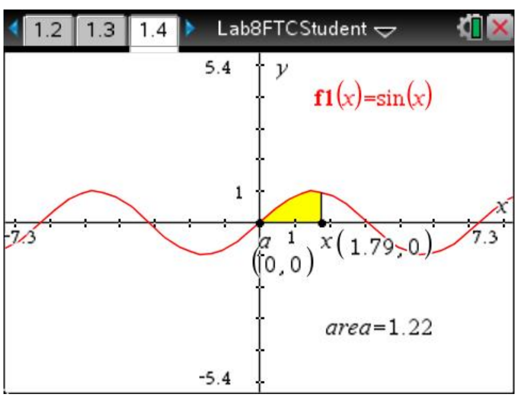
Lab 8: Area Functions and the FTC

Laboratory Experience

As we have seen, we can determine the area under a curve using a definite integral such as $\int_0^5 x^3 dx$. This gives the area under the graph of $y = x^3$ from $x = 0$ to $x = 5$. What if we also wanted to find the area under the curve from $x = 0$ to $x = 6$ or $x = 7$ or $x = 8$ and so on. For each of these endpoints we will have a new numerical value for the area. In this activity we will develop the concept of an "area function" where the output of the function depends on the upper limit of the definite integral.

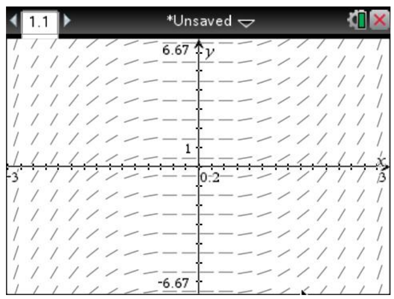
1. Consider the function $f(x) = \sin x$. We are going to look at the relationship between the upper limit of the definite integral and the area under the curve starting at $x = 0$. To do this, we use the integral command on the Graphs page with $f(x) = \sin(x)$. Labeling the lower limit of integration as *a* and the upper limit as *x*, we get the following graph.
    a. On the **Graphs** page (1.4) of your Nspire document, grab the upper limit of integration and drag it along the *x*-axis. What do you notice about the numerical value of the area? When is it increasing and when is it decreasing? On top of the axis system below, sketch your guess for the shape of the area function as *x* varies. You might want to plot several points for various *x*-coordinates using the *area* value given on the screen as the *y*-coordinate.



b. The graph on page 1.7 of the Nspire document has a point plotted, P, to represent the area under the curve as a function of x. Drag the point x and watch the movement of P. Does its movement match the graph you predicted in question 1a? If not, discuss the differences. Give a guess of a function that would describe the path of the "area function" point.

c. In order to see the graph of the area function, we can show the locus of the point, P, as x moves. To do this, on the previous Graphs page, select the **Locus** command under the **Construction** menu and then click on the point P followed by the point x. You should see a graph of the area function appear. What function appears to fit the graph? Make a sketch of your graph.

2. Up to now we have spent a great deal of time in calculus thinking about slopes of graphs. Is there any connection between the idea of area under a curve and the idea of slopes of functions? Suppose we think of the function $f(x)=\sin x$ as defining the slope of some function. In other words, if we plug in a value, say $x=1$, and get that $f(1)\approx .84147$ this could be the slope of some other function. If we did this for many *x* values and then plotted short segments (mini tangent lines) to represent the numerical slope values in a graphical form we would generate what is called a *slope field*. For example, if we used a function $g(x)=x^2$ as the slopes of some other function, $G(x)$, we would get slopes of 9, 4, 1, 0, 1, 4, and 9 for corresponding *x* values of –3, -2, -1, 0, 1, 2, and 3 respectively. If we did this for many *x* values and plotted short slanted segments for each slope we would get an image like:



Now consider the function $f(x)=\sin x$. We begin this process by defining a function, $slope(x)$, on the next page that represents the slopes of some original function at any *x*-coordinate, *x*. Here we will start by using the $\sin(x)$ function as our slope function.

Page 2.4 of the Nspire document takes the derivative function, called $slope(x)$, and generates a *slope field* indicating the family of *antiderivatives* (the functions whose derivatives are the $slope(x)$ function). To change the slope field, simply edit the slope function on the graphing page or re-enter the slope function on a calculator page. To change the length of slope field segments, set the graph type to **Diff Eq** and then bring up the entry line at the top of the page by pressing ctrl G or double clicking in the white space and move to y1'. Clicking on the $\boxed{\cdots}$ icon, tab down to **Field Resolution** and edit the number. Larger numbers give more segments in the slope field.

a. After looking at the slope field for the function $slope(x)$, how does it compare to the locus of your area function from question 1? Give a rough sketch to help illustrate your observations.

b. On page 2.7 of the Nspire document, we define the area function as the area from a fixed point, a, to a variable, x and plot it as a function $f1(x)=\int_a^x slope(t)dt$. Although for a single fixed value, a, we get a specific function, we can also drag a around to see the effects of changing the lower limit of integration. Describe the behavior you see as you manipulate a and x. Give a rough sketch to help illustrate your observations.

c. On page 2.9 of the Nspire document, you see a plot of the slope function showing the area under it in the left window and the slope field with the area function on top of it in the right side of the window. Drag the point x in the left window, describe what you notice on the slope field. Give a rough sketch to help illustrate your observations.

d. You have just used the integral to define a function that we call the area function. It can be written as $Area(x) = \int_a^x \sin(t)\,dt$. In general, we can use the integral to define an area function beginning at any point on the x-axis, call it a, and compiling the area under the curve up to x as $Area(x) = \int_a^x f(t)\,dt$. How might the area function you just found be affected if we change the value of a? Again on page 2.9, drag the point, a, in the left window and describe what you notice about the resulting area function and its relationship to the slope field. Give a rough sketch to help illustrate your observations.

e. Summarize your findings from this lab with respect to the relationship between area functions and antiderivatives.