered by \( \Phi \) and \( \Omega_2 \setminus \bigcup_{j \in J} U_j \) has measure zero. Set \( f_j = f \cdot \chi_j \), where \( \chi_j \) is the indicator function of \( U_j \), so that each \( f_j \in L^p(\Omega_2) \) and \( f = \sum_j f_j \). Also, the functions \( \Phi^\# f_j \) have pairwise disjoint supports in \( \Omega_1 \). Therefore we have

\[
\|\Phi^\# f\|_{L^p(\Omega_1, |\det \Phi'|^{-p})} = \sum_{j \in J} \|\Phi^\# f_j\|_{L^p(\Omega_1, |\det \Phi'|^{-p})} = |\Gamma| \sum_{j \in J} \|f_j\|_{L^p(\Omega_2)}^p = |\Gamma| \cdot \|f\|_{L^p(\Omega_2)}^p
\]

To complete the proof, we need to show that \( \Phi^\# \) is surjective in both (3.5) and (3.6). Let \( g \in \left[ L^p \left( \Omega_1, |\det \Phi'|^{-2p} \right) \right]^{\Gamma} \). Let \( \{U_j\}_{j \in J} \) be as in the previous paragraph, and set \( g_j = g \cdot \chi_{\Phi^{-1}(U_j)} \), where \( \chi_{\Phi^{-1}(U_j)} \) is the indicator function of \( \Phi^{-1}(U_j) \). Notice that \( g_j \in \left[ L^p \left( \Omega_1, |\det \Phi'|^{-2p} \right) \right]^{\Gamma} \). Let
Let \( \Omega_1, \Omega_2 \) be domains in \( \mathbb{C}^n \) and let \( \Phi : \Omega_1 \to \Omega_2 \) be a proper holomorphic map of quotient type with group \( \Gamma \subset \text{Aut}(\Omega_1) \). Then the following diagram commutes:

\[
\begin{array}{ccc}
L^2(\Omega_2) & \xrightarrow{\Phi^\sharp} & [L^2(\Omega_1)]^\Gamma \\
\downarrow B_{\Omega_2} & & \downarrow B_{\Omega_1} \\
A^2(\Omega_2) & \xrightarrow{\Phi^\sharp} & [A^2(\Omega_1)]^\Gamma
\end{array}
\]  

(3.9)

Remark 3.10. This is a refinement (for the class of proper holomorphic maps of quotient type) of a classic result of Bell (see [Bel81, Theorem 1], [Bel82, Equation 2.2]).

Proof. By Proposition 3.4, the \( \Phi^\sharp \) represented by the top (resp. bottom) horizontal arrow is a homothetic isomorphism from the Hilbert space \( L^2(\Omega_2) \) (resp. \( A^2(\Omega_2) \)) onto the Hilbert space \([L^2(\Omega_1)]^\Gamma\) (resp. \([A^2(\Omega_1)]^\Gamma\)). Therefore, \( \Phi^\sharp \) preserves angles and in particular orthogonality. Now consider the map \( P : [L^2(\Omega_1)]^\Gamma \to [A^2(\Omega_1)]^\Gamma \) defined by

\[
P = \Phi^\sharp \circ B_{\Omega_2} \circ (\Phi^\sharp)^{-1},
\]

(3.11)

which, being a composition of continuous linear maps, is a continuous linear mapping of Hilbert spaces. Notice that

\[
P^2 = \Phi^\sharp \circ B_{\Omega_2} \circ (\Phi^\sharp)^{-1} \circ \Phi^\sharp \circ B_{\Omega_2} \circ (\Phi^\sharp)^{-1} = \Phi^\sharp \circ B_{\Omega_2} \circ (\Phi^\sharp)^{-1} = P,
\]

so \( P \) is a projection in \([L^2(\Omega_1)]^\Gamma\), with range contained in \([A^2(\Omega_1)]^\Gamma\). Since \((\Phi^\sharp)^{-1}\) and \(\Phi^\sharp|_{A^2(\Omega)}\) are isomorphisms, and \(B_{\Omega_2}\) is surjective, it follows that \( P \) is a projection onto \([A^2(\Omega_1)]^\Gamma\). We claim that \( P \) is in fact the orthogonal projection on to \([A^2(\Omega_1)]^\Gamma\), i.e., the kernel of \( P \) is \( ([A^2(\Omega_1)]^\Gamma)^{\perp} \), the orthogonal complement of \([A^2(\Omega_1)]^\Gamma\) in \([L^2(\Omega_1)]^\Gamma\). Since in formula (3.11), the maps \((\Phi^\sharp)^{-1}\) and \(\Phi^\sharp\) are isomorphisms, it follows that \( f \in \ker P \) if and only if \((\Phi^\sharp)^{-1} f \in \ker B_{\Omega_2}\). But \(\ker B_{\Omega_2} = A^2(\Omega_2)^{\perp} \), since the Bergman projection is orthogonal. It follows that \( \ker P = \Phi^\sharp(A^2(\Omega_2)^{\perp}) \). Notice that \(\Phi^\sharp\), being a homothetic isomorphism of Hilbert spaces, preserves orthogonality, and maps \( A^2(\Omega_2) \) to \([A^2(\Omega_1)]^\Gamma\) isomorphically, therefore \(\Phi^\sharp((A^2(\Omega_2))^{\perp}) = ([A^2(\Omega_1)]^\Gamma)^{\perp}\), which establishes the claim.
Therefore we have shown that $P = \Phi^4 \circ B_{\Omega_2} \circ (\Phi^2)^{-1}$ is the orthogonal projection from $[L^2(\Omega_1)]^\Gamma$ to the subspace $[A^2(\Omega_1)]^\Gamma$. To complete the proof, we only need to show that the restriction of the Bergman projection $B_{\Omega_1}$ to the $\Gamma$-invariant subspace $[L^2(\Omega_1)]^\Gamma$ is also the orthogonal projection from $[L^2(\Omega_1)]^\Gamma$ onto $[A^2(\Omega_1)]^\Gamma$. But this follows from Proposition 2.6 above.

Thus $P = B_{\Omega_1}|_{[L^2(\Omega_1)]^\Gamma}$, and the commutativity of (3.9) follows. □

3.4. Transformation of the Bergman projection in $L^p$-spaces. The following result will be our main tool in studying $L^p$-regularity of the Bergman projection:

**Theorem 3.12.** Let $\Omega_1, \Omega_2$ be bounded domains in $\mathbb{C}^n$, let $\Phi : \Omega_1 \to \Omega_2$ be a proper holomorphic map of quotient type with group $\Gamma \subset \text{Aut}(\Omega_1)$. Let $p \geq 1$. The following two assertions are equivalent:

1. The Bergman projection $B_{\Omega_2}$ gives rise to a bounded operator mapping $L^p(\Omega_2) \to A^p(\Omega_2)$.

2. The Bergman projection $B_{\Omega_1}$ gives rise to a bounded operator mapping

$$
[L^p(\Omega_1, |\det \Phi|^2)]^\Gamma \to [A^p(\Omega_1, |\det \Phi|^2)]^\Gamma.
$$

If one of the conditions (1) or (2) holds (and therefore both hold), then the following diagram commutes, where $B_{\Omega_j}, j = 1, 2$ denote the extension by continuity of the Bergman projection:

$$
\begin{array}{ccc}
L^p(\Omega_2) & \xrightarrow{\Phi^4} & [L^p(\Omega_1, |\det \Phi|^2)]^\Gamma \\
\downarrow B_{\Omega_2} & & \downarrow B_{\Omega_1} \\
A^p(\Omega_2) & \xrightarrow{\Phi^4} & [A^p(\Omega_1, |\det \Phi|^2)]^\Gamma
\end{array}
$$

(3.13)

**Remark 3.14.** A few comments are in order. First, statement (1) in Theorem 3.12 means more precisely the following: the restriction of the Bergman projection to a dense subspace of $L^p(\Omega_2)$ given by

$$
B_{\Omega_2} : L^2(\Omega_2) \cap L^p(\Omega_2) \to A^2(\Omega_2)
$$

is bounded in the $L^p$-norm, i.e., there is a $C > 0$ such that for all $f \in L^2(\Omega_2) \cap L^p(\Omega_2)$,

$$
\|B_{\Omega_2} f\|_{L^p(\Omega_2)} \leq C \|f\|_{L^p(\Omega_2)}.
$$

By continuity $B_{\Omega_2}$ extends to a bounded linear operator from $L^p(\Omega_2)$ to $A^p(\Omega_2)$.

Similarly, statement (2) means the following: the restriction of the Bergman projection to the dense subspace of $[L^p(\Omega_1, |\det \Phi|^2)]^\Gamma$ given by

$$
B_{\Omega_1} : [L^2(\Omega_1)]^\Gamma \cap [L^p(\Omega_1, |\det \Phi|^2)]^\Gamma \to A^2(\Omega_1)
$$

is bounded in the $L^p(\Omega_1, |\det \Phi|^2)$-norm, i.e., there is a $C > 0$ such that for all $f \in [L^2(\Omega_1)]^\Gamma \cap [L^p(\Omega_1, |\det \Phi|^2)]^\Gamma$,

$$
\|B_{\Omega_1} f\|_{L^p(\Omega_1, |\det \Phi|^2)} \leq C \|f\|_{L^p(\Omega_1, |\det \Phi|^2)}.
$$

We now see by Proposition 2.6 that

$$
B_{\Omega_1} \left( [L^p(\Omega_1, |\det \Phi|^2)]^\Gamma \right) \subseteq [A^p(\Omega_1, |\det \Phi|^2)]^\Gamma,
$$

where we have used continuity to extend the operator. □
Proof. Proposition 3.4 says that $\Phi^\sharp$ is a homothetic isomorphism, mapping

$$L^p(\Omega_2) \to \left[ L^p(\Omega_1, |\det \Phi'|^{2-p}) \right]^\Gamma,$$

and that it restricts to a homothetic isomorphism on the holomorphic subspaces. Similarly, $(\Phi^\sharp)^{-1}$ has the same properties with the domains and ranges switched.

First assume statement (2). From the diagram (3.9), we write

$$B_{\Omega_2} = (\Phi^\sharp)^{-1} \circ B_{\Omega_1} \circ \Phi^\sharp. \quad (3.15)$$

By hypothesis, $B_{\Omega_1}$ is a bounded linear operator mapping

$$\left[ L^p(\Omega_1, |\det \Phi'|^{2-p}) \right]^\Gamma \to \left[ A^p(\Omega_1, |\det \Phi'|^{2-p}) \right]^\Gamma.$$

Consequently, this composition maps $L^p(\Omega_2)$ boundedly into $A^p(\Omega_2)$, giving statement (1). A similar argument shows that (1) implies (2).

For the commutativity of the diagram, rewrite (3.15) and see that on the subspace $L^p(\Omega_2) \cap L^2(\Omega_2)$ of $L^p(\Omega_2)$ we have the relation

$$\Phi^\sharp \circ B_{\Omega_2} = B_{\Omega_1} \circ \Phi^\sharp. \quad (3.16)$$

Using the hypothesis (for the $B_{\Omega_1}$) and Proposition 3.4 (for $\Phi^\sharp$) we see that each of the four maps in the diagram (3.13) extends to the respective domain in that diagram and is continuous. By continuity, (3.16) continues to hold for the extended maps. This shows that the diagram (3.13) is commutative.

Remark 3.17. Diagram (3.9) is a special case of diagram (3.13) for $p = 2$. $\square$

4. Preliminaries to the Proofs of Theorems 1.5 and 1.8

4.1. $L^p$ unboundedness on Reinhardt domains. Notice that the domains $\mathcal{R}_k$ and $\mathcal{K}_k$ defined in (1.3) and (1.4) are Reinhardt, a fact that plays a significant role in what follows. Recall some elementary facts about holomorphic function theory on Reinhardt domains (which are always assumed to be centered at the origin). Let $\Omega \subset \mathbb{C}^n$ be a Reinhardt domain, every holomorphic function $f \in \mathcal{O}(\Omega)$ admits a unique Laurent expansion

$$f = \sum_{\alpha \in \mathbb{Z}^n} a_\alpha(f)e_\alpha, \quad (4.1)$$

where for $\alpha \in \mathbb{Z}^n$

$$e_\alpha(z) = z_1^{\alpha_1} \cdots z_n^{\alpha_n} \quad (4.2)$$

is a so-called Laurent monomial, and where $a_\alpha(f) \in \mathbb{C}$ is the $\alpha$-th Laurent coefficient. The Laurent series of $f$ converges absolutely and uniformly to $f$ on every compact subset of $\Omega$.

When $f$ lies in the Bergman space $A^2(\Omega)$, we can say more about the series (4.1): it is actually an orthogonal series converging in the Hilbert space $A^2(\Omega)$, and the family of monomials

$$\left\{ \frac{e_\alpha}{\|e_\alpha\|_{L^2}} : e_\alpha \in L^2(\Omega) \right\} \quad (4.3)$$

forms an orthonormal basis of $A^2(\Omega)$. In particular, if $f \in A^2(\Omega)$ then the Laurent series (4.1) can have $a_\alpha(f) \neq 0$ only when $e_\alpha \in L^2(\Omega)$. It is possible to generalize some of these results to the spaces $A^p(\Omega)$; see [CEM19].

We now give an easily checkable condition which shows $L^p$-Bergman unboundedness on any Reinhardt domain.
Lemma 4.4. Let $\Omega$ be a bounded Reinhardt domain in $\mathbb{C}^n$, and let $p \geq 2$. Suppose that there is a multi-index $\beta \in \mathbb{Z}^n$ such that

$$e_\beta \in L^2(\Omega) \setminus L^p(\Omega).$$

(4.5) Then the Bergman projection $B_\Omega$ fails to map $L^p(\Omega) \to L^p(\Omega)$.

Proof. Define subsets $J_\beta, K_\beta \subset \{1, 2, \cdots, n\}$ with

$$J_\beta = \{ j : \beta_j \geq 0 \}, \quad K_\beta = \{ k : \beta_k < 0 \},$$

and let

$$f(w) = \prod_{j \in J_\beta} u_j^{\beta_j} \times \prod_{k \in K_\beta} (\bar{w_k})^{-\beta_k}.$$ 

This is a bounded function on $\Omega$, and therefore $f \in L^p(\Omega)$. We now show that $B_\Omega f = Ce_\beta$ for some $C \neq 0$. Since $e_\beta \notin L^p(\Omega)$, this will show that $B_\Omega f$ can be a function in $L^p(\Omega)$. This will imply that $B_\Omega$ is not bounded in the $L^p$-norm, since if it were so, it would extend to a map from the dense subspace $L^p(\Omega) \cap L^2(\Omega)$ to the whole of $L^p(\Omega)$.

In the domain $\Omega$, we make a change of coordinates from the natural coordinates $(w_1, \ldots, w_n)$ to polar coordinates in each variable by setting $w_j = r_j e^{i\theta_j}$. Let $\gamma = (|\beta_1|, \ldots, |\beta_n|) \in \mathbb{N}^n$ be the multi-index obtained by replacing each entry of $\beta$ by its absolute value. Then, in these coordinates

$$f(w) = r_1^{\gamma_1} \cdots r_n^{\gamma_n} e^{i\langle \beta, \theta \rangle},$$

where $\beta \cdot \theta = \sum_{j=1}^n \beta_j \theta_j$, and $r_\gamma = r_1^{\gamma_1} \cdots r_n^{\gamma_n}$. If $\alpha \in \mathbb{Z}^n$, then we also have

$$e_\alpha(w) = w^\alpha = r_\alpha e^{i\langle \alpha, \theta \rangle}.$$ 

Further, denote by $|\Omega|$ the Reinhardt shadow of $\Omega$, i.e., the set $\{(|z_1|, \ldots, |z_n|) : z \in \Omega\}$ in $\mathbb{R}^n$, and let $T^n$ be the unit torus of $n$ dimensions. Then, for each $\alpha \in \mathbb{Z}^n$, we have

$$\langle B_\Omega f, e_\alpha \rangle_{A^2(\Omega)} = \langle f, e_\alpha \rangle_{L^2(\Omega)} = \int_\Omega f \overline{e_\alpha} dV = \int_{|\Omega|} r^{\alpha+\gamma_1} \cdots r_n d\nu \times \int_{T^n} e^{i\langle \beta-\alpha, \theta \rangle} d\theta$$

$$= 0 \quad \text{if } \alpha \neq \beta \quad \text{and} \quad > 0 \quad \text{if } \alpha = \beta.$$ 

Since $\{|\Omega|\}$ is an orthonormal basis of $A^2(\Omega)$, it follows that all the Fourier coefficients of $B_\Omega f$ with respect to this basis vanish, except the $\beta$-th coefficient, which is nonzero. Therefore, $B_\Omega f = Ce_\beta \notin L^p(\Omega)$,

(4.6) for some constant $C \neq 0$. \hfill $\square$

4.2. Estimates on Bergman functions on the polydisc. We state three important integral estimates on the polydisc $\mathbb{D}^n$.

Proposition 4.7. The Bergman projection on the polydisc $\mathbb{D}^n$ is a bounded operator $B_{\mathbb{D}^n} : L^p(\mathbb{D}^n) \to A^p(\mathbb{D}^n)$ for all $1 < p < \infty$.

Proof. For the polydisc $\mathbb{D}^n$, the Bergman projection has the well-known integral representation

$$B_{\mathbb{D}^n} f(z) = \int_{\mathbb{D}^n} K(z, w) f(w) dV(w), \quad f \in L^2(\mathbb{D}^n),$$

where $K$ is the Bergman kernel of the polydisc, which is easily shown to be

$$K(z, w) = \frac{1}{\pi^n} \prod_{j=1}^n \frac{1}{(1 - z_j \bar{w_j})^2}.$$
The case \( n = 1 \) of Proposition 4.7 is by now a staple result in Bergman theory, going back to [ZJ64], where it was proved using the \( L^p \)-boundedness of a Calderon-Zygmund singular integral operator. Another approach, based on Schur’s test for \( L^p \)-boundedness of an integral operator, was used in [FR75]. An alternative proof of the main estimate needed in this method can be found in [Axl88] and in the monograph [DS04].

Since \( \mathbb{D}^n \) is a product domain, the theorem in higher dimensions follows from a textbook application of Fubini’s theorem to the case \( n = 1 \).

**Proposition 4.8.** Let \( 1 \leq p < \infty \), and let \( n \geq 1 \). For \( 1 \leq j \leq n \), let \( \phi_j > 0 \) be a function on the unit disc \( \mathbb{D} \) such that \( \phi_j \in L^1(\mathbb{D}) \) and there is a \( 0 < r < 1 \) such that \( \phi_j \in L^\infty(\mathbb{A}(r)) \), where \( \mathbb{A}(r) = \{ r < |z| < 1 \} \) is an annulus. For \( z \in \mathbb{D}^n \), let

\[
\phi(z) = \prod_{j=1}^n \phi_j(z_j)
\]

be the tensor product of the \( \phi_j \)'s. Then there is a \( C > 0 \) such that for any \( f \in A^p(\mathbb{D}^n) \) we have

\[
\int_{\mathbb{D}^n} |f|^p \, \phi \, dV \leq C \int_{\mathbb{D}^n} |f|^p \, dV, \tag{4.9}
\]

**Proof.** Throughout this proof, \( C \) will denote a constant that depends only on \( p \) and \( \phi \). The actual value of \( C \) may change from line to line.

Proceed by induction on the dimension \( n \). First consider the base case \( n = 1 \). We have, by the Bergman inequality (cf. [DS04, Theorem 1]) that there is a \( C > 0 \) such that

\[
\sup_{|z| < r} |f(z)| \leq C \| f \|_{L^p(\mathbb{D})}
\]

for all \( f \in A^p(\mathbb{D}) \). Therefore, for \( f \in A^p(\mathbb{D}) \) we have the estimate

\[
\int_{|z| < r} |f(z)|^p \phi(z) dV(z) \leq \sup_{|z| < r} |f(z)|^p \cdot \int_{\mathbb{D}} \phi(z) dV(z) < \left( C \| \phi \|_{L^1(\mathbb{D})} \right) \cdot \| f \|_{L^p(\mathbb{D})}^p. \tag{4.10}
\]

On the other hand,

\[
\int_{r \leq |z| < 1} |f(z)|^p \phi(z) dV(z) \leq \| \phi \|_{L^\infty(\mathbb{A}(r))} \cdot \| f \|_{L^p(\mathbb{D})}^p. \tag{4.11}
\]

Adding (4.10) and (4.11), the estimate (4.9) follows in the case \( n = 1 \).

For the general case, assume the result established in \( n - 1 \) dimensions. Write the coordinates of \( \mathbb{C}^n \) as \( z = (z', z_n) \), where \( z' = (z_1, \ldots, z_{n-1}) \). Also let \( w(z') = \prod_{j=1}^{n-1} \phi_j(z_j) \), so that \( \phi(z) = w(z') \phi_n(z_n) \). Then,

\[
\int_{\mathbb{D}^n} |f(z', z_n)|^p \phi(z', z_n) dV(z', z_n) = \int_{\mathbb{D}^{n-1}} w(z') \left( \int_{\mathbb{D}} |f(z', z_n)|^p \phi_n(z_n) dV(z_n) \right) dV(z')
\]

\[
\leq C \int_{\mathbb{D}^{n-1}} w(z') \left( \int_{\mathbb{D}} |f(z', z_n)|^p dV(z_n) \right) dV(z')
\]

\[
\leq C \int_{\mathbb{D}} \left( \int_{\mathbb{D}^{n-1}} |f(z', z_n)|^p w(z') dV(z') \right) dV(z_n)
\]

\[
= C \int_{\mathbb{D}^n} |f(z', z_n)|^p dV(z', z_n),
\]

which proves the result. \( \square \)
The maximum principle now implies the actual value of $C$. We need only to prove the case in which $e_\lambda$ is as in (4.2).

**Proof.** We need only to prove the case in which $\lambda = (1, 0, \ldots, 0)$, so that $e_\lambda(z) = z_1$. Once this special case is established, the general result follows by repeatedly applying it and permuting the coordinates.

In what follows, $C$ will denote some positive constant that depends only on $p$ and $\lambda$, where the actual value of $C$ may change from line to line. First consider the one dimensional case, so that we have to show that for a holomorphic function $f$ on the disc we have

$$\int_{\mathbb{D}} |f(z)|^p dV(z) \leq C \int_{\mathbb{D}} |zf(z)|^p dV(z),$$

where the left hand side is assumed to be finite (and therefore the right hand side is finite.) First note that we obviously have

$$\int_{\frac{1}{2} \leq |z| < 1} |f(z)|^p dV(z) \leq 2^p \int_{\frac{1}{2} \leq |z| < 1} |zf(z)|^p dV(z). \quad (4.14)$$

On the other hand, if $|z| = \frac{1}{2}$, we have

$$|f(z)|^p = 2^p |zf(z)|^p \leq 2^p \sup_{|w| \leq \frac{1}{2}} |wf(w)|^p \quad \text{(maximum principle)}$$

$$\leq C \int_{\mathbb{D}} |wf(w)|^p dV(w) \quad \text{(Bergman’s inequality)}$$

The maximum principle now implies

$$\sup_{|z| \leq \frac{1}{2}} |f(z)|^p \leq C \int_{\mathbb{D}} |zf(z)|^p dV(z),$$

so that we have

$$\int_{|z| \leq \frac{1}{2}} |f(z)|^p \leq C \int_{\mathbb{D}} |zf(z)|^p dV(z). \quad (4.15)$$

Combining (4.14) and (4.15) the result follows for $n = 1$.

For the higher-dimensional case, denote the coordinates of $C^n$ as $(z_1, z')$ where $z' = (z_2, \ldots, z_n)$. Then for $f \in A^p(\mathbb{D}^n)$ we have

$$\int_{\mathbb{D}^n} |f(z_1, z')|^p dV(z_1, z') = \int_{\mathbb{D}^{n-1}} \left( \int_{\mathbb{D}} |f(z_1, z')|^p dV(z_1) \right) dV(z')$$

$$\leq C \int_{\mathbb{D}^{n-1}} \left( \int_{\mathbb{D}} |z_1 f(z_1, z')|^p dV(z_1) \right) dV(z')$$

$$= C \int_{\mathbb{D}^n} |z_1 f(z_1, z')|^p dV(z_1, z').$$

□

**Remark 4.16.** In our applications, Propositions 4.8 and 4.12 will allow us to circumvent the need to obtain directly estimates for the Bergman projection in weighted spaces as in the downwards arrow on the right-hand-side of the diagram (3.13). Traditionally, such $L^p$-estimates have been established using much more sophisticated technology from harmonic analysis, such as the theory of Muckenhoupt $A_p$-weights used in [BB78, Bek82, CKY20]. It would be interesting to see in what...
generality the elementary methods above apply to the general problem of $L^p$-boundedness of the Bergman projection.

\[ \diamond \]

5. $L^p$-mapping properties of the Bergman projection on $\mathcal{H}_k$

We are ready to apply the general theory laid out in the previous sections to the models

For $k \in \mathbb{Z}_n$ we have

\[ \mathcal{H}_k = \left\{ z \in \mathbb{D}^n : |z_1|^{k_1} < \prod_{j=2}^{n} |z_j|^{k_j} \right\}. \]

5.1. Allowable multi-indices and unbounded monomials. We establish the unbounded range first. We calculate the set of multi-indices $\alpha \in \mathbb{Z}_n$ so that $e_\alpha \in A^p(\mathcal{H}_k)$.

Lemma 5.1. Let $1 \leq p < \infty$ and $\alpha \in \mathbb{Z}_n$. The monomial $e_\alpha \in A^p(\mathcal{H}_k)$ if and only if

\[ \begin{align*}
& p\alpha_1 + 2 > 0, \\
& k_1(p\alpha_j + 2) + k_j(p\alpha_1 + 2) > 0, \quad \text{for } 2 \leq j \leq n.
\end{align*} \tag{5.2} \]

If the conditions in (5.2) are satisfied, then

\[ \|e_\alpha\|_{L^p(\mathcal{H}_k)} = (2\pi)^n k_1^{n-1} \frac{(2\pi)^n k_1^{n-1}}{(p\alpha_1 + 2) \cdot \prod_{j=2}^{n} (k_1(p\alpha_j + 2) + k_j(p\alpha_1 + 2))}. \tag{5.3} \]

Proof. Notice that the Reinhardt shadow of $\mathcal{H}_k$ is given by

\[ |\mathcal{H}_k| = \left\{ r \in \mathbb{R}^n : 0 \leq r_1 < r_2^{k_2/k_1} \cdots r_n^{k_n/k_1}, 0 \leq r_j < 1 \right\}. \]

We then have the following integral computation, where the integrand is positive, and the integrals are allowed to have the value $+\infty$:

\[ \|e_\alpha\|^p = \int_{\mathcal{H}_k} |e_\alpha|^p dV \\
= (2\pi)^n \int_{|\mathcal{H}_k|} (r^\alpha)^p r_1 \cdots r_n dV(r) \\
= (2\pi)^n \int_{0}^{1} \cdots \int_{0}^{1} \left\{ \int_{0}^{r_2^{k_2/k_1} \cdots r_n^{k_n/k_1}} \int_{0}^{r_1^{\alpha_1 p+1}} \cdot \int_{0}^{r_2^{\alpha_2 p+1} \cdots r_n^{\alpha_n p+1}} dr_1 \cdots 
\right. \\
= (2\pi)^n \prod_{j=2}^{n} \int_{0}^{r_j^{\alpha_j p+2}} \frac{2\pi)^n k_1^{n-1}}{(p\alpha_1 + 2) \cdot \prod_{j=2}^{n} (k_1(p\alpha_j + 2) + k_j(p\alpha_1 + 2))}. \tag{5.5} \]

In the above computations, we get a finite result if and only if each of the $n$ nested integrals is finite. This is equivalent to condition (5.2). \[ \square \]

Proposition 5.6. Let $k = (k_1, \cdots, k_n) \in (\mathbb{Z}^+)^n$. The Bergman projection $B_{\mathcal{H}_k}$ fails to map $L^p(\mathcal{H}_k) \rightarrow L^p(\mathcal{H}_k)$ for any

\[ p \geq \min_{2 \leq j \leq n} \frac{2(k_1 + k_j)}{k_1 + k_j - \gcd(k_1, k_j)}. \tag{5.7} \]
Proof. Lemma 4.4 says it is sufficient to show that there is a multi-index \( \beta \in \mathbb{Z}^n \) such that \( e_\beta \in A^2(\mathcal{H}_k) \setminus \mathcal{P}(\mathcal{H}_k) \), for \( p \) satisfying (5.7). To construct \( \beta \), let \( 2 \leq J \leq n \) be such that

\[
\frac{2(k_1 + k_J)}{k_1 + k_J - \gcd(k_1, k_J)} = \min_{2 \leq J \leq n} \frac{2(k_1 + k_J)}{k_1 + k_J - \gcd(k_1, k_J)}.
\] (5.8)

Choose an arbitrary \( n \)-tuple of integers \( \gamma = (\gamma_1, \ldots, \gamma_n) \), and let \( \beta_1, \beta_J \) be integers such that

\[
k_1(\beta_J + 1) + k_J(\beta_1 + 1) = \gcd(k_1, k_J).
\]

Let the integer \( t \) be so large that \( \beta_1 + k_1 t \geq 0 \), and for each \( j \neq J, 2 \leq j \leq n \), we have

\[
k_1(\gamma_j + 1) + k_j(\beta_1 + k_1 t + 1) > 0.
\]

Now define the multi-index \( \beta \) by setting

\[
\beta_1 = \beta_1 + k_1 t, \quad \text{and} \quad \beta_J = \beta_1 - k_J t,
\]

and

\[
\beta_j = \gamma_j, \quad j \neq 1, j \neq J, 1 \leq j \leq n.
\]

We now have,

\[
\begin{cases}
\beta_1 \geq 0, \\
k_1(\beta_J + 1) + k_J(\beta_1 + 1) > 0, & \quad 2 \leq j \leq n, \\
k_1(\beta_J + 1) + k_J(\beta_1 + 1) = \gcd(k_1, k_J).
\end{cases}
\]

The first two inequalities above show that \( e_\beta \in L^2(\mathcal{H}_k) \), since the conditions (5.2) are satisfied with \( p = 2, \alpha = \beta \). To complete the proof, we need to show that \( e_\beta \notin L^p(\mathcal{H}_k) \). It suffices to show that the equation corresponding to \( j = J \) in (5.2) does not hold if \( \alpha = \beta \) and \( p \) is greater than or equal to the quantity in (5.8). We have,

\[
k_1(p\beta_J + 2) + k_J(p\beta_1 + 2) = p(k_1(\beta_J + 1) + k_J(\beta_1 + 1)) + (2 - p)(k_1 + k_J)
\]
\[
= p \cdot \gcd(k_1, k_J) + (2 - p)(k_1 + k_J)
\]
\[
= 2(k_1 + k_J) - p((k_1 + k_J) - \gcd(k_1, k_J))
\]
\[
\leq 0.
\]

It follows that \( e_\beta \) is in \( L^2 \) but not in \( L^p \), as needed. \( \square \)

5.2. A proper map \( \Phi : \mathbb{D} \times (\mathbb{D}^*)^{n-1} \to \mathcal{H}_k \) of quotient type. We proceed by transferring to problem from \( \mathcal{H}_k \) to \( \mathbb{D} \times (\mathbb{D}^*)^{n-1} \) via Theorem 3.12. In order to apply this theorem, we need a proper map \( \Phi \) and group \( \Gamma \).

**Proposition 5.9.** Let \( k = (k_1, \ldots, k_n) \) be an \( n \)-tuple of positive integers. Set \( K = \text{lcm}(k_1, \ldots, k_n) \), \( \ell_j = K/k_j \) and define

\[
\Phi(w) = \left( (w_1 \ldots w_n)^{\ell_1}, w_2^{\ell_2}, \ldots, w_n^{\ell_n} \right).
\] (5.10)

Then,

1. \( \Phi : \mathbb{D} \times (\mathbb{D}^*)^{n-1} \to \mathcal{H}_k \) is a proper holomorphic map of quotient type.
2. Let \( z_j = e^{2\pi i / \ell_j} \) be a primitive \( \ell_j \)-th root of unity. The deck-transformation group \( \Gamma \subset \text{Aut}(\mathbb{D} \times (\mathbb{D}^*)^{n-1}) \) associated with \( \Phi \) consists of the linear automorphisms given by the diagonal matrices

\[
\Gamma = \{ \sigma_a = \text{diag}(\zeta_1^{a_1}, \zeta_2^{a_2}, \ldots, \zeta_n^{a_n}) : \ a \in \mathbb{Z}^n \}.
\] (5.11)
Proof. Write the component functions of (5.10) as \( \Phi = (\Phi_1, \Phi_2, \ldots, \Phi_n) \). To see that \( \Phi \) maps \( \mathbb{D} \times (\mathbb{D}^*)^{n-1} \to \mathcal{H}_k \), notice that 

\[
\Phi(w) \in \mathcal{H}_k \iff \{|\Phi_1(w)|^{\epsilon_1} < |\Phi_2(w)|^{\epsilon_2} \cdots |\Phi_n(w)|^{\epsilon_n} \} \cap \{0 < |\Phi_j(w)| < 1 : j = 2, \ldots, n\} \\
\iff \{|w_1 w_2 \cdots w_n|^{\epsilon_K} < |w_2|^{\epsilon_K} \cdots |w_n|^{\epsilon_K} \} \cap \{0 < |w_j| < 1 : j = 2, \ldots, n\} \\
\iff \{||w_1| < 1 \} \cap \{0 < |w_j| < 1 : j = 2, \ldots, n\} \\
\iff w \in \mathbb{D} \times (\mathbb{D}^*)^{n-1}.
\]

To see the properness of \( \Phi \), represent it as \( \Phi = L \circ \Phi_1 \) where \( \Phi_1 : \mathbb{D} \times (\mathbb{D}^*)^{n-1} \to \mathcal{H}_{(1, \ldots, 1)} \) is the biholomorphic map

\[ \Phi_1(w) = (w_1 \ldots w_n, w_2, \ldots, w_n), \]

and \( L : \mathcal{H}_{(1, \ldots, 1)} \to \mathcal{H}_k \) is the proper map

\[ L(z) = (z_1^{\epsilon_1}, \ldots, z_n^{\epsilon_n}). \]

Now let \( \sigma_a \in \Gamma \) be as in (5.11). We claim that \( \Phi \circ \sigma_a = \Phi \). Writing

\[
\sigma_a(w) = \sigma_a(w_1, w_2, \ldots, w_n) = (\zeta_1^{a_1} w_1, \zeta_2^{a_2} w_2, \ldots, \zeta_n^{a_n} w_n),
\]

it is seen that

\[
\Phi(\sigma_a(w)) = ((\zeta_1^{a_1} w_1 \ldots w_n)^{\ell_1}, (\zeta_2^{a_2} w_2)^{\ell_2}, \ldots, (\zeta_n^{a_n} w_n)^{\ell_n}) = \Phi(w).
\]

Suppose now there is some \( \eta \in \mathbb{D} \times (\mathbb{D}^*)^{n-1} \) such that \( \Phi(\eta) = \Phi(w) \). In other words,

\[ ((\eta_1 \ldots \eta_n)^{\ell_1}, \eta_2^{\ell_2}, \ldots, \eta_n^{\ell_n}) = ((w_1 \ldots w_n)^{\ell_1}, w_2^{\ell_2}, \ldots, w_n^{\ell_n}). \]

This says, for \( j = 2, \ldots, n, \)

\[ \eta_j = \zeta_j^{a_j} w_j, \]

where (as before) \( \zeta_j \) is a primitive \( \ell_j \)-th root of unity and \( a_j \) in an integer. This now implies

\[ \eta_2^{a_2} \ldots \zeta_n^{a_n} = w_1^{a_1}. \]

Equations (5.14) and (5.15) show that \( \eta = \sigma_a(w) \) for an appropriate choice of multi-index \( a \). This shows that for every \( z \in \mathcal{H}_k, \) \( \Gamma \) acts transitively on the fiber \( \Phi^{-1}(z) \). \( \square \)

Remark 5.16. It is clear that the group \( \Gamma \) consisting of the matrices \( \sigma_a \) in (5.11) is isomorphic to the product

\[ \Gamma \cong \mathbb{Z}/\ell_1 \mathbb{Z} \times \cdots \times \mathbb{Z}/\ell_n \mathbb{Z}, \]

by the map

\[ \sigma_a \mapsto (a_1 \mod \ell_1, \ldots, a_n \mod \ell_n). \]

\( \diamond \)

5.3. Elements of the proof of Theorem 1.5

Proposition 5.17. Let \( \Gamma \) be the group of automorphisms of \( \mathbb{D}^n \) given by (5.11), and let \( g \) be the monomial

\[ g(z) = z_1^{\ell_1-1} \prod_{j=2}^n z_j^{\gcd(\ell_1, \ell_j)-1}. \]

Then each function \( f \in [\mathcal{O}(\mathbb{D}^n)]^\Gamma \) can be written as

\[ f = g \cdot h, \]

for some \( h \in \mathcal{O}(\mathbb{D}^n) \).
Proof. Since each function in \(\mathcal{O}(\mathbb{D}^n)\) can be represented as a Taylor series, it suffices to consider the case where the function \(f\) is a monomial, say \(f = e_\lambda\). Using the notation of (5.11), we see that for each \(a \in \mathbb{Z}^n\) we have
\[
\sigma^a e_\lambda = e_\lambda \circ \sigma_a \cdot \det \sigma_a' = e^{(\lambda_1+1)a_1} e^{(\lambda_2-\lambda_1)a_2} \cdots e^{(\lambda_n-\lambda_1)a_n} e_\lambda = e_\lambda.
\]
Since \(a\) is arbitrary and \(\zeta_j\) is a primitive \(\ell_j\)-th root of unity, we conclude that
\[
\begin{cases}
\lambda_1 + 1 & \equiv 0 \pmod{\ell_1} \\
\lambda_j - \lambda_1 & \equiv 0 \pmod{\ell_j} \text{ for } 2 \leq j \leq n.
\end{cases}
\]
This implies that there is a \(b \in \mathbb{Z}^n\) such that
\[
\begin{cases}
\lambda_1 = -1 + b_1 \ell_1 \\
\lambda_j = -1 + b_1 \ell_1 + b_j \ell_j \text{ for } 2 \leq j \leq n.
\end{cases}
\]
Therefore, there is a \(c \in \mathbb{Z}^n\) such that
\[
\begin{cases}
\lambda_1 = -1 + c_1 \ell_1 \\
\lambda_j = -1 + c_j \gcd(\ell_1, \ell_j) \text{ for } 2 \leq j \leq n.
\end{cases}
\]
Since each \(\lambda_j \geq 0\), it follows that we must have \(c_j \geq 1\) for \(1 \leq j \leq n\). Writing \(c_j = 1 + d_j\) we see that
\[
\lambda = (\ell_1 - 1, \gcd(\ell_1, \ell_2) - 1, \ldots, \gcd(\ell_1, \ell_n) - 1) + (d_1 \ell_1, d_2 \cdot \gcd(\ell_1, \ell_2), \ldots, d_n \cdot \gcd(\ell_1, \ell_n)).
\]
The representation (5.19) of \(e_\lambda\) follows. \(\square\)

Remark 5.20. Note that while the representation (5.19) is a necessary condition for \(f \in \{\mathcal{O}(\mathbb{D}^n)\}^\Gamma\), not all functions of this form are in \(\{\mathcal{O}(\mathbb{D}^n)\}^\Gamma\). A necessary and sufficient condition is easy to write down, but is not needed here.

\[\Diamond\]

Proposition 5.21. Let
\[
2 \leq p < \min_{2 \leq j \leq n} \frac{2(k_1 + k_j)}{k_1 + k_j - \gcd(k_1, k_j)}.
\]
Then
\[
[A^p(\mathbb{D}^n)]^\Gamma = [A^p(\mathbb{D}^n, |\det \Phi'|^{2-p})]^\Gamma
\]
as sets, and as Banach spaces (with the natural norms) these are isomorphic with equivalent norms. Here, \(\Phi\) is the map from (5.10).

Proof. First note that a computation shows
\[
\det \Phi'(z) = (\ell_1 \cdots \ell_n) z_1^{\ell_1-1} z_2^{\ell_1+\ell_2-1} \cdots z_n^{\ell_1+\ell_n-1}.
\]
Since \(p \geq 2\), there is some \(C > 0\) such that on \(\mathbb{D}^n\) we have
\[
|\det \Phi'|^{2-p} \geq C > 0,
\]
so that for any function \(f\) on \(\mathbb{D}^n\) we have
\[
\int_{\mathbb{D}^n} |f|^p dV \leq \frac{1}{C} \int_{\mathbb{D}^n} |f|^p |\det \Phi'|^{2-p} dV.
\]
We therefore conclude that there is a continuous inclusion map
\[
[A^p(\mathbb{D}^n, |\det \Phi'|^{2-p})]^\Gamma \hookrightarrow [A^p(\mathbb{D}^n)]^\Gamma.
\]
Now let \( g \) be as in (5.18). Then
\[
|g(z)^p| |\det \Phi'(z)|^{2-p} = \left| z^{\ell_1-1} \prod_{j=2}^{n} z_j^{\gcd(\ell_j, \ell_1)-1} \right|^p \left| (\ell_1 \ldots \ell_n) z_1^{\ell_1-1} \prod_{j=2}^{n} z_j^{\ell_j-1} \right|^{2-p} := (\ell_1 \ldots \ell_n)^{2-p} \prod_{j=1}^{n} \phi_j(z_j),
\]
where \( \phi_j(z) = |z|^{\alpha_j} \), with
\[
\alpha_j = \begin{cases} 
2(\ell_1 - 1), & \text{for } j = 1 \\
2(\ell_j + \ell_1) - p(\ell_1 + \ell_j - \gcd(\ell_j, \ell_1)) - 2, & \text{for } 2 \leq j \leq n.
\end{cases}
\]
By (5.22), we have for all \( 2 \leq j \leq n \)
\[
p < \frac{2(k_1 + k_j)}{k_1 + k_j - \gcd(k_1, k_j)} = \frac{2 \left( \frac{K}{\ell_1} + \frac{K}{\ell_j} \right)}{\frac{K}{\ell_1} + \frac{K}{\ell_j} - \gcd(\frac{K}{\ell_1}, \frac{K}{\ell_j})}
= \frac{2(\ell_j + \ell_1)}{\ell_j + \ell_1 - \gcd(\ell_j, \ell_1)}.
\]
It follows that for each \( 1 \leq j \leq n \), we have \( \alpha_j > -2 \). Therefore, for each \( j \) we have \( \phi_j \in L^1(\mathbb{D}) \)
and if \( \frac{1}{2} < |z| < 1 \) then \( \phi_j(z) \leq 2^{-\alpha_j} \), so the hypotheses of Proposition 4.8 are satisfied. It now follows from Proposition 4.8 that for each \( h \in \mathcal{O}(\mathbb{D}^n) \) we have
\[
\int_{\mathbb{D}^n} |h|^p |g(z)^p| |\det \Phi'(z)|^{2-p} \, dV \leq C \int_{\mathbb{D}^n} |h|^p \, dV,
\]
where the inequality holds trivially if \( h \notin A^p(\mathbb{D}^n) \). Now let \( f \in [A^p(\mathbb{D}^n)]^\Gamma \), so that by Proposition 5.17 we see that \( f = gh \) where \( g \) is as in (5.18) and \( h \in \mathcal{O}(\mathbb{D}^n) \). Therefore
\[
\int_{\mathbb{D}^n} |f|^p |\det \Phi'|^{2-p} \, dV = \int_{\mathbb{D}^n} |h|^p \left( |g|^p |\det \Phi'|^{2-p} \right) \, dV
\leq C \int_{\mathbb{D}^n} |h|^p \, dV \quad \text{using (5.24)}
\leq C \int_{\mathbb{D}^n} |gh|^p \, dV \quad \text{using Proposition 4.12}
= C \int_{\mathbb{D}^n} |f|^p \, dV.
\]
We conclude that the spaces \( [A^p(\mathbb{D}^n)]^\Gamma \) and \( \left[ A^p \left( \mathbb{D}^n, |\det \Phi'|^{2-p} \right) \right]^\Gamma \) are equal as sets and that the norms are equivalent. \( \square \)

5.4. Conclusion of the proof of Theorem 1.5 Only a small part of the proof remains. From the Hölder-symmetric nature of the interval of Bergman \( L^p \)-boundedness (see [EM16, CZ16]), it suffices to show that

(i) the Bergman projection is \( L^p \)-bounded if \( p \geq 2 \) is in the interval (5.22), and
(ii) the Bergman projection is not \( L^p \)-bounded if \( p \geq 2 \) is outside this interval.
Statement (ii) has been verified by Proposition 5.2. To prove (i), we invoke Theorem 3.12 applied to the map \( \Phi : \Omega_1 \rightarrow \Omega_2 \) in (5.10), with \( \Omega_1 = \mathbb{D} \times (\mathbb{D}^*)^{n-1} \) and \( \Omega_2 = \mathcal{H}_k \). We need to show that the Bergman projection is a bounded operator on the weighted space

\[
B_{\Omega_1} : \left[ L^p \left( \Omega_1, |\det \Phi|^2 - p \right) \right]^\Gamma \rightarrow \left[ A^p \left( \Omega_1, |\det \Phi|^2 - p \right) \right]^\Gamma,
\]

where the group \( \Gamma \) is given in (5.11).

Since \( p \geq 2 \), the function \( |\det \Phi|^2 - p \) is bounded from below on \( \Omega_1 \). It follows that there is a continuous inclusion map

\[
\iota : \left[ L^p \left( \Omega_1, |\det \Phi|^2 - p \right) \right]^\Gamma \hookrightarrow \left[ L^p (\Omega_1) \right]^\Gamma.
\]

Notice that the Bergman projection on \( \Omega_1 = \mathbb{D} \times (\mathbb{D}^*)^{n-1} \) coincides with the Bergman projection \( B_{\mathbb{D}^n} \) of the unit polydisc, and Proposition 4.7 says that \( B_{\mathbb{D}^n} \) is bounded in the \( L^p \)-norm for \( 1 < p < \infty \). Note also that by Proposition 2.6, \( B_{\mathbb{D}^n} \) restricts to a bounded map

\[
B_{\mathbb{D}^n} : \left[ L^p (\mathbb{D}^n) \right]^\Gamma \rightarrow \left[ A^p (\mathbb{D}^n) \right]^\Gamma.
\]

Finally, note that by Proposition 5.21, the spaces \( \left[ A^p (\mathbb{D}^n) \right]^\Gamma \) and \( \left[ A^p (\mathbb{D}^n, |\det \Phi|^2 - p) \right]^\Gamma \) coincide for \( p \) in the range given by (5.22), and the identity map

\[
\text{id} : \left[ A^p (\mathbb{D}^n) \right]^\Gamma \rightarrow \left[ A^p (\mathbb{D}^n, |\det \Phi|^2 - p) \right]^\Gamma.
\]

is an isomorphism of Banach spaces, if each side is given its natural norm. Therefore, the Bergman projection \( B_{\Omega_1} \), restricted to \( \left[ L^p \left( \Omega_1, |\det \Phi|^2 - p \right) \right]^\Gamma \) can be represented as the composition of bounded maps

\[
B_{\Omega_1} \big| \left[ L^p (\Omega_1, |\det \Phi|^2 - p) \right]^\Gamma = \text{id} \circ B_{\mathbb{D}^n} \circ \iota,
\]

and is therefore bounded as in (5.25). Now Theorem 3.12 with \( \Omega_2 = \mathcal{H}_k \) shows that \( B_{\mathcal{H}_k} \) is a bounded operator from \( L^p (\mathcal{H}_k) \) to \( A^p (\mathcal{H}_k) \) if and only if

\[
\max_{2 \leq j \leq n} \frac{2(k_1 + k_j)}{k_1 + k_j + \gcd(k_1, k_j)} < p < \min_{2 \leq j \leq n} \frac{2(k_1 + k_j)}{k_1 + k_j - \gcd(k_1, k_j)}.
\]

\[ \square \]

Remark 5.26. It is interesting to note that if \( K = \text{lcm}(k_1, \ldots, k_n) \) and \( \ell_j = K/k_j \) denote the exponents appearing in the proper map \( \Phi : \mathbb{D} \times (\mathbb{D}^*)^{n-1} \rightarrow \mathcal{H}_k \) defined in (5.10),

\[
\frac{2(k_1 + k_j)}{k_1 + k_j - \gcd(k_1, k_j)} = \frac{2(\ell_j + \ell_1)}{\ell_j + \ell_1 - \gcd(\ell_1, \ell_j)}.
\]

\[ \diamond \]

6. \( L^p \)-mapping properties of the Bergman projection on \( \mathcal{H}_k \)

The second family of domains to illustrate the general theory is

\[
\mathcal{H}_k = \left\{ z \in \mathbb{D}^n : |z_1|^{k_1} < |z_2|^{k_2} < \cdots < |z_n|^{k_n} < 1 \right\}.
\]

The \( L^p \)-mapping range of the Bergman projection on \( \mathcal{H}_k \) will be studied using the same methods applied in the previous section. Given an \( n \)-tuple of positive integers \( k \), we note the following two facts about its counterpart multi-index \( \ell \) defined in (1.7):

**Proposition 6.1.** Given an \( n \)-tuple \( k = (k_1, \ldots, k_n) \) of positive integers, let \( K = \text{lcm}(k_1, \ldots, k_n) \) and define the multi-index \( \ell \) of positive integers by setting \( \ell_j = K/k_j \). Then

1. \( \gcd(\ell_1, \ldots, \ell_n) = 1 \)
2. \( K = \text{lcm}(\ell_1, \ldots, \ell_n) \cdot \gcd(k_1, \ldots, k_n) \).
Proof. If \( m \in \mathbb{Z}^+ \) and \( p \) a prime, define \( \nu_p(m) \geq 0 \) to be the number of factors equal to \( p \) in the prime factorization of \( m \), i.e., \( p^{\nu_p(m)} \) divides \( m \) but \( p^{\nu_p(m)+1} \) does not. Then for any prime \( p \)

\[
\nu_p(\gcd(\ell_1, \ldots, \ell_n)) = \min_j \nu_p(\ell_j)
\]

\[
= \min_j (\nu_p(K) - \nu_p(k_j))
\]

\[
= \nu_p(K) - \max_j \nu_p(k_j)
\]

\[
= 0,
\]

establishing (1). For (2), notice that for any prime \( p \)

\[
\nu_p(\text{lcm}(\ell_1, \ldots, \ell_n) \cdot \gcd(k_1, \ldots, k_n)) = \max_j \nu_p(\ell_j) + \min_j \nu_p(k_j)
\]

\[
= \nu_p(K) - \min_j \nu_p(k_j) + \min_j \nu_p(k_j)
\]

\[
= \nu_p(K),
\]

6.1. Allowable multi-indices and unbounded monomials on \( \mathcal{I}_k \).

**Lemma 6.2.** Let \( 1 \leq p < \infty \) and \( \alpha \in \mathbb{Z}^n \). The monomial \( e_\alpha \in A^p(\mathcal{I}_k) \) if and only if

\[
\sum_{m=1}^{d} \ell_m(p^{\alpha_m} + 2) > 0, \quad \text{for } d = 1, 2, \ldots, n. \tag{6.3}
\]

If the conditions in (6.3) are satisfied, then

\[
\|e_\alpha\|^p_{L^p(\mathcal{I}_k)} = (2\pi)^n \prod_{d=1}^{n} \left( \frac{\ell_d}{\sum_{m=1}^{d} \ell_m(p^{\alpha_m} + 2)} \right). \tag{6.4}
\]

**Proof.** Calculate using polar coordinates in each variable:

\[
\|e_\alpha\|^p_{L^p(\mathcal{I}_k)} = (2\pi)^n \int_{0}^{1} \cdots \int_{0}^{1} \int_{0}^{r_2/\ell_2} \cdots \int_{0}^{r_2/\ell_2} r_1^{p_{\alpha_1}+1} r_2^{p_{\alpha_2}+1} \cdots r_n^{p_{\alpha_n}+1} \, dr_1 \, dr_2 \cdots dr_n
\]

\[
= \frac{(2\pi)^n}{(\alpha_1 + 2)} \int_{0}^{1} \cdots \int_{0}^{1} \int_{0}^{r_2/\ell_2} \cdots \int_{0}^{r_2/\ell_2} r_2^{p_{\alpha_2}+1} \cdots r_n^{p_{\alpha_n}+1} \, dr_2 \cdots dr_n
\]

\[
= \frac{(2\pi)^n}{(\alpha_1 + 2)} \frac{\ell_2}{\ell_1(p_1 + 2) + \ell_2(p_2 + 2)} \int_{0}^{1} \cdots \int_{0}^{1} \int_{0}^{r_3/\ell_3} \cdots \int_{0}^{r_3/\ell_3} r_3^{p_{\alpha_3}+2} \cdots \cdots r_n^{p_{\alpha_n}+1} \, dr_3 \cdots dr_n
\]

\[
= \frac{(2\pi)^n}{(\alpha_1 + 2)} \prod_{d=1}^{n} \left( \frac{\ell_d}{\sum_{m=1}^{d} \ell_m(p^{\alpha_m} + 2)} \right),
\]

which is the claimed formula. Starting from equation (6.5), the convergence of the innermost integral in each line is equivalent to the condition (6.3). \( \square \)
Proposition 6.6. Let \( k = (k_1, \ldots, k_n) \in (\mathbb{Z}^+)^n \) be integers. The Bergman projection \( B_{\mathcal{S}_k} \) fails to map \( L^p(\mathcal{S}_k) \to L^p(\mathcal{S}_k) \) for any

\[
p \geq \frac{2\Lambda}{\Lambda - 1},
\]

where \( \Lambda = \sum_{j=1}^n \ell_j \).

Proof. Lemma 4.4 says it is sufficient to show that there is a multi-index \( \beta \in \mathbb{Z}^n \) such that \( e_\beta \in A^2(\mathcal{S}_k) \setminus A^p(\mathcal{S}_k) \). Construct the multi-index \( \beta \) as follows. For \( 1 \leq j \leq n \), choose integers \( \beta_j^* \) so that

\[
\ell_1(\beta_j^* + 1) + \ell_2(\beta_2^* + 1) + \cdots + \ell_n(\beta_n^* + 1) = 1.
\]

This is possible because by Proposition 6.1, \( \gcd(\ell_1, \ldots, \ell_n) = 1 \). At this point, there is no indication of the sign on any of the \( \beta_j^* \). For \( j = 2, \ldots, n \), choose positive integers \( t_j \) so that the numbers

\[
\beta_2 := \beta_2^* - t_2\ell_1 < -1
\]

\[
\vdots
\]

\[
\beta_n := \beta_n^* - t_n\ell_1 < -1.
\]

Now set

\[
\beta_1 := \beta_1^* + t_2\ell_2 + t_3\ell_3 + \cdots + t_n\ell_n
\]

and let \( \beta = (\beta_1, \ldots, \beta_n) \). Notice that

\[
\ell_1(\beta_1 + 1) + \ell_2(\beta_2 + 1) + \cdots + \ell_n(\beta_n + 1) = 1.
\]

Since \( \beta_j + 1 < 0 \) for \( j = 2, \ldots, n \), we see that

\[
\sum_{j=1}^d \ell_j(\beta_j + 1) = 1 - \sum_{d+1}^n \ell_j(\beta_j + 1) > 0
\]

for all choices of \( d = 1, 2, \ldots, n \). Thus, (6.3) (with \( p = 2 \)) shows that the monomial \( e_\beta \in A^2(\mathcal{S}_k) \).

Now for \( p \geq \frac{2\Lambda}{\Lambda - 1} \), consider the sum

\[
\sum_{j=1}^n \ell_j(p\beta_j + 2) = (p - 2)\sum_{j=1}^n \ell_j\beta_j + 2\sum_{j=1}^n \ell_j(\beta_j + 1)
\]

\[
= (p - 2)(1 - \Lambda) + 2
\]

\[
= p(1 - \Lambda) + 2\Lambda
\]

\[
\leq 0.
\]

Using (6.3) again, this shows that \( e_\beta \notin A^p(\mathcal{S}_k) \), completing the proof.

6.2. A proper map \( \Phi : \mathbb{D} \times (\mathbb{D}^*)^{n-1} \to \mathcal{S}_k \) of quotient type. Our goal is to again use Theorem 3.12 to establish the \( L^p \)-boundedness range of the Bergman projection on \( \mathcal{S}_k \). First we find such a map \( \Phi \) together with a group \( \Gamma \).

Proposition 6.7. Let \( k = (k_1, \ldots, k_n) \) be an \( n \)-tuple of coprime positive integers. Set \( K = \text{lcm}(k_1, \ldots, k_n) \), \( \ell_j = K/k_j \) and define

\[
\Phi(w_1, \ldots, w_n) = ( (w_1 \ldots w_n)^{\ell_1}, (w_2 \ldots w_n)^{\ell_2}, \ldots, w_n^{\ell_n} ) \quad \text{(6.8)}
\]

Then,

(1) \( \Phi : \mathbb{D} \times (\mathbb{D}^*)^{n-1} \to \mathcal{S}_k \) is a proper holomorphic map of quotient type.
(2) Let $\zeta_j = e^{2\pi i/\ell_j}$ be a primitive $\ell_j$-th root of unity. The deck-transformation group $\Gamma \subset \text{Aut}(\mathbb{D} \times (\mathbb{D}^*)^{n-1})$ associated with $\Phi$ consists of the linear automorphisms given by the diagonal matrices
\[ \Gamma = \{ \sigma_a \in \text{Diag}(n \times n) : a \in \mathbb{Z}^n, (\sigma_a)(j) = \zeta_j^{a_j} \zeta_{j+1}^{a_{j+1}}, 1 \leq j \leq n-1, (\sigma_a)_{nn} = \zeta_n^n \}. \] (6.9)

Proof. The $j$-th component of the map $\Phi$ is given by
\[ \Phi_j(w) = \left( \prod_{k=j}^n w_k \right)^{\ell_j}. \]
To see that this map sends $\mathbb{D} \times (\mathbb{D}^*)^{n-1} \to \mathcal{J}_k$, note
\[ \Phi(w) \in \mathcal{J}_k \iff \{|\Phi_1(w)|^{\ell_1} < |\Phi_2(w)|^{\ell_2} < \cdots < |\Phi_n(w)|^{\ell_n} < 1\} \]
\[ \iff \{|w_1w_2 \cdots w_n|^K |w_2w_3 \cdots w_n|^K < \cdots < |w_n|^K < 1\} \]
\[ \iff \{0 < |w_j| < 1 : j = 2, \ldots, n\} \]
\[ \iff w \in \mathbb{D} \times (\mathbb{D}^*)^{n-1}. \]
To see the properness of $\Phi$, note that we may represent $\Phi$ as a composition $\Phi = L \circ \Phi_1$, where $\Phi_1 : \mathbb{D} \times (\mathbb{D}^*)^{n-1} \to \mathcal{J}_{(1,\ldots,1)}$ is the map
\[ w \mapsto (w_1 \cdots w_n, w_2 \cdots w_n, \ldots, w_n), \]
where the $j$-th component is $\prod_{k=j}^n w_k$. This is easily seen to be a biholomorphism. The map $L : \mathcal{J}_{(1,\ldots,1)} \to \mathcal{J}_k$ is given by $L(z) = (z_1^{\ell_1}, \ldots, z_n^{\ell_n})$, and is easily seen to be proper. Therefore the composition $\Phi$ is proper.

Now let $\sigma_a \in \Gamma$ be as defined by (6.9). We will show that $\Phi \circ \sigma_a = \Phi$. Indeed,
\[ \sigma_a(w) = \sigma_a(w_1, w_2, \ldots, w_n) = (\zeta_1^{a_1} \zeta_2^{a_2} \cdots \zeta_n^{a_n} w_1, \zeta_2^{a_2} \cdots \zeta_n^{a_n} w_2, \ldots, \zeta_n^{a_n} w_n), \]
and therefore,
\[ \Phi(\sigma_a(w)) = ((\zeta_1^{a_1} w_1 \cdots w_n)^{\ell_1}, (\zeta_2^{a_2} w_2)^{\ell_2}, \ldots, (\zeta_n^{a_n} w_n)^{\ell_n}) = \Phi(w). \]
Now see that all preimages of $\Phi(w)$ must be of this form. Suppose there is some $\eta \in \mathbb{Z}^n$ such that $\Phi(\eta) = \Phi(w)$. Then comparing the $n$-th slots
\[ \eta_n^{\ell_n} = w_n^{\ell_n} \iff \eta_n = \zeta_n^{a_n} w_n \]
for some integer $a_n$. Now comparing the $(n-1)^{th}$ coordinate
\[ (\eta_{n-1} \eta_n)^{\ell_{n-1}} = (w_{n-1} w_n)^{\ell_{n-1}} \iff \eta_{n-1} \eta_n = \zeta_n^{a_n-1} w_{n-1} w_n \]
\[ \iff \eta_{n-1} = \zeta_{n-1}^{a_{n-1}} w_{n-1} \]
for some integer $a_{n-1}$. Continuing in this fashion we see that $\eta$ is necessarily of the form $\sigma_a(w)$. This shows that for every $z \in \mathcal{J}_k$, $\Gamma$ acts transitively on the fiber $\Phi^{-1}(z)$. \qed

Remark 6.10. As is the case with $\mathcal{H}_k$, it is clear that the group $\Gamma$ consisting of the matrices $\sigma_a$ in (6.9) is isomorphic to the product
\[ \Gamma \cong \mathbb{Z} / \ell_1 \mathbb{Z} \times \cdots \times \mathbb{Z} / \ell_n \mathbb{Z}, \]
by the map
\[ \sigma_a \mapsto (a_1 \mod \ell_1, \ldots, a_n \mod \ell_n). \]
However, the action of this group on the fibers $\Phi^{-1}(z)$ in the $\mathcal{J}_k$ case is substantially different than the $\mathcal{H}_k$ case. \diamond
6.3. Weighted $L^p$-boundedness on the polydisc and proof of Theorem [1.8] We now establish the boundedness of the map represented by the vertical downwards arrow on the right hand side of diagram 3.13:

**Proposition 6.11.** Let $2 \leq p < \frac{2\Lambda}{\Lambda-1}$ and set $\Omega_1 = \mathbb{D} \times (\mathbb{D}^*)^{n-1}$. The Bergman projection restricts to a bounded operator

$$B_{\Omega_1} : \left[ L^p(\Omega_1, |\det \Phi'|^{2-p}) \right]^\Gamma \rightarrow \left[ A^p(\Omega_1, |\det \Phi'|^{2-p}) \right]^\Gamma.$$  

**Proof.** It will be sufficient to show that $B_{\Omega_1}$ is a bounded operator from $L^p(\Omega_1, |\det \Phi'|^{2-p})$ to $A^p(\Omega_1, |\det \Phi'|^{2-p})$, since Proposition 2.6 says the Bergman projection maps $\Gamma$-invariant functions to $\Gamma$-invariant functions.

From the definition of $\Phi$ in (6.8), a computation shows that

$$\det \Phi'(w) = (\ell_1 \ldots \ell_n) \cdot w_1^{\ell_1-1}w_2^{\ell_2-1} \cdots w_n^{\ell_n-1} \sum_{j=1}^n w_j^{\lambda_j-1},$$

where $\lambda_j = \sum_{k=1}^j \ell_k$. Therefore,

$$|\det \Phi'(w)|^{2-p} = (\ell_1 \ldots \ell_n)^{2-p} \prod_{j=1}^n |w_j|^{|\lambda_j-1|(2-p)}.$$  

Since $p \geq 2$ the function $|\det \Phi'|^{2-p}$ is bounded from below on $\Omega_1$. It follows that there is a continuous inclusion map

$$\iota : L^p \left( \Omega_1, |\det \Phi'|^{2-p} \right) \hookrightarrow L^p(\Omega_1).$$  

Next, we claim that the spaces $A^p(\Omega_1)$ and $A^p(\Omega_1, |\det \Phi'|^{2-p})$ (which are both Banach spaces of holomorphic functions on $\Omega_1$) are equal as sets and have equivalent norms. To see this, note that

$$|\det \Phi'(w)|^{2-p} = (\ell_1 \ldots \ell_n)^{2-p} \prod_{j=1}^n \phi_j(w_j),$$

with $\phi_j(z) = |z|^{(\lambda_j-1)(2-p)}$, where $\lambda_j = \sum_{k=1}^j \ell_k$. Now we have for $1 \leq j \leq n$ that

$$p < \frac{2\Lambda}{\Lambda-1} = \frac{2\Lambda_n}{\lambda_n-1} \leq \frac{2\lambda_j}{\lambda_j-1},$$

since the function $x \mapsto \frac{x}{x-1} = 1 + \frac{1}{x-1}$ is decreasing if $x > 1$. But $p < \frac{2\lambda_j}{\lambda_j-1}$ is easily seen to be equivalent to $(\lambda_j - 1)(2-p) > -2$ which implies that each $\phi_j \in L^1(\mathbb{D})$. Further, since each $\phi_j$ is obviously bounded outside any neighborhood of 0, Proposition 4.8 now shows the existence of a constant $C$ such that for all $f \in A^p(\Omega_1)$,

$$\int_{\Omega_1} |f|^p \cdot |\det \Phi'|^{(2-p)} dV \leq C \int_{\Omega_1} |f|^p dV.$$  

This shows $A^p(\Omega_1) \subset A^p(\Omega_1, |\det \Phi'|^{2-p})$. The other direction is trivial, since $|\det \Phi'|^{(2-p)}$ is bounded from below. This establishes our claim, and shows that the identity map

$$\text{id} : A^p(\Omega_1) \rightarrow A^p(\Omega_1, |\det \Phi'|^{2-p})$$

is an isomorphism of Banach spaces. Therefore the restricted Bergman projection

$$B_{\Omega_1} : L^p \left( \Omega_1, |\det \Phi'|^{2-p} \right) \rightarrow A^p \left( \Omega_1, |\det \Phi'|^{2-p} \right)$$
may now be represented as the composition of bounded maps
\[ B_{\Omega} : L^p(\Omega_1, |\det \Phi|^{|2-p|}) = \text{id} \circ B_{D^n} \circ \iota, \]
where the Bergman projection \( B_{\Omega} \) on the polydisc is bounded by Proposition 6.11 as a mapping \( L^p(D^n) \to A^p(D^n) \). The restricted Bergman projection is therefore bounded, completing the proof.

**Proof of Theorem 1.8** Proposition 6.11 and Theorem 3.12 together show that the Bergman projection
\[ B_{\mathcal{H}_k} : L^p(\mathcal{H}_k) \to A^p(\mathcal{H}_k) \]
is a bounded operator for all \( 2 \leq p < \frac{4A}{A-1} \). Hölder symmetry of the interval of \( L^p \)-boundedness immediately implies that it is bounded for \( \frac{4A}{A-1} < p < \frac{4A}{A-2} \). Proposition 6.6 along with the Hölder symmetry of the interval of \( L^p \)-boundedness shows that the Bergman projection is not bounded for \( p \) outside this interval. This completes the proof of Theorem 1.8.

**Remark 6.12.** The proof of Theorem 1.8 requires less delicacy than that of Theorem 1.5. Notice that in the proof of Proposition 6.11 the boundedness of the Bergman projection on the \( \Gamma \)-invariant subspace \( [L^p(\Omega_1, |\det \Phi'|^{2-p})] \) deduced from its boundedness on the full space \( L^p(\Omega_1, |\det \Phi'|^{2-p}) \), and this range turned out to be sharp, where \( \Phi \) and \( \Gamma \) are as in Proposition 6.7. On the other hand, on \( \mathcal{H}_k \), with \( \Phi \) and \( \Gamma \) as in Proposition 5.9, it happens that the range of \( L^p \)-boundedness of the Bergman projection on the \( \Gamma \)-invariant subspace \( [L^p(\Omega_1, |\det \Phi'|^{2-p})] \) is strictly larger than the range of \( L^p \)-boundedness of the Bergman projection on the full weighted space \( L^p(\Omega_1, |\det \Phi'|^{2-p}) \). Therefore we were able to complete the proof of Theorem 1.8 without the need to carry out more careful analysis of the structure of the \( \Gamma \)-invariant holomorphic functions à la Proposition 5.17.

**References**


