Douglas A. Lapp, Marie Ermete, Natasha Brackett, and Karli Powell

Linked Representations in Algebra: Developing Symbolic Meaning

Connecting Research to Teaching appears in alternate issues of *Mathematics Teacher* and brings research insights and findings to the journal's readers. Manuscripts for the department should be submitted via http://mt.msubmit.net. For more information on the department and guidelines for submitting a manuscript, please visit http://www.nctm.org/publications/content.aspx?id=10440#connecting.

Edited by ***Margaret Kinzel,*** *mkinzel@boisestate.edu*
Boise State University, Boise, ID

Laurie Cavey, *lauriecavey@boisestate.edu*
Boise State University, Boise, ID

The use of symbols provides mathematics with enormous power. From the learner's perspective, however, symbols can present enormous obstacles. Using symbols to encapsulate big ideas allows readers to see various stages of arguments within one field of view. However, for many learners, unpacking symbolization poses a significant hurdle in the development of conceptual understanding. Here we compare and contrast symbolic reasoning approaches that algebra students used when solving equations.

What is a root of an equation, and how is it related to various representations? More generally, what does it mean for a value to be a solution to an equation? These are standard questions that we expect students to be able to address and discuss. Although many students may be able to solve equations, far too many have limited conceptual understanding and rely primarily on procedural knowledge of the equation-solving process.

AARON'S SOLUTION TO A QUADRATIC EQUATION

Consider the following situation in which a student, Aaron, tried to solve the quadratic equation below during an interview:

$$(x - 2)(x + 3) = 6 \text{ (no graph given)}$$

Aaron began by setting each factor equal to 6 (see **fig. 1**). Here he applied a memorized procedure that he overgeneralized from previous experiences of solving quadratic equations by setting each factor equal to zero. After finding solutions, Aaron said, "I don't know if those approaches are right . . . I'd have to know exactly where we are at," indicating that he was unsure of his process. When questioned further, he stated that his process might change depending on "if we're trying to find where it's going to cross on the *x*-axis." In elaborating, Aaron stated that if he were trying to find points on a graph, then he would view the process differently, and he referred to another problem that included a graph along with an equation (see **fig. 2**). However, when he described how his process would change, he repeated the same algebraic steps and changed only the number that he would set each factor equal to (4 instead of 6) (see **fig. 3**).

As Aaron continued to solve the new problem, he noticed an inconsistency:

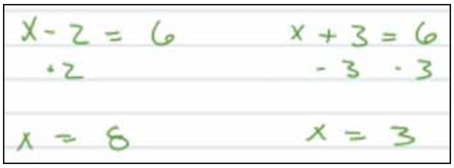

Fig. 1 Aaron shows his algebraic approach for solving $(x - 2)(x + 3) = 6$, setting each factor equal to 6.

Aaron: Uh-oh, something's not right . . . the solutions [don't] match . . . [*pressed to explain*] [I] got 5 and 2, and I would expect them to match –2 and 1 [*pointing to the x-intercepts on the graph*]. Um, just because that's where they cross.

At first, it appeared that Aaron had caught his conceptual mistake, but he had simply replaced an algebraic rule with a graphical one (looking for the points where the graph crosses the *x*-axis). Aaron continued to try to resolve his algebraic solutions with his expectations that these solutions should correspond to the *x*-intercepts from the graph, but he was unsuccessful and gave up.

Why was Aaron unable to see the error in his solution process even though he did attempt to use multiple representations when a graph was provided? The teacher had made it a point to use multiple representations in solving equations during class, so it is not surprising that Aaron would seek confirmation when the graph was present. However, the use of multiple representations alone is not enough to help students make connections; students need to see how these representations connect.

OUR STUDY WITH COLLEGE ALGEBRA STUDENTS

Our proposal for curricular enhancement is based on a study, conducted over two semesters with college algebra students, that examined how students developed algebraic concepts using various types of technology. During the fall semester, fifteen students participated in the study; four of them also took part in two interview sessions conducted during the last three weeks of the course. During the spring semester, thirty-four students participated in the study; seven of them took part in similar interview sessions. Data were collected during

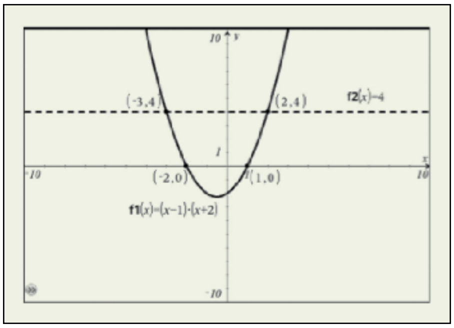

Fig. 2 Given the graph of the function as shown, students were asked to find all solutions to $(x - 1)(x + 2) = 4$ and justify their answer.

both semesters through interviews, researcher field notes, and student artifacts. Because the content of the typical college algebra course is roughly the same as that of a high school second-year algebra course, our results maintain some applicability to the secondary school curriculum.

In this particular study, students in the first semester were taught with standard graphing calculators (the TI-84, which does not allow dynamically linked representations) along with occasional demonstrations by the instructor using a platform that allows dynamically linked representations (TI-Nspire CAS). By *dynamically linked*, we mean representations that update in real time as each is manipulated. During the second semester, each student was issued a TI-Nspire CAS to use during the entire semester, and the same teacher served as a control for instructor influence with respect to general approaches to instruction. The teacher for both semesters was an expert on the use of technology in the teaching and learning of mathematics who has delivered many hours of in-service professional development.

Aaron participated in the first semester of the study. As a result, he was not able to see a real-time linkage between the key features of the graph and their corresponding algebraic counterparts in the symbolic world.

DEVELOPMENT OF SYMBOLIC MEANING

The theoretical framework of this study is based on research with microcomputer-based laboratory (MBL) technology (Lapp and Cyrus 2000) as well as Kaput, Carraher, and Blanton's (2008) model

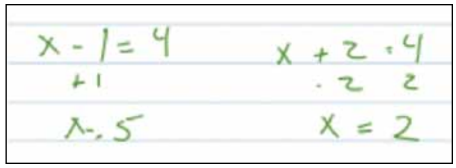

Fig. 3 Aaron's algebraic approach to solving $(x - 1)(x + 2) = 4$ mirrored his algebraic approach to solving $(x - 2)(x + 3) = 6$.

for the development of symbol systems. In the research with MBL, we have seen that real-time changes in linked representations that are simultaneously visible help students make connections among corresponding features of the various representations (Lapp and Cyrus 2000; Beichner 1990; Brasell 1987).

Linking representations deepens understanding of mathematical concepts, but we still must ask how the use of symbol systems influences our learning and, further, our development of new ideas. Kaput, Blanton, and Moreno (2008) propose a model that describes how we develop our understanding of existing ideas as well as how we create new concepts. The key to this model is the communication and analysis between two worlds. One is the "real world" (i.e., the world of broader mathematical or physical experiences), and the other is the world of the symbols that we use to represent these real-world experiences. In this model, the learner initially creates raw representations from experiences; in turn, these representations and understanding of the real world evolve as the student interacts with both worlds. The evolution of the student's view of the real world is relative to the symbolic worlds of *B* and *C*, which are represented by A_B and A_C (see **figs. 4a** and **4b**). In this sense, Kaput, Blanton, and Moreno's model is consistent with Tall and Vinner's (1981) view of an evolving "concept image."

This process, although focusing on representations, is consistent with Sfard's (1991) process of interiorization, condensation, and reification, in which concepts become internalized through interaction with the various natures of the mathematical ideas. In some ways, the use of technology here reduces the length of time for Sfard's interiorization and facilitates the condensation process. Alleviating the computational load can

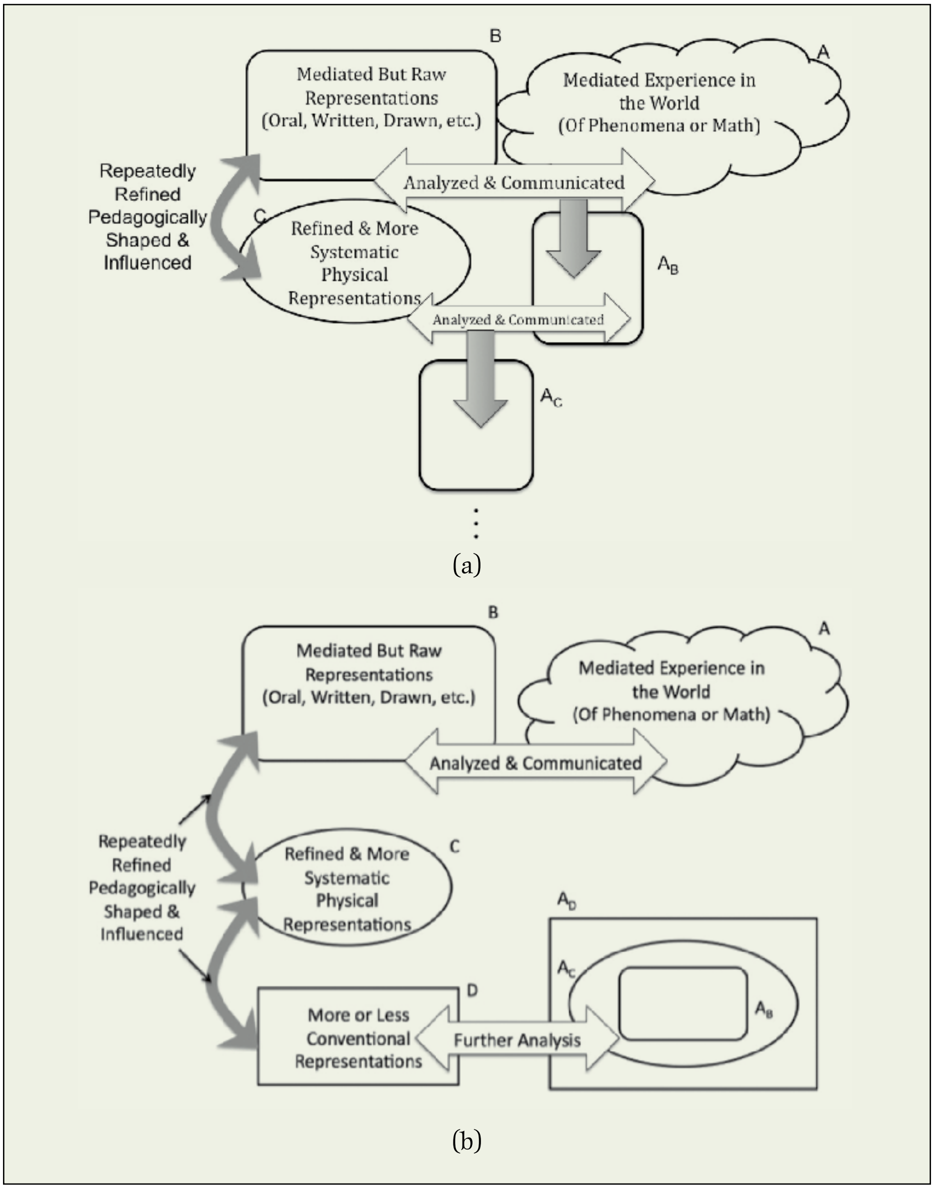

Fig. 4 Kaput, Blanton, and Moreno (2008, pp. 30 and 31) describe a model that explains how symbolic meaning evolves (**a**) and how symbol systems are nested as they evolve (**b**).

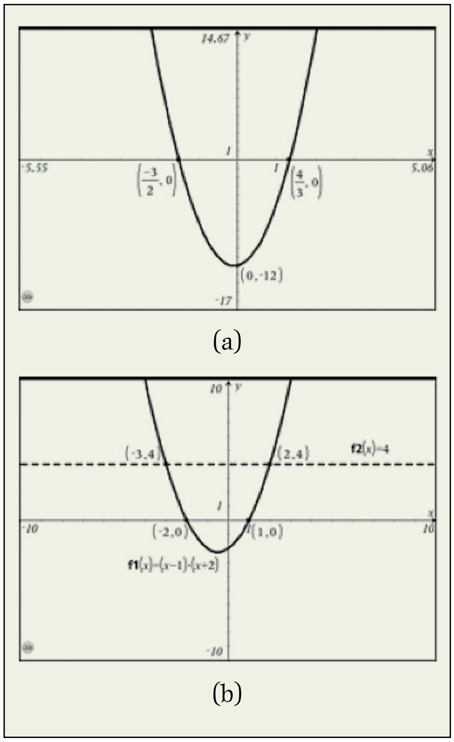

Fig. 5 Jon was able to use his solutions to these problems to help him solve $(x - 2)(x + 3) = 6$. These are the graphs for problems 6 and 7 in Jon's earlier interview.

also keep the lack of procedural fluency from becoming a cognitive obstacle to reaching condensation or reification, as observed by Lapp, Nyman, and Berry (2010).

Although Aaron had access to the use of multiple representations and the teacher routinely used graphical, numerical, and algebraic representations, they were not dynamically linked. Thus, Aaron may have had more difficulty making a connection between the meaning of the graphical representation of x-intercepts and the zero property of multiplication as a technique for equation solving. It is also important to note that Aaron's approach was typical of all students interviewed during the first semester of the study.

JON'S APPROACH

How do we create experiences that allow students to move fluidly among various representations while maintaining an isomorphic mapping of meaning between various worlds used to represent the ideas involved? One way is to use technology that allows for linked representations.

In contrast to the students interviewed in the fall semester, who used only the basic graphing technology, all the students interviewed in the spring semester showed an ability to move among representations as a means of justifying their reasoning. To illustrate, we examine Jon's response to the same question posed to Aaron—solving the equation $(x - 2)(x + 3) = 6$. Unlike students from the previous semester, Jon realized that he must get the equation set equal to zero to make the comparison to zeros on the graph of a function. In the following exchange, Jon referred to his solutions to other problems (problems 6 and 7; see **fig. 5**).

***Interviewer*:** So what do you think is different about this? Why is it important that you get the 6 to the other side?

***Jon*:** Because I think if you were to graph it, you're trying to find a completely different line, if it's a 6 or 0 and if it's 0 . . . You're trying to find y. And then if y was 0 . . . I don't know.

***Interviewer*:** So you're thinking of it from a graphical standpoint?

***Jon*:** Yeah, I'm thinking about it from a graph. If it's equal to 6, it's going to be up higher and it's gonna be . . . You're not looking for the zeros [*points to problems 6 and 7 from the earlier questions*].

***Interviewer*:** So kind of like the

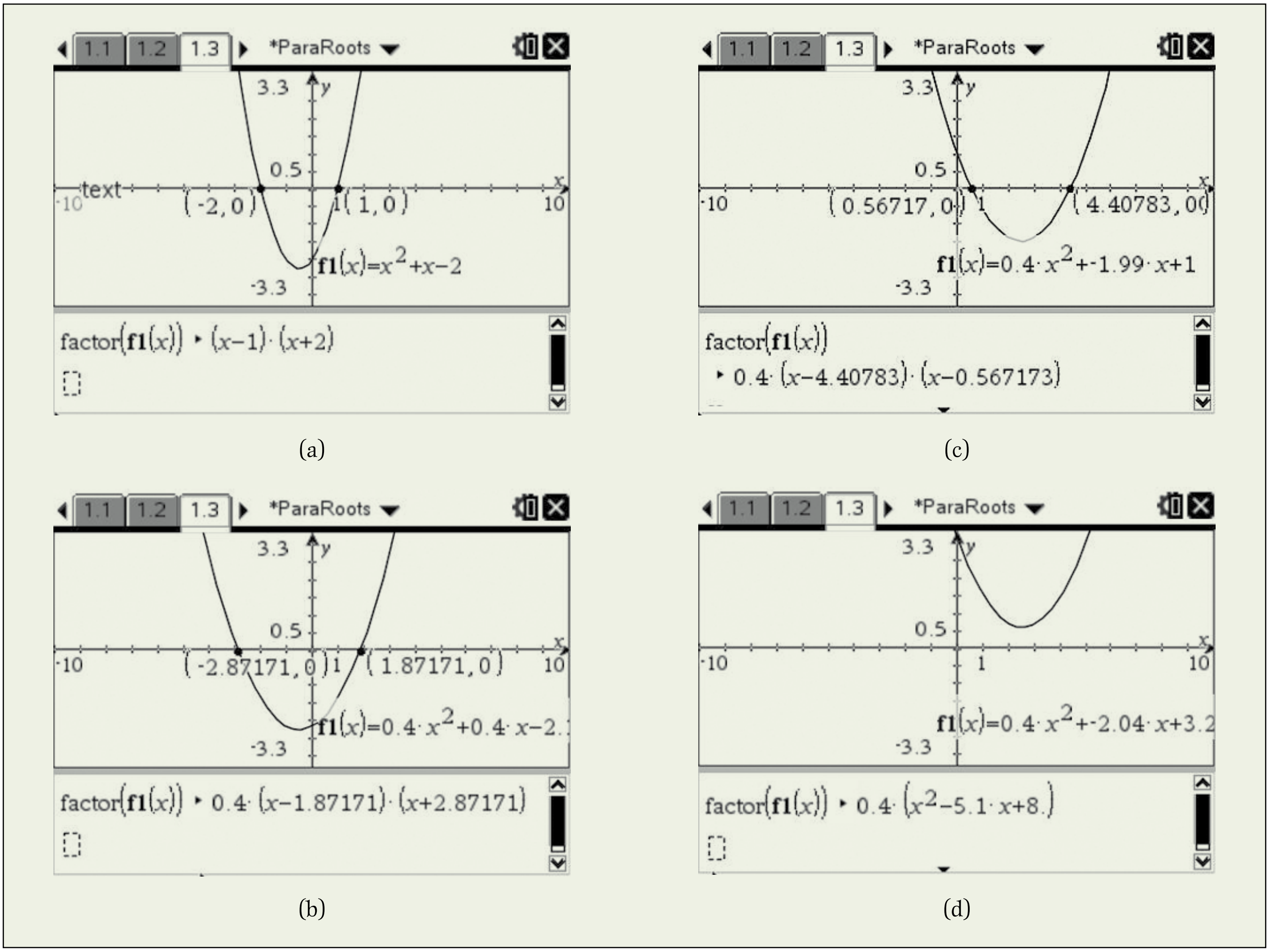

Fig. 6 These are still images from the dynamic experience of moving a graph and seeing the factored form of the function change in real time. Manipulating the graph enabled Jon to see the relationships between roots and x-intercepts.

difference in those earlier problems?

***Jon*:** Basically, I took number 7 and tried to turn it more into, like, number 6. Like, I tried . . . I just moved it down, so that way I would be able to solve for it. I visualized it as more like bringing it down, and I just solved for the vertex, which I know how to do. When it comes to that, it's more complicated to get to that point.

***Interviewer*:** Oh, the x-intercepts you mean, as opposed to the vertex?

***Jon*:** Yeah, 'cause when it's at equals 6, it's like the whole thing got raised 6. So I just made it . . . I brought it down to the vertex . . . and then I just solved for it there because that's easier for me to do algebraically.

Jon had first been exposed to dynamically linked representations through technology-rich investigations in class. In one investigation, manipulating a graph gave real-time change in the algebraic representation of the quadratic function as well as the factored form of the function. Jon was asked to label the x-intercepts on the graph, and, as he manipulated the graph, he noticed a connection between the numerical values of the x-coordinates of the x-intercepts and the numerical values, r_1 and r_2, found in the factored form of the function $a(x - r_1)(x - r_2)$ (see **fig. 6**).

In the past, we have seen that students tend to focus on aspects of a situation that are invariant across representations (Lapp 1997). In this case, the student noticed that, no matter how the graph was manipulated, the numerical values of the x-coordinates in both the labeled x-intercepts and the r_1 and r_2 values in the factored form of the function remained the same. The classroom investigation also included questions designed to focus the student's attention on the effects on the function's output for entering each x-coordinate of the zeros into the factored form of the function as well as the values of each factor. Our research suggests that this combination—communication and analysis between symbolic worlds—enabled the student to articulate the reason for the use of factoring as a technique for equation solving. For this reason, it is imperative that the equation is set equal to zero before factoring.

Students in both semesters were exposed to the zero property for multiplication as a reason for solution by factoring, but students in the second semester, using CAS, were allowed to interact with linked representations and

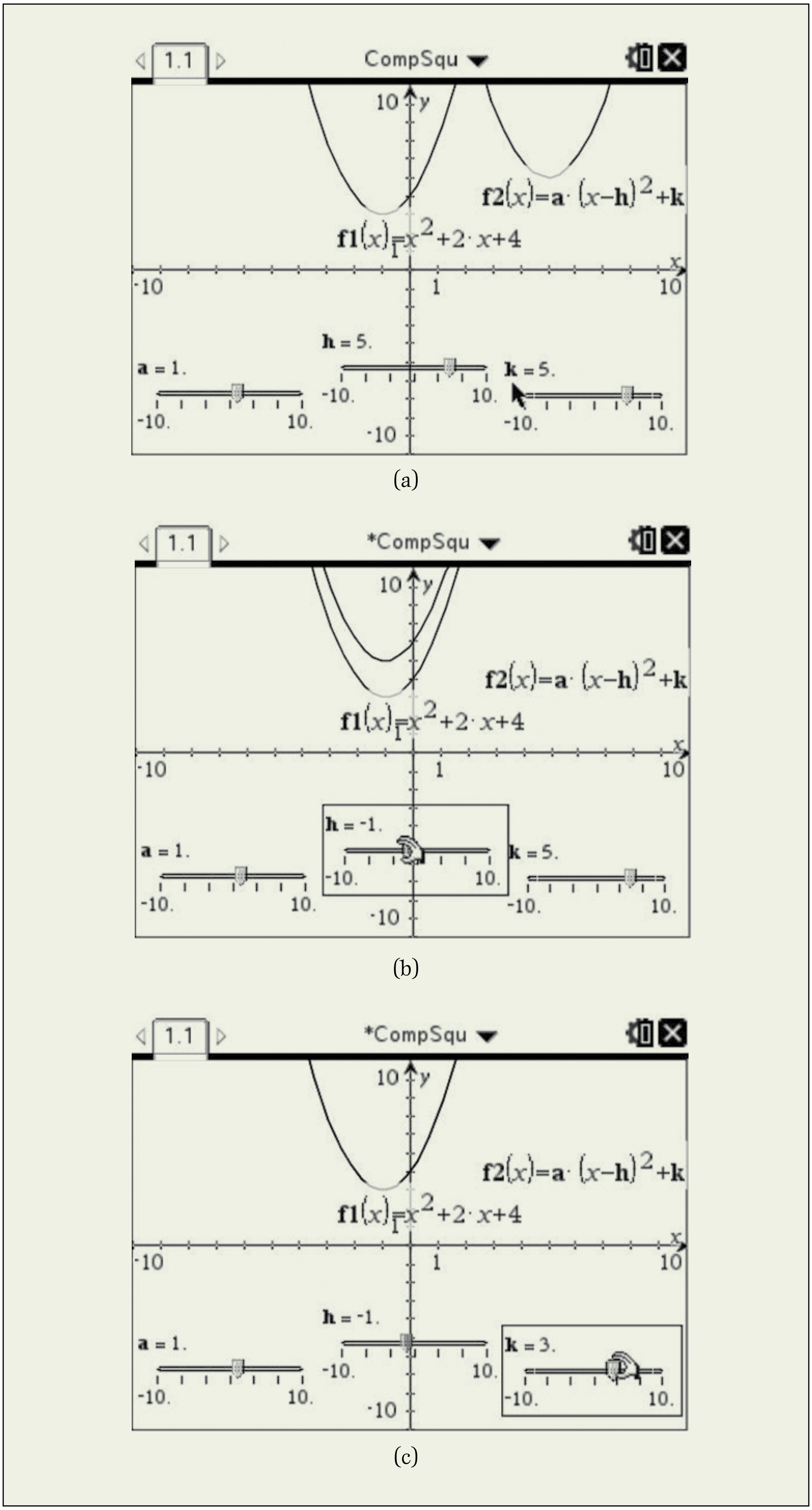

Fig. 7 Students use sliders to manipulate a parabola given in vertex form and see its equivalent expression in standard form. In this way, they were able to make more connections when the quadratic was expressed in vertex form.

were required to justify their reasoning in a lab experience. Students in the first semester were told by the teacher about this reason but simply watched the teacher point out the fact on a single graphical representation that was not linked algebraically.

In another investigation, Completing the Square (see **fig. 7**), students developed a connection between algebraic and graphical representations for the process of completing the square and observed relationships among various parameters found in the algebraic expressions. Here students further experienced the connection between algebraic and graphical transformations.

This experience likely led to Jon's strategy of transforming the graph of a parabola and its intersection with a horizontal line above the x-axis into one in which the parabola was shifted down until the horizontal line was superimposed on the x-axis. In his desire to transform an equation by subtracting 6 from both sides, Jon expressed a graphical understanding linked to the algebraic transformation of subtracting 6. He stated that he was essentially moving the dotted line (see **fig. 5**) down to coincide with the x-axis so that he could use his technique of factoring, which required the equation to be set equal to zero. In this instance, he was able to justify his process and not just execute it procedurally.

The influence of the Completing the Square investigation can also be seen in Jon's reference to the movement of the parabola's vertex during his verbal description of his graphical understanding. In this investigation, students were specifically asked to follow the movement of the vertex.

Jon clearly demonstrated an understanding of the concept of root and its connection to factoring as an equation-solving technique. He explained why he could not simply set each factor equal to zero in the equation $(x - 2)(x + 3) = 6$:

***Jon*:** If I plug 2 into the first one, 2 minus 2 . . . that would give me 0, and 0 times anything would equal 0, not 6, so that's wrong. Then if I plug –3 in, –3 minus 2 would be –1, and then the second one [*referring to*

> *the second factor*] would still be 0, so −1 doesn't equal 6 either [*meaning entering −1 into the equation*], so that would be wrong too.

Jon could articulate an understanding of the root-solving process using the zero property of multiplication along with his verbal reference to the vertex movement. This fact indicates an influence of both these investigations on his mathematical understanding and its relationship to the symbolic world that describes these ideas.

DYNAMIC LINKING IS THE KEY

From these contrasting examples, we see that, as we teach algebra, conceptual understanding can go hand-in-hand with procedural ability. Our research challenges the conventional wisdom that students must first become procedurally fluent before they can understand the concepts that we teach. Heid (1988) challenged this position more than two decades ago, arguing that we should rethink the sequencing of skills and concepts in calculus by using computer algebra systems to develop concepts before teaching procedures.

Here we see this same principle applied to high school algebra concepts. The difference between our study and Heid's is that we suggest that it is not just the computer algebra system that influences how students see connections but rather the dynamic linking of various representations. Students in the first semester of this study used the TI-84 and were introduced to concepts before skills; however, they did not have the ability to manipulate various representations and see real-time effects among representations.

A second aspect of concept development that we noticed involves student control of the environment. As Lapp and Cyrus (2000) suggest from research on the use of data collection devices, the student's ability to manipulate the environment plays a significant role in making connections. During the first semester of this study, the teacher merely demonstrated some of the dynamically linked representations using a computer during class; the students did not have use of this technology individually. Results of our interviews showed that none of these students could articulate connections among various representations. However, during the second semester, each student had a TI-Nspire CAS device and used it during student-centered investigations. This ability to communicate and analyze through dynamically connected representations between the symbolic world and the world of mathematical ideas allowed students to make connections between concepts and procedures.

As technology that links representations has become readily available, there is no reason we should not take advantage of it to better develop students' mathematical understanding. However, technology alone cannot make these connections for students. Kaput, Blanton, and Moreno (2008) as well as Sfard (2008) suggest that students' negotiation of discourse between the symbolic world and the world of mathematical ideas plays a key role in the development of symbolic meaning. Therefore, as teachers, we need to use appropriate technology to engage students in investigations that allow them to make mathematically meaningful observations and justify them.

ACKNOWLEDGMENTS

This research was supported in part with funding from the National Science Foundation (NSF) grant no. DMS 06-36528. The opinions expressed are those of the authors and do not necessarily represent the views of the National Science Foundation.

REFERENCES

Beichner, Robert J. 1990. "The Effect of Simultaneous Motion Presentation and Graph Generation in a Kinematics Lab." *Journal of Research in Science Teaching* 27 (8): 803–15.

Brasell, Heather. 1987. "The Effect of Real-Time Laboratory Graphing on Learning Graphic Representations of Distance and Velocity." *Journal of Research in Science Teaching* 24 (4): 385–95.

Heid, M. Kathleen. 1988. "Resequencing Skills and Concepts in Applied Calculus Using the Computer as a Tool." *Journal for Research in Mathematics Education* 19 (1): 3–25.

Kaput, James J., Maria L. Blanton, and Luis Moreno. 2008. "Algebra from a Symbolization Point of View." In *Algebra in the Early Grades*, pp. 19–55. New York: Lawrence Erlbaum Associates, National Council of Teachers of Mathematics.

Lapp, Douglas A. 1997. "A Theoretical Model for Student Perception of Technological Authority." In *Third International Conference on Technology in Mathematics Teaching*, vol. 29. Koblenz, Germany.

Lapp, Douglas A., and Vivian Flora Cyrus. 2000. "Using Data-Collection Devices to Enhance Students' Understanding." *Mathematics Teacher* 93 (6): 504–10.

Lapp, Douglas A., Melvin A. Nyman, and John S. Berry. 2010. "Student Connections of Linear Algebra Concepts: An Analysis of Concept Maps." *International Journal of Mathematical Education in Science and Technology* 41 (1): 1–18.

Sfard, Anna. 1991. "On the Dual Nature of Mathematical Conceptions: Reflections on Processes and Objects as Different Sides of the Same Coin." *Educational Studies in Mathematics* 22 (1): 1–36.

———. 2008. *Thinking as Communicating: Human Development, the Growth of Discourses, and Mathematizing*. Cambridge: Cambridge University Press.

Tall, David, and Shlomo Vinner. 1981. "Concept Image and Concept Definition in Mathematics with Particular Reference to Limits and Continuity." *Educational Studies in Mathematics* 12 (2): 151–69.




DOUGLAS A. LAPP, lapp1da@cmich.edu, is professor of mathematics and mathematics education at Central Michigan University (CMU) in Mt. Pleasant. He is interested in the use of technology for the development of algebraic thinking and symbolic meaning. **MARIE ERMETE,** ermet1mn@gmail.com, a senior at CMU, plans to teach high school mathematics. She is interested in the role of dynamic representations in students' development of mathematical concepts and in mathematical fiber works. **NATASHA BRACKETT,** brack2nd@cmich.edu, also a senior at CMU, plans to teach middle school mathematics. She is interested in symbol development in middle school students. **KARLI POWELL,** kpowell@lindenschools.org, teaches at Linden High School in Linden, Michigan. She continues to integrate technology into her classroom to help students form connections across different representations and create a student-centered learning environment. Ermete, Brackett, and Powell were participants in the NSF-funded Long-term Undergraduate Research Experience (LURE) program at CMU.

more4U

To view and interact with some of the student investigations, download one of the free apps for your smartphone and then scan this tag to access **www.nctm.org/mt044**.