Lab 1: Linear Combinations & Spanning

Laboratory Experience

As we have seen by the operations we have defined for vectors, we can create new vectors using combinations of scalar multiplication and addition of existing vectors. In this lab you will explore both the algebraic and geometric combination of such vectors and develop an intuition for how algebraic operations relate to geometric representations of the same ideas.

1. Consider the two vectors $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. On page 1.1 of the document LinearComb.tns, these vectors have been defined for you. On page 1.2, you will see two sliders representing scalars ($a$ and $b$) along with the geometric representation of the linear combination of these vectors using the scalar values (i.e. $a\vec{v} + b\vec{w}$).

\begin{center}
\includegraphics[width=0.5\textwidth]{linear_combination.png}
\end{center}

a. On page 1.2, adjust the sliders to find scalars $a$ and $b$ so that the linear combination of the vectors $\vec{v}$ and $\vec{w}$ will be the same as the vector $\begin{bmatrix} 9.4 \\ 6.4 \end{bmatrix}$.

Record the values of the scalars you find giving a rough sketch of your linear combination.
b. On page 1.1, perform the computation \( a \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix} \) for the values of \( a \) and \( b \) that you found and discuss what you notice relative to your experience on the graphing page.

c. Is it possible to find a different pair of values \( a \) and \( b \) that will also result in the linear combination \( a\vec{v} + b\vec{w} \) being equal to the vector \( \begin{bmatrix} 9.4 \\ 6.4 \end{bmatrix} \)? If so, what values for \( a \) and \( b \) did you find? If not, explain why it was not possible.
d. Now grab the point \((9.4,6.4)\) on page 1.2 representing the vector \[
\begin{bmatrix}
9.4 \\
6.4
\end{bmatrix}
\] and move it to another location in the plane. Find a new set of scalars \(a\) and \(b\) so that the linear combination of the vectors \(\vec{v}\) and \(\vec{w}\) will be the same as your new vector. Record the new vector you choose along with the scalars \(a\) and \(b\) that work for the linear combination giving a rough sketch of your linear combination.

e. Repeat part (d) for at least two more vectors recording your new vectors and the scalars \(a\) and \(b\) that work for the linear combination giving a rough sketch of your linear combinations.

If we start with a certain set of fixed vectors and look at all of the vectors we can get (or reach) with just linear combinations of these vectors \((a\vec{v} + b\vec{w})\), it is like asking what objects can we grab with our arms. Thus we call the set of all vectors that we can reach with linear combinations of a certain set of vectors the Span of those vectors (analogous to our “arm span”). In this case we would denote the set of all vectors that we can obtain from linear combinations of \(\vec{v}\) and \(\vec{w}\) by \(Span\{\vec{v},\vec{w}\}\).
2. Choose two new vectors to use for vectors $\vec{v}$ and $\vec{w}$ redefining them on page 1.1 (be careful to choose entries that will allow the vectors to fit into the graphing window). While on page 1.1, you can choose menu option **1:Actions** followed by **1:Define** to redefine the vectors. To enter the desired new vectors, press the $\text{tab}$ key to get the math palette and select the $2 \times 1$ vector option (see below).

![Math palette screenshot](image)

**a.** Repeat what you did in question 1(d) recording the third vector and the scalars $a$ and $b$ that work for the linear combination giving a rough sketch of your linear combination.

**b.** Now place your target vector back to $\begin{bmatrix} 9.4 \\ 6.4 \end{bmatrix}$ and redefine the vector $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Now try to define another vector for $\vec{w}$ so that it is not possible to reach the vector $\begin{bmatrix} 9.4 \\ 6.4 \end{bmatrix}$ with any linear combination of vectors $\vec{v}$ and $\vec{w}$. What do you notice about the geometric relationship between $\vec{v}$ and $\vec{w}$?

**c.** Based on your experiences, given any two vectors in $\mathbb{R}^2$, can you always find a linear combination of them that will give you any vector in $\mathbb{R}^2$ you might desire? Explain your observations.
At this point we have examined linear combinations of two vectors to get another vector in $\mathbb{R}^2$. What if we used linear combinations of three vectors? For the next part of the lab, consider three original vectors $\vec{v}$, $\vec{w}$, and $\vec{u}$ using linear combination to obtain a fourth vector.

3. On page 2.1 of the *LinearComb.tns* document, you can see three vectors $\vec{v}$, $\vec{w}$, and $\vec{u}$ defined.

   a. Go to page 2.2 and adjust the scalars $a$, $b$, and $c$ so that you get vector $\begin{bmatrix} 7.9 \\ 8.7 \end{bmatrix}$ as a linear combination of vectors $\vec{v}$, $\vec{w}$, and $\vec{u}$. Record your values for $a$, $b$, and $c$ giving a rough sketch of your linear combination.

   b. Can you find a different set of scalars $a$, $b$, and $c$ so that you also get vector $\begin{bmatrix} 7.9 \\ 8.7 \end{bmatrix}$ as a linear combination of vectors $\vec{v}$, $\vec{w}$, and $\vec{u}$? If so, record your different set of values for $a$, $b$, and $c$ so that work giving a rough sketch of your linear combination. If not, explain why it can’t happen.
c. Compare/contrast the “uniqueness” of the linear combination in the case of three vectors to what you found in question 1(c) for two vectors.

4. Based on what you found in question 3, is it possible to replace the three-vector situation with a unique linear combination of two vectors? To help answer this question, try leaving \( a \cdot \vec{v} \) alone and finding a unique second vector that could be used instead of \( b\vec{w} + c\vec{u} \) so that when added to \( a\vec{v} \) would give the vector \[
\begin{bmatrix}
7.9 \\
8.7
\end{bmatrix}
\]
giving a rough sketch of your linear combination.

5. Based on your observations, what do you think would be the minimum number of vectors needed so that you could reach any desired vectors in \( \mathbb{R}^2 \) as a linear combination of the vectors? What relationship must there be among these vectors so that any vector in \( \mathbb{R}^2 \) can be obtained as a linear combination of them?