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Multiple Representations for Pattern Exploration with the Graphing Calculator and Manipulatives

To teach mathematics as a connected system of concepts, we must have a "shift in emphasis from a curriculum dominated by memorization of isolated facts and procedures and by proficiency with paper-and-pencil skills to one that emphasizes conceptual understandings, multiple representations and connections, mathematical modeling, and mathematical problem solving," according to the NCTM's *Curriculum and Evaluation Standards for School Mathematics* (1989, 125). Too often, educators spend time teaching skills. As a result, little time remains to concentrate on concepts that are essential to understanding mathematics. Students should not be forced to play with symbols on a piece of paper; rather, they should be allowed to play with ideas that lead to conceptual understanding. Manipulatives and technology encourage discovery. Later, some of the desirable skills can be addressed in a remarkably shorter time frame than traditional instruction usually requires (Heid 1992).

Many times, as a first introduction to proof by induction, students are asked to prove

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

Why would students feel compelled to prove this summation? It likely holds little meaning for them. To motivate students, a teacher could discuss the following problem, used by Thompson (1985):

> Ten blocks are needed to make a staircase of four steps [as shown in **fig. 1**]. How many blocks are needed to make ten steps? How many blocks are needed to make fifty steps?

Pólya (1957) suggests that a first step in solving a problem is to be certain that the problem is understood. Manipulatives make a great starting point from which students can examine the process used in figuring the number of blocks needed to build a staircase of a certain size. The act of placing blocks or color tiles to build the figure can be of great benefit in understanding a process by which the number of blocks can be determined. For example, in building a staircase of height 3, the student obtains a figure containing six blocks (see **fig. 2a**). To then continue and build a staircase of height 4, all that he or she needs is a row of four blocks to place on the bottom (see **fig. 2b**). The act of placing the blocks helps the student understand the process.

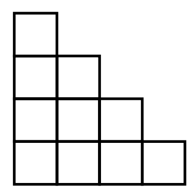

Fig. 1
Four-step staircase

Why would students feel compelled to prove this summation?

Once the pattern of $1 + 2 + 3 + \cdots + n$ as an expression for the number of blocks in a staircase of height n is recognized, the "brute force" process can be applied to calculate the number of blocks needed for a staircase of height, say, 1000. Of course, this approach is not necessarily a desirable method for finding a solution.

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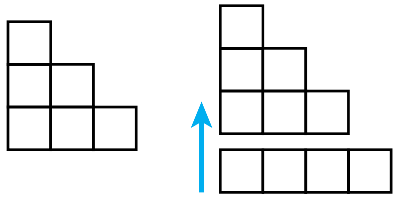


(a) (b)

Fig. 2
Extending the three-step staircase to four steps

The important aspect of the investigation is merging representa-tions

At this point, technology, such as the TI-83 graphing calculator, can be used to experiment with different limits of the summation (see **fig. 3a–e**). However, the machine registers an error for a fairly small upper index of 1000 (see **fig. 3e**). Even a powerful machine like the TI-92 takes quite long to evaluate

$$\sum_{i=1}^{100\,000} i.$$

This slowness gives the teacher the opportunity to pose the problem of how to find a nice formula for the sum of the first n positive integers, thus eliminating the long wait for the answer.

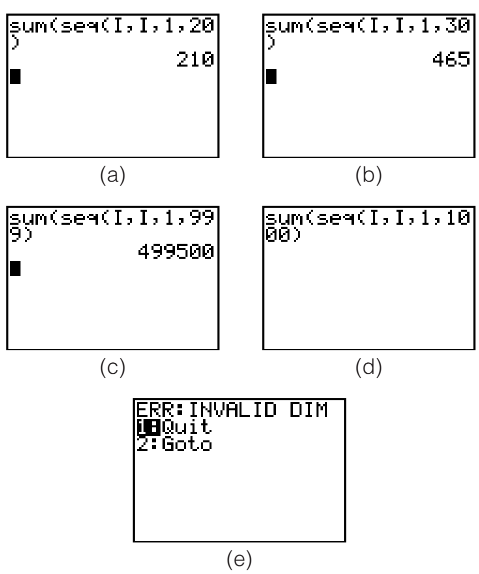
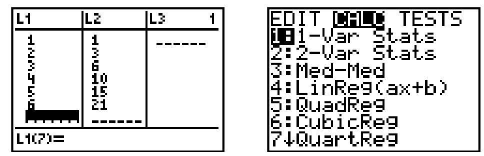


Fig. 3
Using the TI-83 to find the number of blocks in a staircase of height 1000

Teachers can motivate students to discover many different representational forms for solving problems. Lapp (1995) suggests that confirming an answer across several representational forms yields better acceptance in the mind of the student. Manipulatives can help students discover the process that leads to an answer. However, in looking for mathematical relationships, students often find a graphical approach useful. From a more graphical approach, the curve-fitting capabilities of graphing calculators—such as the TI-83, the TI-82, or even the TI-80—can supply another mode of exploration. For example, consider the list of data in **figure 4a,** where **L1** represents the height of the staircase and **L2** represents the number of blocks needed. The typical graphing calculator contains several regression options from which to choose for curve fitting (see **fig. 4b**).

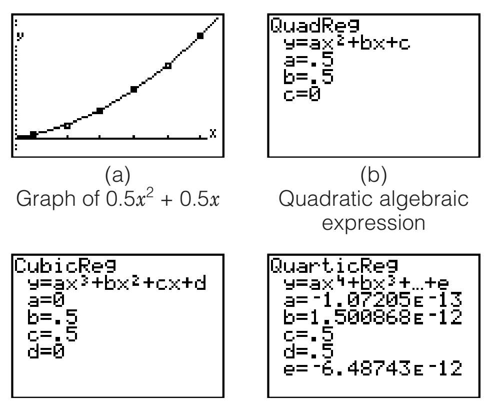


(a) Data table

(b) Regression options

Fig. 4

For example, the quadratic regression command yields the expression $0.5x^2 + 0.5x$ for the general staircase of height x. The graph **(fig. 5a)** and the algebraic expression **(fig. 5b)** of this function are given as displayed on the TI-83. Further investigation with a cubic **(fig. 5c)** and a quartic **(fig. 5d)** regression yields algebraic expressions for the functions. Note that the cubic and quartic expressions contain x^4, x^3, and constant coefficients that are either zero or essentially zero. If these terms are ignored, the machine gives the same expression for



(a) Graph of $0.5x^2 + 0.5x$

(b) Quadratic algebraic expression

(c) Cubic algebraic expression

(d) Quartic algebraic expression

Fig. 5
Regression options for staircase of height x

all instances of curve fitting, namely, $0.5x^2 + 0.5x$, that is,

$$\frac{x(x+1)}{2}.$$

Once an apparent pattern emerges, the students can be asked to prove that it will always hold. Not all students will be ready to give a formal proof by mathematical induction or some other means. At the early stages, a convincing geometric argument alone may suffice. Letting the students revisit the manipulatives is often helpful in getting them to construct a reasonable argument. Consider a specific case in which the staircase has height 4. By constructing two such staircases **(fig. 6a)** and fitting them together to form a rectangle **(fig. 6b),** we can argue that the dimensions in the nth case will always be n on one side and $n + 1$ on the other side, which gives the total number of blocks in the rectangle as $n(n + 1)$. But since only half of this rectangle is desired, we get

$$\frac{n(n+1)}{2}.$$

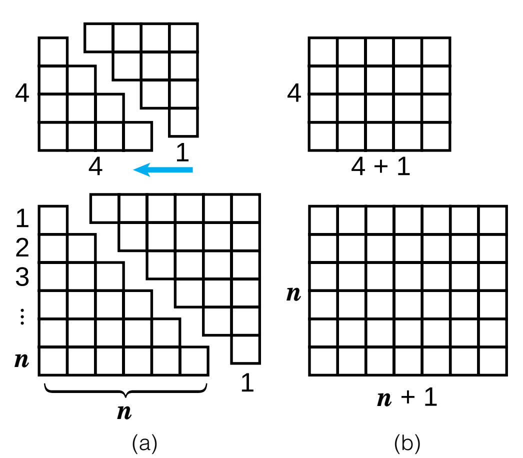


Fig. 6
Geometric argument

Another explanation frequently given by students is the "triangle approach." In looking at a staircase as a triangle with jagged edges **(fig. 7),** we can argue that the "area" of the staircase, which corresponds to the number of blocks, can be determined by the area of the triangle,

$$\frac{1}{2}n \bullet n = \frac{1}{2}n^2,$$

along with half the blocks along the diagonal that were cut off to form the triangle. Since n blocks are along the diagonal, the number of blocks needed to construct a staircase of height n would be given by

$$\frac{1}{2}n^2 + \frac{1}{2}n = \frac{n(n+1)}{2}.$$

The important aspect of this investigation is the merging of representations. The graphical representation afforded by technology aids in the search for patterns. However, the use of other representations, such as physical or iconic models, can play an important role in the construction of a logical argument that explains the observed pattern.

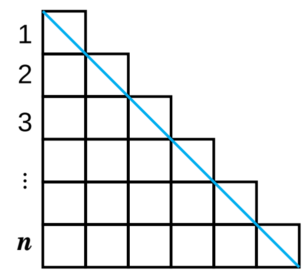


Fig. 7
Triangle approach

It is important to consider the other representational forms for solving this particular problem permitted by such new technologies as the TI-92. A representation for extending this investigation to the sum of squares uses matrices. Since the sum of the first n positive integers yields a quadratic closed-form formula, might the sum of squares yield a cubic closed-form formula? To investigate, we can sum to several upper indices of

$$\sum_{i=1}^{n} i^2$$

and generate data points of (10, 385), (15, 1240), (20, 2870), and (25, 5525). If we assume that the formula is of the form $ax^3 + bx^2 + cx + d$, we can construct a linear system in which the new "variables" are the coefficients of the general cubic.

$$\begin{aligned} a \bullet 1000 + b \bullet 100 + c \bullet 10 + d &= 385 \\ a \bullet 3375 + b \bullet 225 + c \bullet 15 + d &= 1240 \\ a \bullet 8000 + b \bullet 400 + c \bullet 20 + d &= 2870 \\ a \bullet 15625 + b \bullet 625 + c \bullet 25 + d &= 5525 \end{aligned}$$

Using a matrix representation for the system, we get

$$\begin{bmatrix} 1000 & 100 & 10 & 1 \\ 3375 & 225 & 15 & 1 \\ 8000 & 400 & 20 & 1 \\ 15625 & 625 & 25 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 385 \\ 1240 \\ 2870 \\ 5525 \end{bmatrix}.$$

We next use the matrix capabilities of the TI-92 to solve the system. Storing the coefficient matrix in A and the column matrix on the right-hand side of the equation in B, we get a solution by evaluating $A^{-1} \bullet B$ **(fig. 8).** The result suggests that the cubic we are seeking is given by

$$\frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{6}x.$$

Using the factor command **(fig. 9),** we get the form more commonly expressed as

$$\frac{x(x+1)(2x+1)}{6}.$$

The beauty of the matrix treatment of this problem is that it allows the student to extend polynomial curve fitting beyond the normal choices offered by most graphing calculators. The TI-92 and TI-83 calculators have “canned” polynomial curve-fitting capabilities up through a general quartic equation. The use of matrices allows students to try higher degree polynomials provided that the number of data points is sufficient to produce a nonsingular coefficient matrix. Even though other calculators, such as the TI-83, also allow the use of matrices, the advantage of the TI-92 is that it keeps solutions in rational form rather than as the decimal approximations used by less powerful machines.

Discovering a formula modeled through an investigative method gives the student a compelling reason to seek a proof confirming the formula. As with the previous example of

$$\sum_{i=1}^{n} i,$$

we wish to motivate students to produce a proof by manipulatives of our discovered formula,

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

One method, proposed by Siu (1984), is given in **figure 10.** This approach is more complicated than the physical proof offered previously for

$$\sum_{i=1}^{n} i.$$

As a result, students may have a more difficult time “discovering” it on their own. Some prompting by the teacher may be necessary. However, the proof given by Siu does use strategies found in both the “triangle” approach and the “two-staircase” approach presented previously for

$$\sum_{i=1}^{n} i.$$

Students do not often have an opportunity to experience what many mathematicians do regularly, that is, look for concepts on the basis of patterns, and this approach will allow them to see the mathematical process firsthand. This investigation introduces the students to the process of conjecture

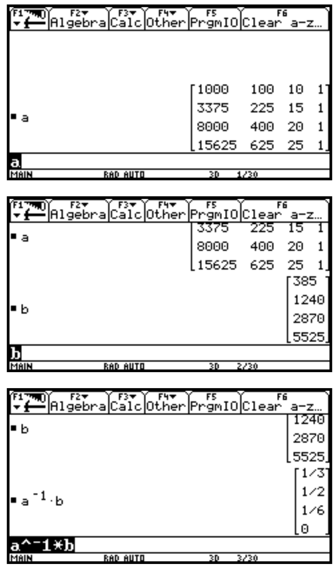


Fig. 8
Using the matrix capabilities of the TI-92

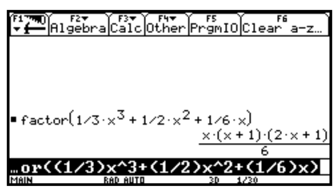


Fig. 9
Using the factor command on the TI-92

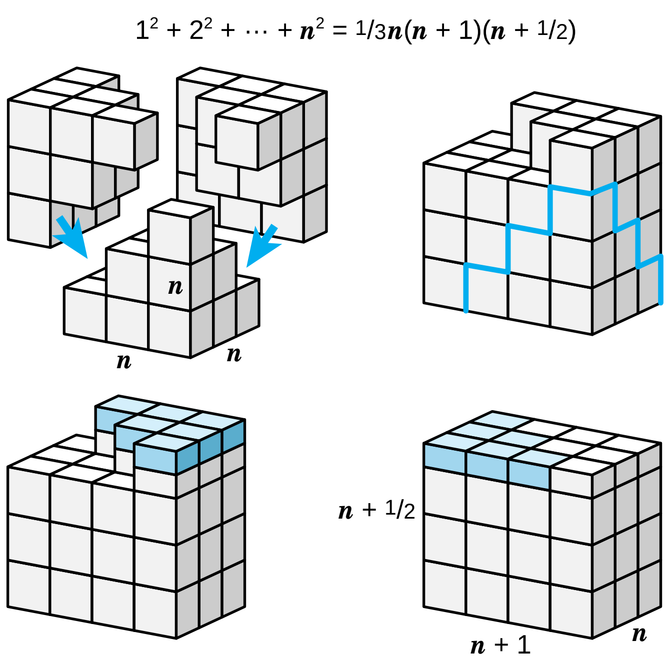


Fig. 10
Proof without words (Siu 1984)

followed by proof, and the combination of manipulatives and technology allows this process to be approached earlier than traditionally thought appropriate.

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