Harmonic Analysis techniques in Several Complex Variables

Loredana Lanzani
Syracuse University

Abstract. This talk concerns the applications of relatively classical tools from real harmonic analysis (namely, the $T(1)$-theorem for spaces of homogenous type) to the novel context of several complex variables. Specifically, I will present recent joint work with E. M. Stein on the extension to higher dimension of Calderón's and Coifman-McIntosh-Meyer's seminal results about the Cauchy integral for a Lipschitz planar curve (interpreted as the boundary of a Lipschitz domain $D \subset \mathbb{C}$). From the point of view of complex analysis, a fundamental feature of the 1-dimensional Cauchy kernel:

$$H(w,z) = \frac{1}{2\pi i}\frac{dw}{w-z}$$

is that it is holomorphic (that is, analytic) as a function of $z \in D$. In great contrast with the one-dimensional theory, in higher dimension there is no obvious holomorphic analogue of $H(w,z)$. This is because of geometric obstructions (the Levi problem) that in dimension 1 are irrelevant.

A good candidate kernel for the higher dimensional setting was first identified by Jean Leray in the context of a C^∞-smooth, convex domain D: while these conditions on D can be relaxed a bit, if the domain is less than C^2-smooth (never mind Lipschitz!) Leray's construction becomes conceptually problematic.

In this talk I will present *(a)*, the construction of the Cauchy-Leray kernel and *(b)*, the $L^p(bD)$-boundedness of the induced singular integral operator under the weakest currently known assumptions on the domain's regularity – in the case of a planar domain these are akin to Lipschitz boundary, but in our higher-dimensional context the assumptions we make are in fact optimal. The proofs rely in a fundamental way on a suitably adapted version of the so-called "$T(1)$-theorem technique" from real harmonic analysis.

Time permitting, I will describe applications of this work to complex function theory – specifically, to the Szegő and Bergman projections (that is, the orthogonal projections of L^2 onto, respectively, the Hardy and Bergman spaces of holomorphic functions).

References

[C] Calderón A. P, *Cauchy integrals on Lipschitz curves and related operators*, Proc. Nat. Acad. Sci. **74** no. 4, (1977) 1324-1327.

[CMM] Coifman R., McIntosh A. and Meyer Y., *L'intègrale de Cauchy dèfinit un opèrateur bornè sur L^2 pour les courbes lipschitziennes* Ann. of Math. **116** (1982) no. 2, 361 - 387.

[L] Lanzani, L. *Harmonic Analysis Techniques in Several Complex Variables*, Bruno Pini Mathematical Analysis Seminar 2014, 83-110, Univ. Bologna Alma Mater Studiorum, Bologna.

[LS-1] Lanzani L. and Stein E. M., *The Szegő projection for domains in $\mathbb{C}^n$ with minimal smoothness*, Dule Math. J. **166** no. 1 (2017), 125-176.

[LS-2] Lanzani L. and Stein E. M., *The Cauchy Integral in $\mathbb{C}^n$ for domains with minimal smoothness*, Adv. Math. **264** (2014) 776 – 830.

[LS-3] Lanzani L. and Stein E. M., *Hardy spaces for domains with minimal smoothness* Harmonic Analysis, Partial Differential Equations, Complex Analysis, Banach Spaces, and Operator Theory (Vol. 1): Celebrating Cora Sadosky's life. Vol. 1, AWM-Springer Series, Springer, (2016), 179 – 200. ISBN: 978-3-319-30961-3.