Lab 14: Impersonating Functions

Materials: Computer Algebra System

Goal: This activity explores basic concepts of algebra and calculus involved in using power series to imitate transcendental functions. In particular, the series for $\sin(x)$, $\cos(x)$, and e^x are investigated and connections are made with practical uses of such functions for creating computing algorithms used by calculators and computers to evaluate numerical approximations.

Have you ever wondered how your calculator knows what the $\ln(5)$ or the $\sin(0.7)$ is or how it can find decimals for numbers like $\sqrt{3}$? Did someone program every possible square root, trig value, or logarithm into the machine? Wouldn't that take up a tremendous amount of memory? Well, the answer is that your calculator knows how to use basic operations like addition, subtraction, multiplication, and division to approximate these types of values to the desired decimal accuracy. In this exploration you will first investigate how we can approximate more complicated transcendental functions using simple polynomials. Once we establish this method for approximating functions with polynomials, we will then use it to explore a surprising connection between trigonometry and the complex number systems.

To this point we have seen that if we zoom in far enough, a function looks a great deal like a straight line. But how does this help us impersonate more complicated functions like e^x ?

1. Graph $y = e^x$ along with several possible linear functions that would look like the "linear-looking" graph of $y = e^x$ when you zoom in near $x = 0$. Remember, you need the linear function to have the same slope as $y = e^x$ and also have the same point in common with $y = e^x$ at zero. Record the lines you tried along with brief explanations for why you tried those particular lines. Give your final line that you feel best imitates $y = e^x$ at $x = 0$.

2. Now that you have created a line that looks similar to $y = e^x$ near $x = 0$, graph it along with $y = e^x$ on the same window. Now zoom out and give an approximation of the interval about $x = 0$ where your line is a “fairly close” imitation of $y = e^x$.

3. Suppose you want to imitate $y = e^x$ within a larger interval. Since $y = e^x$ is not linear, we need a function that curves so that it will “hug” the graph of $y = e^x$ through a larger interval. Since quadratics are curved, experiment with your linear function from question 1 by adding on a term of the form ax^2. Record your choices for *a* along with sketches of your new quadratic function and $y = e^x$ on the same axes. Give your final choice of quadratic function of the form $y = ax^2 + bx + c$ that best imitates $y = e^x$ near $x = 0$.

4. You have now been able to extend the size of the interval where $y = e^x$ can be imitated by a polynomial by adding on a higher degree polynomial term. Can this continue? Take the quadratic function you have developed in question 3 and try to extend the interval of imitation further by adding on a third degree term and experimenting with its coefficient. Give your final choice of cubic function of the form $y = ax^3 + bx^2 + cx + d$ that best imitates $y = e^x$ near $x = 0$ and sketch your graph of $y = e^x$ along with your cubic below.

The polynomials you have created to imitate the function $y = e^x$ are called Taylor polynomials named for the mathematician, Brook Taylor. To understand why these particular coefficients work for imitating this function, let's look at one other function ($f(x) = \sin(x)$) and imitate it using a built-in command on your CAS. On most CAS, the Taylor command has similar syntax. On Maple, we use **taylorpoly(sin(x), x=0, 7);** to get a seventh degree polynomial that mimics $f(x) = \sin(x)$ near zero. On a TI-Nspire CX CAS, the syntax is **taylor(sin(x), x, 7, 0)** where the 0 at the end is the point near where you want to imitate the function and the **x** is the variable you would like used in the polynomial (if you omit the value about which you want to imitate the function, it assumes a value of 0). On the TI-Nspire CX CAS you can find the Taylor command under the **Series** submenu of the **Calculus** menu.

5. Use your CAS's Taylor command to generate 1st, 3rd, 5th, 7th, and 9th degree taylor polynomials for $f(x) = \sin(x)$ and record them in the table below. Graph all of these along with $y = \sin(x)$ on the same axes.

Degree of Polynomial	Polynomial
1	
3	
5	
7	
9	

Now in order to understand why these particular polynomials work, let's examine the 9th degree polynomial that you have found. As you learned in Calculus I, we can infer information about the "shape" of a graph by looking at the derivative. More specific, the first derivative tells us the slope of the function at a given point, the second derivative tells us the rate at which the slope is changing, the third derivative tells us the rate at which the rate of the slope is changing and so on. With each higher order derivative, we gain more information about the "shape" of the graph and its behavior. This is the idea behind Taylor polynomials.

6. To explore this relationship, complete the following table where you evaluate the n^{th} derivatives of both your polynomial and $f(x)=\sin(x)$ at $x=0$. State your observations related to your results.

Order, n	Polynomial derivative evaluated at $x=0$	$\frac{d^n}{dx^n}\left(\sin(x)\right)$ evaluated at $x=0$
1		
2		
3		
4		
5		
6		
7		
8		
9		

7. Now find the 8th degree Taylor polynomial for $f(x) = \cos(x)$ using your CAS's Taylor command. Repeat what you did in question 6 and record your results and observations.

Order, n	Polynomial derivative evaluated at $x = 0$	$\frac{d^n}{dx^n}(\cos(x))$ evaluated at $x = 0$
1		
2		
3		
4		
5		
6		
7		
8		
9		

8. Using your observations from questions 6 and 7, state a conjecture about the nth derivative of a Taylor polynomial and the nth derivative of the function it is imitating. Then test your conjecture with the 3rd degree Taylor polynomial for e^x you found earlier.

9. In the special case of e^x near zero, use your Taylor command to generate polynomials of degree 5, 6, 7, and 8 for $f\left(x\right)=e^x$. Describe the pattern you see for the general nth degree Taylor polynomial for $f\left(x\right)=e^x$.

10. Recall from Calculus I, $\frac{d^n}{dx^n}\left(e^x\right)=e^x$ for $n\in\mathbb{Z}^+$. Explain how this fact is related to your results from the nth derivative of your general Taylor polynomial for e^x.

The fact that we can create polynomials to imitate the functions e^x, $\sin(x)$, and $\cos(x)$ is extremely important from a practical sense. Have you ever wondered how your calculator knows the value of something like $\sin(0.7)$? The operations of addition, subtraction, multiplication, and division are basic algorithms that exist as routines in any computer or calculator (note that exponents can be accomplished using repeated multiplication). If we wish to know $\sin(0.7)$, the machine can simply create a finite polynomial that imitates the sine function to the desired decimal accuracy and then compute it using these basic binary operations that make up the polynomial.

11. Now suppose you have only a non-scientific calculator (without sine, cosine, and tangent keys) and need to evaluate the $\sin(1.78)$ where the argument for sine is in radians. Explain how you could accomplish it? What about $\cos(1.78)$? What about $\tan(1.78)$?