Lab 11: Numerical Integration Techniques

Introduction

The purpose of this laboratory experience is to develop fundamental methods for approximating the area under a curve for the definite integral. With this in mind, we will work from specific cases for calculating the sum of trapezoids and parabolic arcs that approximate the area under a curve and explore the relative speeds of these methods for convergence to the definite integral paying particular attention to the error bounds on the approximation.

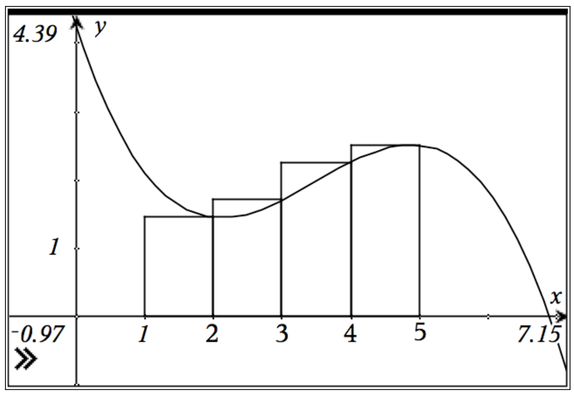


Figure 1

Laboratory Experience

From the Fundamental Theorem of Calculus, we know that if we want to calculate $\int_a^b f(x)\,dx$, we simply have to find an antiderivative F of the function f and find the difference $\int_a^b f(x)\,dx = F(b) - F(a)$. This works well for functions for which we can easily find the antiderivative. But you may recall functions such as $f(x) = e^{x^2}$ give us difficulty when attempting to find an exact solution for $\int_0^1 e^{x^2}\,dx$ since the antiderivative of $f(x) = e^{x^2}$ is elusive. In this lab we will explore several methods for approximating integrals such as this and try to get a handle on the error bounds for our approximation.

One approach for approximating the area under a curve comes from the use of rectangles as rough approximates for portions of the area. When these rectangular areas are added together we can obtain an approximation of the area. You may recall this approach as a Riemann sum. Figure 1 uses a Riemann sum with 4 subintervals of equal length, where the height of each rectangle uses the value of the function at the right-hand endpoints of each subinterval.

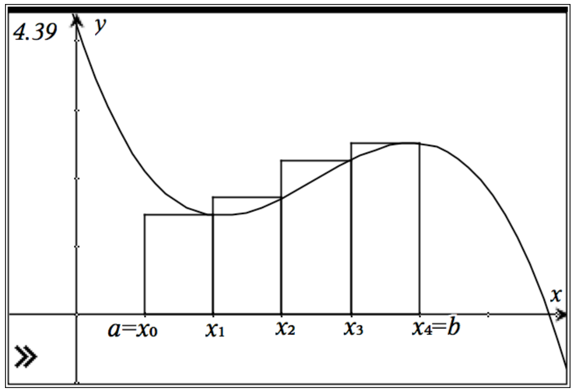


Figure 2: Approximations Using Rectangles

Since our ultimate goal is to understand the error involved in using numerical methods of approximation of integrals, we will begin by using integrals for which we do know the exact value and compare it to the approximated area from various numerical methods. Eventually, we will look at integrals for which we do not know the exact value and try to place a bound on the error of approximation.

1. In this lab, we will begin by comparing several numerical methods for approximating the integral $\int_0^2 4x^3 - 13x^2 + 9x + 3\,dx$ with its exact value. Using your CAS, what is the exact value of $\int_0^2 4x^3 - 13x^2 + 9x + 3\,dx$?

2. In questions 2, 4, and 5, you will be comparing different methods of approximation with the exact value you have just obtained. Therefore use the table at the end of the lab for recording your information. For this question you will use a Riemann sum approximation (rectangles).

    a. Using a Riemann sum with $n = 4$ subintervals, approximate $\int_0^2 4x^3 - 13x^2 + 9x + 3\,dx$. Show all work, writing out the sum in expanded form (e.g. express each term in the sum as a product of two values). Note whether you used left or right sums.

b. Use **Reimann Library** commands (for TI-Nspire CX CAS) to approximate the same integral using $n = 16$ subintervals. Record your data in the given table.

c. Experiment using at least three different values of *n* until you find a value for *n* so that the Riemann sum approximates the integral to within 0.001. Record your trials in the table. State your final value for *n* and the corresponding area.

So far we have used rectangles to approximate the area under a curve, but one of the consequences of this is that we get fairly large errors unless we have a large number of rectangles as you observed in the last question. One way we can reduce the error for "wiggly" functions with the same number of subintervals is to use trapezoids instead of rectangles (see Figure 3).

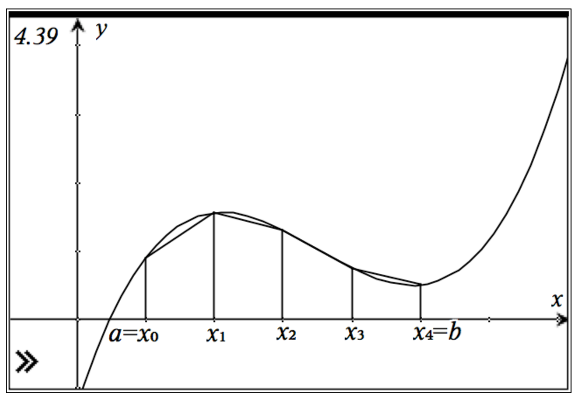
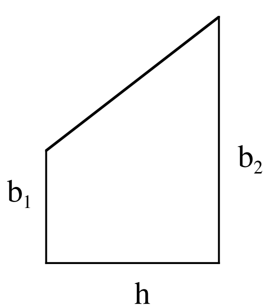


Figure 3: Approximations Using Trapezoids

Recall that for a trapezoid with dimensions of b_1, b_2, and h as shown below, the area is given by $A = \frac{1}{2}(b_1 + b_2)h = \frac{h}{2}(b_1 + b_2)$. One way to visualize this formula is to take two of these trapezoids and flip one and join it with the other like a puzzle to get a larger rectangle. Try it and sketch your image below explaining why the formula makes sense.



3. Since we know how to find the area of a trapezoid, we can use it to find a general form for using trapezoids to approximate areas under curves.

   a. Apply the formula given above four times to the four trapezoids in Figure 3. If we let T_4 denote the sum of the areas of the four trapezoids, using basic algebra and a corresponding labeled sketch of the graph and trapezoids, show that $T_4 = \frac{\Delta x}{2}\left[f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+f(x_4)\right]$ where $\Delta x = \frac{b-a}{4}$. Explain geometrically why there are "2"s in the expression referring to your sketch.

   b. We now want to generalize this expression beyond just 4 subintervals. Suppose we have *n* subintervals, use your process from part (a) to generalize to an expression for T_n. Explain your reasoning for the generalization by describing the relationship between the algebraic expression and the physical aspects of the sketched trapezoids.

c. In order to create a function that will calculate the area approximation using trapezoids (as well as other methods), we will need to find a way to locate the various values of x_k on the x-axis. The value of x_k can be found using $x_k = a + k \cdot \left(\frac{b-a}{n} \right)$. Explain why this expression works.

d. Create a function of three variables, called **trap(*a*,*b*,*n*)**, of three parameters (a, b, and n) on your CAS that calculates the trapezoid approximation for the area under a function assumed to be predefined as *f* on the interval $[a,b]$ using n subintervals. Here you will need to use the sigma (Σ) command found in the symbol palette on the TI-Nspire. Also, you want your **trap(*a*,*b*,*n*)** function to only contain the parameters *f*, a, b, and n. Therefore you will need to use the fact that $\Delta x = \frac{b-a}{n}$ and that the location of $x_k = a + k \cdot \left(\frac{b-a}{n} \right)$. State your function.

e. Now repeat what you did in Question 2 using the trapezoid approach and recording your data in the table.

So far we have use rectangles and trapezoids to approximate areas under curves. However, since most curves are "curved" we may want to use a figure that is itself smooth and curved. A familiar such figure is a parabola. We have been working with parabolas since our early years in algebra and so we might expect them to come in handy as approximating tools for areas. It also happens that Archimedes figured out how to find the area under an arbitrary parabola. From Archimedes' result we know that the area under a parabola passing through points $(x_0, f(x_0))$, $(x_1, f(x_1))$, and $(x_2, f(x_2))$ can be given by $\frac{\Delta x}{3}[f(x_0)+4f(x_1)+f(x_2)]$ where Δx is the length of a subinterval $[x_i, x_{i+1}]$. So if we have three points on a function, we can construct a parabola through the points so that it approximates the original function on the two consecutive subintervals as seen in Figure 4 showing two parabolic arcs along four subintervals. In Figure 4, we have two parabolas each mimicking the original function along two consecutive subintervals. The first parabola passes through the points $(x_0, f(x_0))$, $(x_1, f(x_1))$, and $(x_2, f(x_2))$. The second passes through the points $(x_2, f(x_2))$, $(x_3, f(x_3))$, and $(x_4, f(x_4))$. If we do this for many parabolas under a curve, we can get an approximation for the area under the curve.

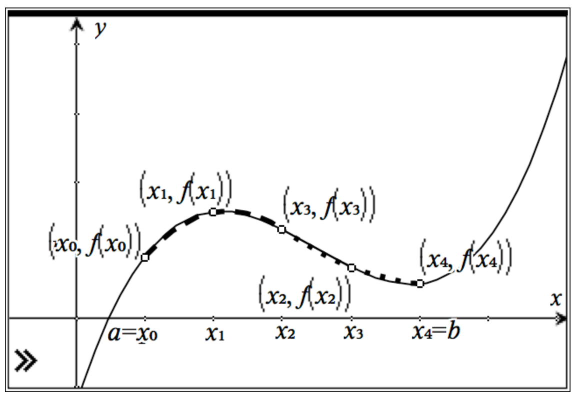


Figure 4: Approximating Using Parabolas

4. We will now use Archimedes' result to approximate the areas under the two parabolas shown in Figure 4.

    a. Let S_4 be the combined areas found under the two parabolas as shown in Figure 4. Using algebra and a corresponding labeled sketch of the graph, show that the formula for S_4 is given by: $S_4 = \frac{\Delta x}{3}[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+f(x_4)]$

b. Let S_n be the sum of the areas found under $\frac{n}{2}$ parabolas (where n is even). Write an expression for S_n. Explain your reasoning for the generalization by describing the relationship between the algebraic expression and the physical aspects of the sketched parabolic pieces.

c. Create a function, called **simp(*a*,*b*,*n*)**, of three parameters (a, b, and n) on your CAS that calculates the parabola approximation for the area under a function predefined as f on the interval $[a,b]$ using n subintervals with n even. State your function. This method of approximation is called *Simpson's Rule*.

5. Now we will compare the method using parabolas, called Simpson's Rule, to the other methods.

    a. Repeat Question 2 using Simpson's Rule, organizing your data in the table.

    b. What value of *n* did you need for each method to get the approximation within the desired accuracy of 0.001? Which method needed the smallest value of *n*? Which needed the largest? Why do you think some methods required fewer subintervals to achieve the same level of accuracy?

6. The previous examples have been a bit contrived since we could easily compute the integral exactly using the Fundamental Theorem of Calculus. The methods we have discussed here are particularly handy when we are dealing with a function for which we cannot find an algebraic representation for the integral. Let's now investigate the integral $\int_0^2 \sqrt{1+25x^4}\, dx$, one that we cannot easily find exactly.

    a. Approximate $\int_0^2 \sqrt{1+25x^4}\, dx$ using the Trapezoid Rule and Simpson's Rule. Use different values of *n* until you feel that your answers are accurate to within 0.001.

    b. Discuss how you decided when your approximations would be within the desired tolerance? Discuss how you were able to get a feel for the accuracy of your answer when you do not have the exact answer to use to compare it?

c. Which method appears to be the fastest at converging on the area to the desired accuracy?

We have now been introduced to several methods for approximating integrals. The comparisons we have made give us some idea of the relative speeds of these methods for approximating integrals to a certain desired accuracy. However, we have not yet been able to determine a value of n **a priori** that will guarantee us a desired accuracy. We will explore this issue later.

Data for $\int_0^2 4x^3 - 13x^2 + 9x + 3\,dx$

Method Used	# Subintervals	Δx	Exact Area	Approx. Area	Error

