Linked Representations in Algebra: Developing Symbolic Meaning

The use of symbols provides mathematics with enormous power. From the learner’s perspective, however, symbols can present enormous obstacles. Using symbols to encapsulate big ideas allows readers to see various stages of arguments within one field of view. However, for many learners, unpacking symbolization poses a significant hurdle in the development of conceptual understanding. Here we compare and contrast symbolic reasoning approaches that algebra students used when solving equations.

What is a root of an equation, and how is it related to various representations? More generally, what does it mean for a value to be a solution to an equation? These are standard questions that we expect students to be able to address and discuss. Although many students may be able to solve equations, far too many have limited conceptual understanding and rely primarily on procedural knowledge of the equation-solving process.

**AARON’S SOLUTION TO A QUADRATIC EQUATION**

Consider the following situation in which a student, Aaron, tried to solve the quadratic equation below during an interview:

\[(x - 2)(x + 3) = 6\] (no graph given)

Aaron began by setting each factor equal to 6 (see fig. 1). Here he applied a memorized procedure that he overgeneralized from previous experiences of solving quadratic equations by setting each factor equal to zero. After finding solutions, Aaron said, “I don’t know if those approaches are right . . . I’d have to know exactly where we are at,” indicating that he was unsure of his process. When questioned further, he stated that his process might change depending on “if we’re trying to find where it’s going to cross on the x-axis.” In elaborating, Aaron stated that if he were trying to find points on a graph, then he would view the process differently, and he referred to another problem that included a graph along with an equation (see fig. 2). However, when he described how his process would change, he repeated the same algebraic steps and changed only the number that he would set each factor equal to (4 instead of 6) (see fig. 3).

As Aaron continued to solve the new problem, he noticed an inconsistency:
connections; students need to see how these is not enough to help students make con-
tions during class, so it is not surprising
multiple representations in solving equa-
tions that Aaron would seek confirmation
he did attempt to use multiple repre-
error in his solution process even though
At first, it appeared that Aaron had
catch his conceptual mistake, but he
had simply replaced an algebraic rule
with a graphical one (looking for the
points where the graph crosses the
x-axis). Aaron continued to try to resolve
his algebraic solutions with his expecta-
tions that these solutions should corre-
spond to the x-intercepts from the graph,
but he was unsuccessful and gave up.
Why was Aaron unable to see the
error in his solution process even though
he did attempt to use multiple repre-
sentations when a graph was provided?
The teacher had made it a point to use multiple representations in solving equations during class, so it is not surprising that Aaron would seek confirmation when the graph was present. However, the use of multiple representations alone is not enough to help students make connections; students need to see how these representations connect.

OUR STUDY WITH COLLEGE ALGEBRA STUDENTS
Our proposal for curricular enhance-
ment is based on a study, conducted
over two semesters with college algebra
students, that examined how students
developed algebraic concepts using vari-
ous types of technology. During the fall
semester, fifteen students participated in
the study; four of them also took part in
two interview sessions conducted dur-
ing the last three weeks of the course.
During the spring semester, thirty-four
students participated in the study; seven
of them took part in similar interview
sessions. Data were collected during

Fig. 1 Aaron shows his algebraic approach for solving \((x - 2)(x + 3) = 6\), setting each factor equal to 6.

Aaron: Uh-oh, something’s not right . . . the solutions [don’t] match . . . [pressed to explain] [I] got 5 and 2, and I would expect them to match –2 and 1 [pointing to the x-intercepts on the graph]. Um, just because that’s where they cross.

For the development of symbol systems. In the research with MBL, we have seen that real-time changes in linked representations that are simultaneously visible help students make connections among corresponding features of the various representations (Lapp and Cyrus 2000; Beichner 1990; Brasell 1987).

Linking representations deepens understanding of mathematical concepts, but we still must ask how the use of symbol systems influences our learning and, further, our development of new ideas. Kaput, Blanton, and Moreno (2008) propose a model that describes how we develop our understanding of existing ideas as well as how we create new concepts. The key to this model is the communication and analysis between two worlds. One is the “real world” (i.e., the world of broader mathematical or physical experiences), and the other is the world of the symbols that we use to represent these real-world experiences. In this model, the learner initially creates raw representations from experiences; in turn, these representations and understanding of the real world evolve as the student interacts with both worlds. The evolution of the student’s view of the real world is relative to the symbolic worlds of \(B\) and \(C\), which are represented by \(A_B\) and \(A_C\) (see figs. 4a and 4b). In this sense, Kaput, Blanton, and Moreno’s model is consistent with Tall and Vinner’s (1981) view of an evolving “concept image.”

This process, although focusing on representations, is consistent with Sfard’s (1991) process of interiorization, condensation, and reification, in which concepts become internalized through interaction with the various natures of the mathematical ideas. In some ways, the use of technology here reduces the length of time for Sfard’s interiorization and facilitates the condensation process. Alleviating the computational load can

Fig. 2 Given the graph of the function as shown, students were asked to find all solutions to \((x - 1)(x + 2) = 4\) and justify their answer.

Fig. 3 Aaron’s algebraic approach to solving \((x - 2)(x + 3) = 6\) mirrored his algebraic approach to solving \((x - 2)(x + 3) = 6\).
also keep the lack of procedural fluency from becoming a cognitive obstacle to reaching condensation or reification, as observed by Lapp, Nyman, and Berry (2010).

Although Aaron had access to the use of multiple representations and the teacher routinely used graphical, numerical, and algebraic representations, they were not dynamically linked. Thus, Aaron may have had more difficulty making a connection between the meaning of the graphical representation of x-intercepts and the zero property of multiplication as a technique for equation solving. It is also important to note that Aaron’s approach was typical of all students interviewed during the first semester of the study.

**JON’S APPROACH**

How do we create experiences that allow students to move fluidly among various representations while maintaining an isomorphic mapping of meaning between various worlds used to represent the ideas involved? One way is to use technology that allows for linked representations.

In contrast to the students interviewed in the fall semester, who used only the basic graphing technology, all the students interviewed in the spring semester showed an ability to move among representations as a means of justifying their reasoning.

To illustrate, we examine Jon’s response to the same question posed to Aaron—solving the equation \((x - 2)(x + 3) = 6\).

Unlike students from the previous semester, Jon realized that he must get the equation set equal to zero to make the comparison to zeros on the graph of a function. In the following exchange, Jon referred to his solutions to other problems (problems 6 and 7; see fig. 5).

**Interviewer:** So what do you think is different about this? Why is it important that you get the 6 to the other side?

**Jon:** Because I think if you were to graph it, you’re trying to find a completely different line, if it’s a 6 or 0 and if it’s 0 . . . You’re trying to find y. And then if y was 0 . . . I don’t know.

**Interviewer:** So kind of like the also keep the lack of procedural fluency from becoming a cognitive obstacle to reaching condensation or reification, as observed by Lapp, Nyman, and Berry (2010).

**Interviewer:** So you’re thinking of it from a graphical standpoint?

**Jon:** Yeah, I’m thinking about it from a graph. If it’s equal to 6, it’s going to be up higher and it’s gonna be . . . You’re not looking for the zeros [points to problems 6 and 7 from the earlier questions].

**Interviewer:** So kind of like the
difference in those earlier problems?

Jon: Basically, I took number 7 and tried to turn it more into, like, number 6. Like, I tried . . . I just moved it down, so that way I would be able to solve for it. I visualized it as more like bringing it down, and I just solved for the vertex, which I know how to do. When it comes to that, it’s more complicated to get to that point.

Interviewer: Oh, the -intercepts you mean, as opposed to the vertex?

Jon: Yeah, ’cause when it’s at equals 6, it’s like the whole thing got raised 6. So I just made it . . . I brought it down to the vertex . . . and then I just solved for it there because that’s easier for me to do algebraically.

Jon had first been exposed to dynamically linked representations through technology-rich investigations in class. In one investigation, manipulating a graph gave real-time change in the algebraic representation of the quadratic function as well as the factored form of the function. Jon was asked to label the -intercepts on the graph, and, as he manipulated the graph, he noticed a connection between the numerical values of the -coordinates of the -intercepts and the numerical values, and , found in the factored form of the function (see fig. 6).

In the past, we have seen that students tend to focus on aspects of a situation that are invariant across representations (Lapp 1997). In this case, the student noticed that, no matter how the graph was manipulated, the numerical values of the -coordinates in both the labeled -intercepts and the , values in the factored form of the function remained the same. The classroom investigation also included questions designed to focus the student’s attention on the effects on the function’s output for entering each -coordinate of the zeros into the factored form of the function as well as the values of each factor. Our research suggests that this combination—communication and analysis between symbolic worlds—enabled the student to articulate the reason for the use of factoring as a technique for equation solving. For this reason, it is imperative that the equation is set equal to zero before factoring.

Students in both semesters were exposed to the zero property for multiplication as a reason for solution by factoring, but students in the second semester, using CAS, were allowed to interact with linked representations and

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Fig. 6 These are still images from the dynamic experience of moving a graph and seeing the factored form of the function change in real time. Manipulating the graph enabled Jon to see the relationships between roots and -intercepts.
were required to justify their reasoning in a lab experience. Students in the first semester were told by the teacher about this reason but simply watched the teacher point out the fact on a single graphical representation that was not linked algebraically.

In another investigation, Completing the Square (see fig. 7), students developed a connection between algebraic and graphical representations for the process of completing the square and observed relationships among various parameters found in the algebraic expressions. Here students further experienced the connection between algebraic and graphical transformations.

This experience likely led to Jon’s strategy of transforming the graph of a parabola and its intersection with a horizontal line above the \(x\)-axis into one in which the parabola was shifted down until the horizontal line was superimposed on the \(x\)-axis. In his desire to transform an equation by subtracting 6 from both sides, Jon expressed a graphical understanding linked to the algebraic transformation of subtracting 6. He stated that he was essentially moving the dotted line (see fig. 5) down to coincide with the \(x\)-axis so that he could use his technique of factoring, which required the equation to be set equal to zero. In this instance, he was able to justify his process and not just execute it procedurally.

The influence of the Completing the Square investigation can also be seen in Jon’s reference to the movement of the parabola’s vertex during his verbal description of his graphical understanding. In this investigation, students were specifically asked to follow the movement of the vertex.

Jon clearly demonstrated an understanding of the concept of root and its connection to factoring as an equation-solving technique. He explained why he could not simply set each factor equal to zero in the equation \((x - 2)(x + 3) = 6:

\[\text{Jon: If I plug 2 into the first one, 2 minus 2... that would give me 0, and 0 times anything would equal 0, not 6, so that's wrong. Then if I plug -3 in, -3 minus 2 would be -1, and then the second one}\]

Fig. 7 Students use sliders to manipulate a parabola given in vertex form and see its equivalent expression in standard form. In this way, they were able to make more connections when the quadratic was expressed in vertex form.
the second factor] would still be 0, so –1 doesn’t equal 6 either [meaning entering –1 into the equation], so that would be wrong too.

Jon could articulate an understanding of the root-solving process using the zero property of multiplication along with his verbal reference to the vertex movement. This fact indicates an influence of both these investigations on his mathematical understanding and its relationship to the symbolic world that describes these ideas.

**DYNAMIC LINKING IS THE KEY**

From these contrasting examples, we see that, as we teach algebra, conceptual understanding can go hand-in-hand with procedural ability. Our research challenges the conventional wisdom that students must first become procedurally fluent before they can understand the concepts that we teach. Heid (1988) challenged this position more than two decades ago, arguing that we should rethink the sequencing of skills and concepts in calculus by using computer algebra systems to develop concepts before teaching procedures.

Here we see the same principle applied to high school algebra concepts. The difference between our study and Heid’s is that we suggest that it is not just the computer algebra system that influences how students see connections but rather the dynamic linking of various representations. Students in the first semester of this study used the TI-84 and were introduced to concepts before skills; however, they did not have the ability to manipulate various representations and see real-time effects among representations.

A second aspect of concept development that we noticed involves student control of the environment. As Lapp and Cyrus (2000) suggest from research on the use of data collection devices, the student’s ability to manipulate the environment plays a significant role in making connections. During the first semester of this study, the teacher merely demonstrated some of the dynamically linked representations using a computer during class; the students did not have use of this technology individually. Results of our interviews showed that none of these students could articulate connections among various representations. However, during the second semester, each student had a TI-Nspire CAS device and used it during student-centered investigations. This ability to communicate and analyze through dynamically connected representations between the symbolic world and the world of mathematical ideas allowed students to make connections between concepts and procedures.

As technology that links representations has become readily available, there is no reason we should not take advantage of it to better develop students’ mathematical understanding. However, technology alone cannot make these connections for students. Kaput, Blanton, and Moreno (2008) as well as Sfard (2008) suggest that students’ negotiation of discourse between the symbolic world and the world of mathematical ideas plays a key role in the development of symbolic

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meaning. Therefore, as teachers, we need to use appropriate technology to engage students in investigations that allow them to make mathematically meaningful observations and justify them.

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REFERENCES

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