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EXAMPLES OF ANOSOV LIE ALGEBRAS

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ABSTRACT. We construct new families of examples of (real) Anosov Lie algebras, starting with algebraic units. We also give examples of indecomposable Anosov Lie algebras (not a direct sum of proper Lie ideals) of dimension 13 and 16, and we conclude that for every $n \geq 6$ with $n \neq 7$ there exists an indecomposable Anosov Lie algebra of dimension n.

1. Introduction.

A diffeomorphism f of a compact differentiable manifold M is called *Anosov* if it has a global hyperbolic behavior; i.e., the tangent bundle $\mathrm{T}M$ admits a continuous invariant splitting $\mathrm{T}M = E^+ \oplus E^-$ such that $\mathrm{d}f$ expands E^+ and contracts E^- exponentially. This kind of diffeomorphism plays an important and beautiful role in dynamics because they give examples of dynamical systems with very nice properties, and it is then a natural problem to understand which are the manifolds supporting them (see [16]).

Up to now, the only known examples are hyperbolic automorphisms on infranilmanifolds (manifolds finitely covered by nilmanifolds), which are called *Anosov automorphisms*, and all their topological conjugates. Moreover, it is conjectured that any Anosov diffeomorphism is topologically conjugate to an Anosov automorphism of a infranilmanifold (see [15]). The conjecture is known to be true in many particular cases; for example, J. Franks [6] and A. Manning [12] proved it for Anosov diffeomorphisms on infranilmanifolds themselves.

We will say that an n-dimensional rational Lie algebra is *Anosov* if it admits a hyperbolic automorphism τ (i.e., none of the eigenvalues of τ are of modulus 1) such that $[\tau]_\beta \in \mathrm{GL}_n(\mathbb{Z})$ for some basis β of $\mathfrak{n}$, where $[\tau]_\beta$ denotes the matrix of τ with respect to β. We say that a real Lie algebra is *Anosov* if it admits a rational form which is Anosov. It is easy to observe that a real Lie algebra $\mathfrak{n}$ is Anosov if it admits a hyperbolic automorphism τ such that $[\tau]_\beta \in \mathrm{GL}_n(\mathbb{Z})$ for some $\mathbb{Z}$-basis β of $\mathfrak{n}$ (i.e., with integer structure constants), because one can always get a $\mathbb{Z}$-basis by scaling a $\mathbb{Q}$-basis. Conversely, if a real Lie algebra $\mathfrak{n}$ admits a hyperbolic automorphism τ

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such that $[\tau]_\beta \in GL_n(\mathbb{Z})$ for some $\mathbb{Z}$-basis β of $\mathfrak{n}$, it is Anosov since the $\mathbb{Q}$-subspace generated by β is an Anosov rational form of $\mathfrak{n}$.

It is well known that any Anosov Lie algebra is necessarily nilpotent, and it is easy to see that the classification of nilmanifolds, which admit an Anosov automorphism, is essentially equivalent to that of Anosov Lie algebras (see [9, 2, 7, 4]).

Therefore, if one is interested in finding those Lie groups, which are simply connected covers of an Anosov infranilmanifold, then the objects to find are real nilpotent Lie algebras $\mathfrak{n}$ supporting an Anosov automorphism.

Concerning the known examples besides the case of free nilpotent Lie algebras (see [2]), there were only sporadic examples of Anosov Lie algebras before [9], where it is proved that $\tilde{\mathfrak{n}} = \mathfrak{n} \oplus \cdots \oplus \mathfrak{n}$ (m times, $m \geq 2$) is a real Anosov Lie algebra for any graded Lie algebra $\mathfrak{n}$ admitting a rational form. Also, in [3] other kinds of examples are given in the context of certain two-step nilpotent Lie algebras attached to graphs. In this way, there are in the literature examples of nonabelian Anosov real Lie algebras for each dimension $n \geq 6$, with the exception of 7 and 13. Moreover, in [3] the existence of an indecomposable n-dimensional 2-step Anosov Lie algebra is proved for $n \geq 6$, except for $n = 7, 9, 12, 13, 16$. We recall that a Lie algebra is said to be *indecomposable* if it cannot be expressed as a direct sum of proper Lie ideals. It is known that there is no 7-dimensional Anosov Lie Algebra [10], and for $n = 9, 12$ there exists an indecomposable Anosov Lie algebra of dimension n (see [9]). In fact, [9] gives a family of indecomposable $3r$-dimensional Anosov Lie algebras, $r \geq 2$.

In this paper we will give explicit families of examples of Anosov (real) Lie algebras to illustrate a general procedure to construct Anosov Lie algebras, and as an application we will give an indecomposable 13-dimensional Anosov Lie algebra. In fact, for each pair of algebraic integers λ, μ of degree p and q respectively which satisfy the following conditions

1. they are units,
2. if we denote by $\{\lambda = \lambda_1, \ldots, \lambda_p\}$ and $\{\mu = \mu_1, \ldots, \mu_q\}$ the conjugates to λ and μ respectively, then $|\lambda_i| \neq 1 \neq |\mu_j|$, and
3. $|\lambda_i \mu_j| \neq 1$,

we will exhibit a type $(pq+p, q)$ Anosov Lie algebra. This first construction is quite easy to extend, and we are able to show examples of 3-step (and in fact of k-step) Anosov Lie algebras; also, in the special case of $p = 2$ we give another example of type $(3q, q+2)$ for any q.

Finally, we also give an example of an indecomposable 16-dimensional Anosov Lie algebra, which allows us to conclude that for $n \geq 6$ ($n \neq 7$), there exists an indecomposable n-dimensional Anosov Lie algebra.

2. **Examples.** Given a nilpotent Lie algebra $\mathfrak{n}$, we call the *type* of $\mathfrak{n}$ to the r-tuple $(n_1, ..., n_r)$, where $n_i = \dim C^{i-1}(\mathfrak{n})/C^i(\mathfrak{n})$ and $C^i(\mathfrak{n})$ is the central descending series. We also consider a decomposition $\mathfrak{n} = \mathfrak{n}_1 \oplus \cdots \mathfrak{n}_r$, a direct sum of vector spaces, such that $C^i(\mathfrak{n}) = \mathfrak{n}_{i+1} \oplus \cdots \oplus \mathfrak{n}_r$ for all i. It is proven in [10] that if $\mathfrak{n}$ is a real Anosov Lie algebra of type $(n_1, ..., n_r)$, then there exists a hyperbolic $A \in \operatorname{Aut}(\mathfrak{n})$ such that

(i) $A\mathfrak{n}_i = \mathfrak{n}_i$ for all $i = 1, ..., r$,
(ii) A is semisimple (in particular, A is diagonalizable over $\mathbb{C}$), and
(iii) For each i, there exists a basis β_i of $\mathfrak{n}_i$ such that $[A_i]_{\beta_i} \in SL_{n_i}(\mathbb{Z})$, where $n_i = \dim \mathfrak{n}_i$ and $A_i = A|_{\mathfrak{n}_i}$.

It is important to mention that the existence of an Anosov automorphism is a really strong condition on an infranilmanifold and also in a Lie algebra, and therefore our approach is to start with a hyperbolic automorphism.

In this context, to show an example of an Anosov Lie algebra, we are going to construct a complex Lie algebra (to be able to work with eigenvalues) in such a way that it admits a hyperbolic automorphism A such that $[A]_\beta \in GL_n(\mathbb{Z})$ for some $\mathbb{Z}$-basis β of $\mathfrak{n}$.

We begin by noting that if λ and μ are algebraic units of degree p and q respectively, and we denote by $\{\lambda = \lambda_1, \ldots, \lambda_p\}$ and $\{\mu = \mu_1, \ldots, \mu_q\}$ the sets of conjugates of λ and μ over $\mathbb{Q}$ respectively, it is not hard to see that $\{\lambda_i\mu_j\}$ are also algebraic units and, moreover, the matrix $\begin{bmatrix} \lambda_1\mu_1 & & \\ & \ddots & \\ & & \lambda_p\mu_q \end{bmatrix}$ is conjugated to a matrix in $GL_{pq}(\mathbb{Z})$ with determinant ± 1.

Bearing this in mind, for each pair of non negative integers $p \neq q$, we take the Lie algebra $\mathfrak{n}$ with basis $\beta = \{X_1, \ldots, X_{pq}, Y_1, \ldots, Y_p, Z_1, \ldots, Z_q\}$, and the non-zero Lie brackets amongst these vectors are given by:

$$[X_{ip+j}, Y_j] = Z_{i+1} \qquad 0 \leq i < q,\ 1 \leq j \leq p. \tag{1}$$

It is clear that $\mathfrak{n}$ is a two-step nilpotent Lie algebra. We take $\{X_i, Y_j : 1 \leq i \leq pq,\ 1 \leq j \leq p\}$ as a basis of $\mathfrak{n}_1$ and $\{Z_k : 1 \leq k \leq q\}$ as a basis of $\mathfrak{n}_2$. Now, let A be an automorphism such that $[A]_\beta = \begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix}$, where

$$A_1 = \begin{bmatrix} \lambda_1\mu_1 & & & & & & & \\ & \ddots & & & & & & \\ & & \lambda_p\mu_1 & & & & & \\ & & & \lambda_1\mu_2 & & & & \\ & & & & \ddots & & & \\ & & & & & \lambda_p\mu_q & & \\ & & & & & & \lambda_1^{-1} & \\ & & & & & & & \ddots \\ & & & & & & & & \lambda_p^{-1} \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_q \end{bmatrix}.$$

We note that β is a basis of eigenvectors of A. Also, if we take λ and μ as above and such that $|\lambda_i| \neq 1, |\mu_j| \neq 1$ and $|\lambda_i\mu_j| \neq 1$ for all i, j, then A is a hyperbolic automorphism.

In what follows, we are going to show that $\mathfrak{n}$ is an Anosov Lie algebra by constructing a $\mathbb{Z}$-basis of $\mathfrak{n}$ preserved by A. In order to make the calculation more clear we will make a small change in the notation. Let $X_{(i,j)}$ be the eigenvector of A corresponding to the eigenvalue $\lambda_i\mu_j$. Note that this is only a reordering of the $\{X_i\}$. In fact, $X_{(i,j)} = X_{(j-1)p+i}$, and therefore we may say that $\beta = \{X_{(i,j)}, Y_k, Z_l : 1 \leq i, k \leq p,\ 1 \leq j, l \leq q\}$ and (1) is now given by

$$[X_{(i,j)}, Y_i] = Z_j. \tag{2}$$

Let $\beta' = \{\mathcal{X}_{(k,l)}, \mathcal{Y}_r, \mathcal{Z}_s : 0 \le r, k < p,\ 0 \le s, l < q\}$ be the new basis of $\mathfrak{n}$ given by

$$\mathcal{X}_{(k,l)} = \sum_{i=1}^{p}\sum_{j=1}^{q}\lambda_i^k\mu_j^l\, X_{(i,j)} \quad 0 \le k < p,\ 0 \le l < q,$$

$$\mathcal{Y}_r = \sum_{k=1}^{p}\lambda_k^{-r} Y_k \qquad 0 \le r < p,$$

$$\mathcal{Z}_s = \sum_{l=1}^{q}\mu_l^s Z_l \qquad 0 \le s < q.$$

To see that this is actually a basis of $\mathfrak{n}$, it is enough to check that the sets $\{\mathcal{X}_{(k,l)}\}$, $\{\mathcal{Y}_r\}$ and $\{\mathcal{Z}_s\}$ are linearly independent over $\mathbb{C}$. Since all the calculations are similar, we are only going to show how to proceed with $\{\mathcal{X}_{(k,l)}\}$. Suppose $a_{kl} \in \mathbb{C}$ such that

$$\begin{aligned} 0 &= \sum_{k=0}^{p-1}\sum_{l=0}^{q-1} a_{kl}\mathcal{X}_{(k,l)} \\ &= \sum_{k=0}^{p-1}\sum_{l=0}^{q-1} a_{kl}\left(\sum_{i=1}^{p}\sum_{j=1}^{q}\lambda_i^k\mu_j^l\, X_{(i,j)}\right) \\ &= \sum_{i=1}^{p}\sum_{j=1}^{q}\left(\sum_{k=0}^{p-1}\sum_{l=0}^{q-1}a_{kl}\lambda_i^k\mu_j^l\right) X_{(i,j)}. \end{aligned}$$

Hence, for $1 \le i \le p,\ 1 \le j \le q$ we have that

$$0 = \sum_{k=0}^{p-1}\left(\sum_{l=0}^{q-1}a_{kl}\mu_j^l\right)\lambda_i^k.$$

This can be seen, for each $1 \le j \le q$ fixed, as a polynomial in λ_i. This polynomial has degree at most $p-1$, and it vanishes on each one of the λ_i, so by our choice of λ it has p different roots and therefore is identically zero. Hence for $1 \le j \le q$ we have that its coefficients are zero. That is, for each $0 \le k < p$

$$0 = \sum_{l=0}^{q-1}a_{kl}\mu_j^l,$$

which is again a polynomial in μ_j of degree at most $q-1$ with q different roots, and therefore we can conclude that $a_{kl} = 0$ for all k, l, as we wanted to show.

If $x^p + a_{p-1}x^{p-1} + \cdots + a_0$ and $x^p + b_{q-1}x^{q-1} + \cdots + b_0$ are the minimal polynomials of λ^{-1} and μ respectively, it is not hard to check that

$$A\mathcal{Y}_r = \begin{cases} \mathcal{Y}_{r+1} & r < p-1 \\ -\sum_{j=0}^{p-1}a_j\mathcal{Y}_j & r = p-1 \end{cases}, \qquad A\mathcal{Z}_s = \begin{cases} \mathcal{Z}_{s+1} & s < q-1 \\ -\sum_{l=0}^{q-1}b_l\mathcal{Z}_l & s = q-1 \end{cases}.$$

Note that a_i and b_j are all integers.

Concerning $\mathcal{X}_{(k,l)}$, by the definition we have that for each i, j,

$$A(\lambda_i^k\mu_j^l\, X_{(i,j)}) = \lambda_i^{k+1}\mu_j^{l+1}\, X_{(i,j)},$$

and therefore, for $k < p-1$ and $l < q-1$, $A\mathcal{X}_{(k,l)} = \mathcal{X}_{(k+1,l+1)}$. In the same line of the calculation done above, we have that

$$A\mathcal{X}_{(k,l)} = \begin{cases} -\sum_{k=1}^{p-1}c_k\,\mathcal{X}_{(k,l+1)} & k = p-1,\ l < q-1 \\ -\sum_{l=1}^{q-1}b_l\,\mathcal{X}_{(k+1,l)} & k < p-1,\ l = q-1 \end{cases},$$

$$A\mathcal{X}_{(p-1,q-1)} = -\sum_{k=1}^{p-1}\sum_{l=1}^{q-1}c_kb_l\,\mathcal{X}_{(k,l)},$$

where $c_j \in \mathbb{Z}$ are the coefficients of the minimal polynomial of λ.

On the other hand, to see that the Lie bracket of any two elements of β' is a linear combination of elements of β' with integer coefficients, it is enough to check it for $[\mathcal{X}_{(k,l)}, \mathcal{Y}_r]$. Using (2), we have that

$$\begin{aligned} [\mathcal{X}_{(k,l)}, \mathcal{Y}_r] &= \sum_{i=1}^{p}\sum_{j=1}^{q}\lambda_i^{k-r}\mu_j^l\,[X_{(i,j)}, Y_i] \\ &= \left(\sum_{i=1}^{p}\lambda_i^{k-r}\right)\left(\sum_{j=1}^{q}\mu_j^l\, Z_j\right) \\ &= M(k,l)\mathcal{Z}_l. \end{aligned} \tag{3}$$

Here $M(k,l) = \operatorname{tr} A_\lambda^{k-l}$, where $A_\lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_p \end{bmatrix}$. Due to our choice of λ, A_λ is conjugated to a matrix in $GL_p(\mathbb{Z})$ and therefore so is A_λ^m for any $m \in \mathbb{N}$. Hence $M(k,l)$ is an integer number for any k, l, as we wanted to show.

Remark 2.1. Note that the Lie algebra $\mathfrak{n}$ we have constructed does not depend on the algebraic numbers λ and μ; it only depends on p and q, and moreover it is easy to see (by looking at the dimension of the center, for example) that the Lie algebra associated to (p,q) is not isomorphic to the one corresponding to (q,p) unless $p = q$. We have obtained in this way two non isomorphic Anosov Lie algebras of dimension n for all $n = p.q + p + q$, for any non negative integers $p, q \ge 2$. It is easy to check that for $p = 2 = q$, we obtain the two step nilpotent Lie algebra $\mathfrak{g}$, of type $(6,2)$ given in [10].

Concerning the existence of algebraic numbers as we need, we refer to [11, pg.1].

We would like to point out, for further use, that $\mathfrak{n}$ can be viewed as $V_0 \oplus V_1 \oplus Z$, where V_0 is the subspace generated by the $\{X_{(i,j)}\}$, V_1 is the one spanned by the $\{Y_k\}$, and Z is the center. In this setup, V_0 acts on $V_1 \oplus Z$, as it is stated in (2).

Example 2.2. As a new example, we can carry out the calculations for $p = 3,\ q = 2$ to obtain the 11-dimensional Lie algebra with basis

$$\beta = \{X_1, \ldots, X_6, Y_1, Y_2, Y_3, Z_1, Z_2\}$$

and Lie bracket among them given by

$$\begin{aligned} &[X_1,Y_1]=Z_1 \quad [X_2,Y_2]=Z_1 \quad [X_3,Y_3]=Z_1 \\ &[X_4,Y_1]=Z_2 \quad [X_5,Y_2]=Z_2 \quad [X_6,Y_3]=Z_2. \end{aligned} \tag{4}$$

The hyperbolic automorphism A is given by $[A]_\beta = \begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix}$, where

$$A_1 = \begin{bmatrix} \mu\lambda_1 &&&&&&&& \\ & \mu\lambda_2 &&&&&&& \\ && \mu\lambda_3 &&&&&& \\ &&& \mu^{-1}\lambda_1 &&&&& \\ &&&& \mu^{-1}\lambda_2 &&&& \\ &&&&& \mu^{-1}\lambda_3 &&& \\ &&&&&& \lambda_1^{-1} && \\ &&&&&&& \lambda_2^{-1} & \\ &&&&&&&& \lambda_3^{-1} \end{bmatrix}, \quad A_2 = \begin{bmatrix} \mu & \\ & \mu^{-1} \end{bmatrix}.$$

In this case we have obtained a Lie algebra of type $(9,2)$; also note that for $p=2$, $q=3$ we obtain a Lie algebra of type $(8,3)$. We would like to point out that here and in general, we can add a non zero constant to the Lie brackets in (4), but it is easy to see that this leads to isomorphic Lie algebras.

Once we have stated the general picture, let us consider an analogous procedure by starting from two algebraic units λ and μ. In this case, by following essentially the same procedure as above, we can construct a two step nilpotent Anosov Lie algebra of type $(p+q, pq)$, where the eigenvalues of the corresponding A_1 are the conjugated numbers to λ and μ, and the ones corresponding to A_2 are all the products among them, $\{\lambda_i\mu_j\}$. It is not hard to see that this algebra is isomorphic to the one associated to a bipartite graph (p,q), which is proved to be Anosov in [3]. In this case, a lot of changes can be made to this procedure to obtain a variety of new examples. Among them we are now going to mention a few more, and since the proofs are essentially the same, they will be omitted.

Example 2.3. As above, one can start by taking three algebraic units λ, μ and ν of degree p, q, and r respectively, such that the conjugate numbers to each of them satisfy $|\lambda_i\mu_j| \neq 1$, $|\lambda_i\nu_k| \neq 1$ and $|\nu_k\mu_j| \neq 1$. It is not hard to see that we can proceed analogously by considering the pair $\lambda\mu$ and ν, with (pq, r) the degree of $\lambda\mu$. In fact, in the proof of the linear independence of the new basis, and also in (3), we only use the fact that we are adding over all the conjugated numbers to λ and μ. Following the lines of the above procedure, we then obtain an Anosov Lie algebra, $\mathfrak{n}_{(pq,r)}$ of type $(pqr+r, pq)$. Moreover, once we have stated this, it is clear that it is also true for $\lambda\nu$ and μ, (pr, q), and in this case our procedure leads to a Lie algebra of type $(prq+q, pr)$.

Now, it is clear that in each of these algebras, $\mathfrak{n}_{(pq,r)}$ and $\mathfrak{n}_{(pr,q)}$, the eigenvalues of the associated automorphism corresponding to $X_{(k,l)}$ are the same, that is, $\lambda_i\mu_j\nu_s$ for some i, j, s. Therefore, the corresponding subspaces V_0 can be identified (see Remark 2.1). In this case, it is easy to see that a new algebra can be constructed from these two by identifying the V_0. Explicitly, if $\mathfrak{n}_{(pq,r)} = V_0 \oplus V_1 \oplus Z_1$ and $\mathfrak{n}_{(pr,q)} = V_0 \oplus V_2 \oplus Z_2$, let $\mathfrak{n}$ be the Lie algebra with vector space $\mathfrak{n} = (V_0 \oplus V_1 \oplus V_2) \oplus (Z_1 \oplus Z_2)$, where the action is as before: $[V_0, V_i] \subset Z_i$, $i = 1, 2$. This is a two step nilpotent Lie algebra of type $(pqr+q+r, pr+pq)$. In this framework, there is a natural way to define an automorphism in $\mathfrak{n}$, using the ones in $\mathfrak{n}_{(pq,r)}$ and $\mathfrak{n}_{(pr,q)}$: $[A]_\beta = \begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix}$,

where

$$A_1 = \begin{bmatrix} \lambda_1\mu_1\nu_1 &&&&&&&& \\ & \ddots &&&&&&& \\ && \lambda_p\mu_q\nu_r &&&&&& \\ &&& \mu_1^{-1} &&&&& \\ &&&& \ddots &&&& \\ &&&&& \mu_q^{-1} &&& \\ &&&&&& \nu_1^{-1} && \\ &&&&&&& \ddots & \\ &&&&&&&& \nu_r^{-1} \end{bmatrix},$$

$$A_2 = \begin{bmatrix} \lambda_1\nu_1 &&&&& \\ & \ddots &&&& \\ && \lambda_p\nu_r &&& \\ &&& \lambda_1\mu_1 && \\ &&&& \ddots & \\ &&&&& \lambda_p\mu_q \end{bmatrix}.$$

It is easy to check that due to our choice of λ, μ and ν, it is hyperbolic. On the other hand, note that in both cases the lattice we have constructed in V_0, $\mathcal{X}_{(k,l)}$ are the same (it is just a matter of notation), and therefore, as with the automorphism, the natural extension of the lattice we have in $\mathfrak{n}_{(pq,r)}$ and $\mathfrak{n}_{(pr,q)}$ is a $\mathbb{Z}$-basis preserved by A; hence $\mathfrak{n}$ is an Anosov Lie Algebra.

In this way we obtain two step Anosov Lie algebras of dimension $n = pqr + pq + pr + q + r$ for any p, q, r. Distinguishing them by the type, it is easy to see that in general, if $p \neq q \neq r$ then the Lie algebra one obtains by interchanging the roll of p, q and r is not isomorphic. The smallest one we can construct corresponds to $p = q = r = 2$, is 18 dimensional, and its type is $(10, 8)$.

It is not hard to see that this procedure extends in a natural way to considering k algebraic units in order to obtain a 2-step Anosov Lie algebra.

Example 2.4. Now we are going to show how to use the procedure to construct three-step Anosov Lie algebras. As before, we take algebraic numbers λ, μ and ν of degree p, q, and r respectively.

In this case we have in mind $A = \begin{bmatrix} A_1 && \\ & A_2 & \\ && A_3 \end{bmatrix}$, where A_1 and A_2 are as in the previous example, and $A_3 = \begin{bmatrix} \lambda_1 && \\ & \ddots & \\ && \lambda_p \end{bmatrix}$. As before, we are going to make a small change in the notation in order to be consistent with the eigenvalues. Let $\mathfrak{n}$ be the Lie algebra with basis

$$\beta = \{X_{(i,j,k)}, Y_j, Z_k, V_{(i,k)}, W_{(i,j)}, U_i : 1 \leq i \leq p,\ 1 \leq j \leq q,\ 1 \leq k \leq r\}$$

and the Lie bracket among them be given by

$$\begin{aligned} &[X_{(i,j,k)}, Y_m] = \delta_{(j,m)} V_{(i,k)} \qquad [X_{(i,j,k)}, Z_n] = \delta_{(k,n)} W_{(i,j)} \\ &[Z_n, V_{(i,k)}] = \delta_{(n,k)} U_i \qquad [Y_m, W_{(i,j)}] = \delta_{(m,j)} U_i. \end{aligned}$$

It is easy to see that $\mathfrak{n}$ is a three-step nilpotent Lie algebra; that is, it satisfies Jacobi identities, and the type of $\mathfrak{n}$ is $(pqr+q+r, pr+pq, p)$. Let A denote the linear transformation of $\mathfrak{n}$ such that $[A]_\beta = \begin{bmatrix} A_1 && \\ & A_2 & \\ && A_3 \end{bmatrix}$; hence, A is a hyperbolic automorphism of $\mathfrak{n}$ and β is a basis of eigenvectors of A. To construct a $\mathbb{Z}$-basis, we proceed similarly as before:

$$\mathcal{X}_{(m,l,s)} = \sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{k=1}^{r} \lambda_i^m \mu_j^l \nu_k^s X_{(i,j,k)} \qquad 0 \le m < p,\ 0 \le l < q,\ 0 \le k < r,$$

$$\mathcal{Y}_l = \sum_{k=1}^{q} \mu_k^{-l}\, Y_k \qquad 0 \le l < q,$$

$$\mathcal{Z}_s = \sum_{l=1}^{r} \nu_l^{-s}\, Z_l \qquad 0 \le s < r,$$

$$\mathcal{V}_{(m,s)} = \sum_{i=1}^{p}\sum_{k=1}^{r} \lambda_i^m \nu_k^s\, V_{(i,k)} \qquad 0 \le m < p,\ 0 \le s < r,$$

$$\mathcal{W}_{(m,l)} = \sum_{i=1}^{p}\sum_{j=1}^{q} \lambda_i^m \mu_j^l\, W_{(i,j)} \qquad 0 \le m < p,\ 0 \le l < q,$$

$$\mathcal{U}_i = \sum_{n=1}^{p} \lambda_n^i\, U_n \qquad 0 \le i < p.$$

Straightforward calculations, using the same techniques as before, shows that this is a $\mathbb{Z}$-basis preserved by A, and therefore $\mathfrak{n}$ is an Anosov Lie algebra. The smallest example we obtain in this way is a three step nilpotent Lie algebra of dimension 20, type $(10,8,2)$.

It is not hard to see that if $\mathfrak{n} = \mathfrak{n}_1 \oplus \cdots \oplus \mathfrak{n}_r$ is a real Anosov Lie algebra of type $(n_1, n_2, \dots, n_r)$, then $\mathfrak{n}/\mathfrak{n}_r$ is also an Anosov Lie algebra (see [14]). Note that in this case, this fact is what we have showed in the previous construction.

Also, it is not hard to prove by induction that this procedure extends in a natural way to the case of considering k-algebraic units to obtain a k-step Anosov Lie algebra. Note that to check the Jacobi identity, it is enough to check with $[X_i, [Y, X_j]]$, where $\{X_l\}$ is a basis of $\mathfrak{n}_1$ and $Y \in \mathfrak{n}$, since it is easy to see that all the other brackets are zero. Also, by the above observation, we have all the quotients one has in between.

Example 2.5. As a last application of our procedure, we are going to consider the special case of $p = 2$. In this case, in addition to $\mathfrak{n}_{(2,q)}$ and $\mathfrak{n}_{(q,2)}$, we can define other Lie algebras, for example by adding Lie brackets among the $\{X_{(i,j)}\}$. That is, we take

$$\beta = \{X_i, Y_k, Z_l : 1 \le i \le 2q\ 1 \le k \le q,\ 1 \le l \le q+2\}$$

as a basis of $\mathfrak{n}$, with the Lie bracket given by

$$\begin{aligned} &[X_{iq+j}, Y_j] = Z_{q+i+1} && i = 0,1 : j = 1 \dots, q \\ &[X_j, X_{j+q}] = Z_j && j = 1 \dots, q. \end{aligned} \tag{5}$$

This is a two-step nilpotent Lie algebra of type $(3q, q+2)$. Let A be the automorphism of $\mathfrak{n}$ such that $[A]_\beta = \begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix}$, where

$$A_1 = \begin{bmatrix} \lambda\mu_1 & & & & & \\ & \ddots & & & & \\ & & \lambda^{-1}\mu_q & & & \\ & & & \mu_1^{-1} & & \\ & & & & \ddots & \\ & & & & & \mu_q^{-1} \end{bmatrix}, \quad A_2 = \begin{bmatrix} \mu_1^2 & & & & \\ & \ddots & & & \\ & & \mu_q^2 & & \\ & & & \lambda & \\ & & & & \lambda^{-1} \end{bmatrix}.$$

Concerning the lattice, we will take $\mathcal{Y}_k$ as before and let

$$\mathcal{X}_i = \begin{cases} \sum_{k=1}^{q} \mu_k^i\, (X_k + X_{q+k}) & 0 \le i < q \\ \sum_{k=1}^{q} \mu_k^{i-q}\, (\lambda X_k + \lambda^{-1} X_{q+k}) & q \le i < 2q \end{cases},$$
$$\mathcal{Z}_l = \begin{cases} (\lambda^{-1} - \lambda)\sum_{k=1}^{q} \mu_k^l\, Z_k & 0 \le l < q \\ Z_q + Z_{q+1} & l = q \\ \lambda Z_q + \lambda^{-1} Z_{q+1} & l = q+1. \end{cases} \tag{6}$$

One can see that this is also a basis of $\mathfrak{n}$ preserved by A, and moreover one can check that

$$\begin{aligned} &[\mathcal{X}_i, \mathcal{X}_j] = 0 && i,j < q \text{ or } i,j \ge q \\ &[\mathcal{X}_i, \mathcal{X}_j] = \mathcal{Z}_{i+j-q} && 0 \le i < q \text{ and } q \le j < 2q. \end{aligned} \tag{7}$$

Also, as in (3), one can see that

$$[\mathcal{X}_{iq+j}, \mathcal{Y}_k] = N(j,k)\mathcal{Z}_{q+i}, \qquad i = 0,1 : 0 \le j < q,$$

where $N(j,k) = \operatorname{tr}(A_\mu^{j-k})$, such that $A_\mu = \begin{bmatrix} \mu_1 & & & \\ & \mu_2 & & \\ & & \ddots & \\ & & & \mu_q \end{bmatrix}$ is an integer number, and therefore we can conclude that $\mathfrak{n}$ is an Anosov Lie algebra. Note that the dimension of $\mathfrak{n}$ is $n = 4q+2$, and the type is $(3q, q+2)$. The smallest one we can obtain corresponds to $q = 2$ and is of type $(6,4)$, dimension 10.

Remark 2.6. Note that the subalgebra generated by the set $\{X_i, Z_l : 1 \le i \le 2q,\ 1 \le l \le q\}$ is an Anosov Lie subalgebra of $\mathfrak{n}$.

3. 13-dimensional example.

This example is rather different. We will take $p = 2$, $q = 3$, and we will split the basis in the center. So let λ and μ be two algebraic numbers of degree 2 and 3 respectively such that $|\lambda_i| \ne 1$, $|\mu_j| \ne 1$ and $|\lambda_i\mu_j| \ne 1$ for all the conjugated numbers of λ and μ. Then we take $\mathfrak{n}$ as the complex vector space with basis

$$\beta = \{X_1, \dots, X_6, Y_1, Y_2, Y_3, Z_1, Z_2, W_1, W_2\},$$

and define the Lie bracket among them by

$$\begin{aligned} &[X_1,Y_1]=Z_1 && [X_4,Y_1]=W_1 \\ &[X_2,Y_2]=Z_2 && [X_5,Y_2]=W_2 \qquad (8) \\ &[X_3,Y_3]=-(Z_1+Z_2) && [X_6,Y_3]=-(W_1+W_2). \end{aligned}$$

This is a two-step nilpotent Lie algebra of type $(9,4)$. Let A be the automorphism of $\mathfrak{n}$ such that $[A]_\beta = \left[\begin{smallmatrix} A_1 & \\ & A_2 \end{smallmatrix}\right]$, where

$$A_1 = \begin{bmatrix} \lambda\mu_1 & & & & & \\ & \ddots & & & & \\ & & \lambda^{-1}\mu_3 & & & \\ & & & \mu_1^{-1} & & \\ & & & & \mu_2^{-1} & \\ & & & & & \mu_3^{-1} \end{bmatrix}, \quad A_2 = \begin{bmatrix} \lambda & & & \\ & \lambda & & \\ & & \lambda^{-1} & \\ & & & \lambda^{-1} \end{bmatrix}.$$

Concerning the lattice, we will take $\{\mathcal{X}_i\}$, $0 \le i \le 5$ and $\{\mathcal{Y}_k\}$, $0 \le k \le 2$, as in the previous example, and let

$$\mathcal{Z}_l = \sum_{i=1}^{2} \left(\mu_i^l - \mu_3^l\right)(Z_i + W_i) \qquad l=-1,1,$$

$$\mathcal{W}_l = \sum_{i=1}^{2} \left(\mu_i^l - \mu_3^l\right)\left(\lambda Z_i + \lambda^{-1} W_i\right) \quad l=-1,1.$$

To see that this is a basis of $\mathfrak{n}$, as we have pointed out before, it is enough to check that each one of $\{\mathcal{X}_i\}$, $\{\mathcal{Y}_j\}$ and $\{\mathcal{Z}_k\ \mathcal{W}_l\}$ are linearly independent sets. We also note that the calculations for the first two sets have already been done, and then we are only going to show how to proceed in the center. Suppose then that

$$\begin{aligned} 0 = &\ a_{-1}\mathcal{Z}_{-1} + a_1\mathcal{Z}_1 + b_{-1}\mathcal{W}_{-1} + b_1\mathcal{W}_1 \\ = &\sum_{i=1}^{2}\Big[\left(\mu_i^{-1}-\mu_3^{-1}\right)(a_{-1}+\lambda b_{-1}) + \left(\mu_i^1-\mu_3^1\right)(a_1+\lambda b_1)\Big] Z_i \\ &+\sum_{i=1}^{2}\Big[\left(\mu_i^{-1}-\mu_3^{-1}\right)\left(a_{-1}+\lambda^{-1}b_{-1}\right) + \left(\mu_i^1-\mu_3^1\right)\left(a_1+\lambda^{-1}b_1\right)\Big] W_i. \end{aligned}$$

Hence, for $i=1,2$ we have that

$$\begin{aligned} 0 &= \left(\mu_i^{-1}-\mu_3^{-1}\right)(a_{-1}+\lambda b_{-1}) + \left(\mu_i^1-\mu_3^1\right)(a_1+\lambda b_1), \\ 0 &= \left(\mu_i^{-1}-\mu_3^{-1}\right)\left(a_{-1}+\lambda^{-1}b_{-1}\right) + \left(\mu_i^1-\mu_3^1\right)\left(a_1+\lambda^{-1}b_1\right). \end{aligned} \qquad (9)$$

If we denote

$$P_\lambda(x) = x^{-1}(a_{-1}+\lambda b_{-1}) + x(a_1+\lambda b_1),$$

then by the first equation we have that $P_\lambda(\mu_i) = P_\lambda(\mu_j) = C$ for $i,j = 1,2,3$. Hence,

$$(a_{-1}+\lambda b_{-1}) + x^2(a_1+\lambda b_1) - Cx \qquad (10)$$

is a degree two polynomial annihilated by each one of the μ_i. Since these are three different algebraic numbers, we have that (10) is identically zero. In particular,

$$0 = (a_{-1}+\lambda b_{-1}) \quad \text{and} \quad 0 = (a_1+\lambda b_1). \qquad (11)$$

We can do the same for the second equation in (9), and we will obtain

$$0 = \left(a_{-1}+\lambda^{-1}b_{-1}\right) \quad \text{and} \quad 0 = \left(a_1+\lambda^{-1}b_1\right). \qquad (12)$$

Finally, from (11) and (12) we can conclude that

$$a_{-1} = b_{-1} = a_1 = b_1 = 0,$$

as was to be shown.

One can also see for $i=1$ or 2 (let's say $i=1$ for simplicity) that

$$\begin{aligned} \mu_1^2 - \mu_3^2 &= (\mu_1-\mu_3)(\mu_1+\mu_3) \\ &= (\mu_1-\mu_3)\left(t_1 - (\mu_1\mu_3)^{-1}\right) \\ &= t_1(\mu_1-\mu_3) + \left(\mu_1^{-1}-\mu_3^{-1}\right), \end{aligned}$$

where $t_j = \operatorname{tr} A_\mu^j$ is an integer number for all $j \in \mathbb{Z}$, and $\mu_2 = (\mu_1\mu_3)^{-1}$. Therefore,

$$\sum_{i=1}^{2}\left(\mu_i^2-\mu_3^2\right)(Z_i+W_i) = t_1\mathcal{Z}_1 + \mathcal{Z}_{-1}.$$

In the same way, we also have that

$$\sum_{i=1}^{2}\left(\mu_i^2-\mu_3^2\right)\left(\lambda Z_i + \lambda^{-1}W_i\right) = t_1\mathcal{W}_1 + \mathcal{W}_{-1}.$$

It is also easy to see that this is valid for $\mu_i^{-2}-\mu_3^{-2}$ as well; that is, we have formulas similar to these ones for

$$\sum_{i=1}^{2}\left(\mu_i^{-2}-\mu_3^{-2}\right)\left(\lambda^j Z_i + \lambda^{-j}W_i\right),$$

for $j=0,1$.

Also, as in the previous examples, it is not hard to see that this is a basis of $\mathfrak{n}$ preserved by A. With all this, one can check that

$$\begin{aligned} &[\mathcal{X}_0,\mathcal{Y}_0]=0 && [\mathcal{X}_3,\mathcal{Y}_0]=0 \\ &[\mathcal{X}_1,\mathcal{Y}_0]=\mathcal{Z}_1 && [\mathcal{X}_4,\mathcal{Y}_0]=\mathcal{W}_1, \\ &[\mathcal{X}_2,\mathcal{Y}_0]=t_1\mathcal{Z}_1+\mathcal{Z}_{-1} && [\mathcal{X}_5,\mathcal{Y}_0]=t_1\mathcal{W}_1+\mathcal{W}_{-1} \\ &[\mathcal{X}_0,\mathcal{Y}_1]=\mathcal{Z}_{-1} && [\mathcal{X}_0,\mathcal{Y}_2]=t_{-1}\mathcal{Z}_1+\mathcal{Z}_1. \end{aligned}$$

Using this and the fact that A is an automorphism, it is easy to prove that this is a $\mathbb{Z}$-basis of $\mathfrak{n}$. For example,

$$\begin{aligned} [\mathcal{X}_2,\mathcal{Y}_3] &= [A\mathcal{X}_1, A\mathcal{Y}_2] \\ &= A[A\mathcal{X}_0, A\mathcal{Y}_1] \\ &= A(A\mathcal{Z}_{-1}) \\ &= A(\mathcal{W}_{-1}) = -a\mathcal{W}_{-1} - \mathcal{Z}_{-1}, \end{aligned}$$

where $x^2 + ax + 1$ is the minimal polynomial of λ. Therefore, we can conclude that this is a Anosov Lie algebra, as desired. In the following we will prove, by using similar arguments as in Lemma 6.6 of [3], that it is also indecomposable.

Proposition 3.1. $\mathfrak{n}$, *defined by the relations given by (8), is indecomposable.*

Proof. Suppose on the contrary that $\mathfrak{n} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ is the sum of two nontrivial ideals of $\mathfrak{n}$. By definition, we have that $\mathfrak{n} = V \oplus W$, where V is the subspace spanned by the set $S = \{X_1, \ldots, X_6, Y_1, Y_2, Y_3\}$ and W is the subspace of $\mathfrak{n}$ spanned by $\{Z_1, Z_2, W_1, W_3\}$. Let $p : \mathfrak{n} \to V$ be the projection onto V with respect to this decomposition $\mathfrak{n} = V \oplus W$, and let $V_1 = p(\mathfrak{m}_1)$ and $V_2 = p(\mathfrak{m}_2)$. For $i = 1, 2$ let

$$S_i = \{v \in S : v + \sum_{v' \in S, v' \neq v} a_{v'} v' \in V_i \text{ for some scalars } a_{v'}\text{'s}\}.$$

Then $S = S_1 \cup S_2$, and since $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are nontrivial ideals, it is easy to see that S_1 and S_2 are nonempty sets. Moreover, we have that $S_2 \setminus S_1$ is empty. In fact, if $S_2 \setminus S_1$ is nonempty, we can either have that $[v, v']$ is zero for all $v \in S_2 \setminus S_1$ and $v' \in S_1$, or there exists $v \in S_2 \setminus S_1$ and $v' \in S_1$ such that $[v, v']$ is nonzero.

In the first situation, as $S = S_1 \cup S_2$, we may assume that Y_1 is contained in S_1. Then there exists nonzero a such that $aY_1 + x \in V_1$, where x is contained in the span of $S \setminus \{Y_1\}$. Since $\mathfrak{m}_1$ is an ideal, and $[aY_1 + x, X_1]$ and $[aY_1 + x, X_4]$ are contained in $\mathfrak{m}_1$, this means that $Z_1, W_1 \in \mathfrak{m}_1$. We notice that not all Y_i's are contained in S_1 because if all Y_i's are contained in S_1, then (by our assumption) all X_j's are contained in S_1, and hence $S_2 \setminus S_1$ is empty. Now either $Y_2 \in S_1$ or $Y_2 \in S_2 \setminus S_1$. If $Y_2 \in S_1$ (a similar argument works for the other case), then Y_3 must be contained in $S_2 \setminus S_1$. In that case Z_2 and W_2 are contained in $\mathfrak{m}_1$ (by considering Lie brackets with X_2 and X_5), and similarly $Z_1 + Z_2$ and $W_1 + W_2$ are contained in $\mathfrak{m}_2$. This is a contradiction.

On the other hand, if there exists $v \in S_2 \setminus S_1$ and $v' \in S_1$ such that $[v, v']$ is nonzero, it is easy to see that if $v \in \{Y_1, Y_2, Y_3\}$, then $v' \in \{X_1, \ldots X_6\}$, and if $v \in \{X_1, \ldots X_6\}$, then $v' \in \{Y_1, Y_2, Y_3\}$. So since it is entirely equivalent, we can assume that $v \in \{Y_1, Y_2, Y_3\}$ and $v = Y_1$. Moreover, either $v' = X_1$ or $v' = X_4$, and therefore we may assume that $v' = X_1$. From our definition of S_i, there exist nonzero scalars s and t such that $sX_1 + x$ is contained in V_1 and $tY_1 + y$ is contained in V_2, where x is in the subspace of V spanned by $Y_1, Y_2, Y_3, X_2, \ldots, X_6$ and y is in the subspace of V spanned by $Y_2, Y_3, X_1, X_2, \ldots, X_6$. Hence $[sX_1 + x, Y_1] \in \mathfrak{m}_1$ and $[tY_1 + y, X_1], [tY_1 + y, X_4] \in \mathfrak{m}_2$, since $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are Lie ideals of $\mathfrak{n}$. This implies that $sZ_1 + s'W_1 \in \mathfrak{m}_1$, where s' is a scalar and $Z_1, W_1 \in \mathfrak{m}_2$. This is a contradiction because s is nonzero, and thus we can conclude that $S_2 \setminus S_1$ is empty.

Therefore we have that $S = S_1$, and moreover we can see that Z_1, Z_2, W_1, W_2 are contained in $\mathfrak{m}_1$. Hence $[\mathfrak{m}_1, \mathfrak{m}_1] = [\mathfrak{n}, \mathfrak{n}]$, and from this one has that $\mathfrak{m}_2$ is in the center of $\mathfrak{n}$.

On the other hand, it is easy to see that the center is equal to $[\mathfrak{n}, \mathfrak{n}] = W$, and hence $\mathfrak{m}_2$ is contained in $\mathfrak{m}_1$, contradicting our assumption that $\mathfrak{m}_2$ is nontrivial. Hence $\mathfrak{n}$ cannot be seen as a sum of two proper ideals, as we wanted to show. $\square$

4. 16-dimensional example.

Let (S, E) denote the complete bipartite graph on a set S of 5 elements partitioned into subsets S_1 and S_2 of 2 and 3 elements, respectively. Following for example [3], we can define from this (and any graph) a 2-step nilpotent Lie algebra. Let $\mathcal{N} = V \oplus W$ denote the 2-step nilpotent Lie algebra associated with (S, E), where V is the vector space with a basis S and W is the subspace of $\wedge^2 V$ spanned by $\{\alpha \wedge \beta : \alpha \in S_1, \beta \in S_2\}$. The nonzero Lie brackets are given by $[\alpha, \beta] = \alpha \wedge \beta$ for all $\alpha \in S_1$ and $\beta \in S_2$. We recall that in this case we obtain an Anosov Lie algebra of type $(5, 6)$ (see [3]). Using this algebra, we are going to construct a 16-dimensional Anosov Lie algebra as follows.

Let $\mathfrak{n}$ be that Lie algebra with linear space $\mathfrak{n} = V \oplus V \oplus W$ and Lie bracket be defined by $[(x_1, x_2, w), (y_1, y_2, w')] = [x_1, y_1] + [x_2, y_2]$, where x_i's and y_i's are vectors in V, $w, w' \in W$ and $[x_i, y_i]$ denotes the Lie bracket in $\mathcal{N}$. To see that it is an Anosov Lie algebra, let us consider Φ, the additive subgroup of $\mathfrak{n}$ generated by the elements of the type $(v, 0, 0), (0, v', 0), (0, 0, [\gamma, \delta])$, where $v, v', \gamma, \delta \in S$. It is easy to see that Φ is a $\mathbb{Z}$-subalgebra of $\mathfrak{n}$ (i.e., Φ is the set of all $\mathbb{Z}$-linear combinations of the basis of $\mathfrak{n}$ with integer structure coefficients), and moreover that $\mathfrak{n}$ admits a hyperbolic automorphism τ such that $\tau(\Phi) = \Phi$. In fact, if Φ' is a subgroup of $\mathcal{N}$ generated by $S \cup \{[v, v'] : v, v' \in S\}$, then $\mathcal{N}$ admits a hyperbolic automorphism τ' such that $\tau'(\Phi') = \Phi'$ (see [3], Theorem 1.1). The matrix of $\tau'|_V$ with respect to the basis $S_1 \cup S_2$ is given by

$$A = \begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix},$$

where $A_1 \in GL(2, \mathbb{Z})$ and $A_2 \in GL(3, \mathbb{Z})$ are hyperbolic such that the pairwise products $\lambda\mu$, λ and μ are the eigenvalues of A_1 and A_2, respectively, and are not of absolute value 1. A can be extended to an Anosov automorphism τ' of $\mathcal{N}$ such that $\tau'(\Phi') = \Phi'$. We take τ to be the natural extension of τ' to $\mathfrak{n}$. Hence $\mathfrak{n}$ is an Anosov Lie algebra.

Proposition 4.1. $\mathfrak{n}$, *defined as above, is indecomposable.*

Proof. Let $S_1 = \{\alpha, \beta\}$ and $S_2 = \{\gamma, \delta, \eta\}$. Suppose that $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are two proper ideals of $\mathfrak{n}$ such that $\mathfrak{n} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$. Seeing as $[\mathfrak{n}, \mathfrak{n}] = [\mathfrak{m}_1, \mathfrak{m}_1] \oplus [\mathfrak{m}_2, \mathfrak{m}_2]$ and $[\mathfrak{n}, \mathfrak{n}]$ is 6-dimensional, we may assume that $\dim [\mathfrak{m}_1, \mathfrak{m}_1] \leq 3$. Let $X = (v, v', w) \in \mathfrak{m}_1$, where $v, v' \in V$ and $w \in W$. Let $v = \sum_{\zeta \in S} a_\zeta \zeta$. Because $\mathfrak{m}_1$ is an ideal, $[X, (\xi, 0, 0)] \in \mathfrak{m}_1$ for all $\xi \in S$. Hence $a_\gamma \gamma \wedge \xi + a_\delta \delta \wedge \xi + a_\eta \eta \wedge \xi$ and $a_\alpha \alpha \wedge \zeta + a_\beta \beta \wedge \zeta$ are contained in $[\mathfrak{m}_1, \mathfrak{m}_1]$ for all $\xi \in S_1$ and $\zeta \in S_2$. As $\dim [\mathfrak{m}_1, \mathfrak{m}_1] \leq 3$, we see that either $a_\xi = 0$ for all $\xi \in S_1$ or $a_\zeta = 0$ for all $\zeta \in S_2$. Let V_1 (respectively V_2) denote the subspace of V spanned by S_1 (respectively S_2). Then by the above observation, $v \in V_1$ or $v \in V_2$.

Suppose $v \in V_1$. We will prove that $\mathfrak{m}_1$ is contained in $V_1 \oplus V \oplus W$. Suppose that $v \neq 0$. Then the vectors $a_\alpha \alpha \wedge \zeta + a_\beta \beta \wedge \zeta \in [\mathfrak{m}_1, \mathfrak{m}_1]$ for all $\zeta \in S_2$ are linearly independent. Hence $\dim [\mathfrak{m}_1, \mathfrak{m}_1] = 3$. Suppose $(v_1, v_1', w_1) \in \mathfrak{m}_1$ be such that $v_1 \in V_2$. Let $v_1 = a_\gamma' \gamma + a_\delta' \delta + a_\eta' \eta$. Now as $\dim [\mathfrak{m}_1, \mathfrak{m}_1] = 3$, and $a_\gamma' \gamma \wedge \alpha + a_\delta' \delta \wedge \alpha + a_\eta' \eta \wedge \alpha$ and $a_\gamma' \gamma \wedge \beta + a_\delta' \delta \wedge \beta + a_\eta' \eta \wedge \beta$ are contained in $[\mathfrak{m}_1, \mathfrak{m}_1]$, $a_\gamma' = a_\delta' = a_\eta' = 0$. Hence $v_1 = 0$. Thus we have proved that if $v \in V_1$, then $\mathfrak{m}_1$ is contained in $V_1 \oplus V \oplus W$. Similarly we prove that if $v \in V_2$, then $\mathfrak{m}_1$ is contained in $V_2 \oplus V \oplus W$. Suppose $v \in V_2$ and $v \neq 0$. Let $(v_1, v_1', w_1) \in \mathfrak{m}_1$ be such that $v_1 \in V_1$, and write $v_1 = a_\alpha' \alpha + a_\beta' \beta$. We note that $a_\alpha' \alpha \wedge \zeta + a_\beta' \beta \wedge \zeta \in [\mathfrak{m}_1, \mathfrak{m}_1]$ for all $\zeta \in S_2$. If the vectors $a_\alpha' \alpha \wedge \zeta + a_\beta' \beta \wedge \zeta$ are linearly independent for all $\zeta \in S_2$, then $\dim [\mathfrak{m}_1, \mathfrak{m}_1] = 3$. This is a contradiction since $a_\gamma \gamma \wedge \alpha + a_\delta \delta \wedge \alpha + a_\eta \eta \wedge \alpha$ and $a_\gamma \gamma \wedge \beta + a_\delta \delta \wedge \beta + a_\eta \eta \wedge \beta$ are contained in $[\mathfrak{m}_1, \mathfrak{m}_1]$. Hence $a_\alpha' = a_\beta' = 0$, and so $v_1 = 0$. Thus if $v \in V_2$, then $\mathfrak{m}_1$ is contained in $V_2 \oplus V \oplus W$. Similarly we can prove that $\mathfrak{m}_1$ is contained in $V \oplus V_1 \oplus W$ or $V \oplus V_2 \oplus W$. Hence $\mathfrak{m}_1$ is contained in $V_i \oplus V_j \oplus W$ for some $i, j \in \{1, 2\}$. But then $[\mathfrak{m}_1, \mathfrak{n}_1] = 0$, which is a contradiction. This proves the proposition. $\square$

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THE ATTRACTORS FOR WEAKLY DAMPED NON-AUTONOMOUS HYPERBOLIC EQUATIONS WITH A NEW CLASS OF EXTERNAL FORCES

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ABSTRACT. For weakly damped non-autonomous hyperbolic equations, we introduce a new concept Condition (C*), denote the set of all functions satisfying Condition (C*) by $L^2_{c*}(\mathbb{R};X)$ which are translation bounded but not translation compact in $L^2_{loc}(\mathbb{R};X)$, and show that there are many functions satisfying Condition (C*); then we study the uniform attractors for weakly damped non-autonomous hyperbolic equations with this new class of time dependent external forces $g(x,t) \in L^2_{c*}(\mathbb{R};X)$ and prove the existence of the uniform attractors for the family of processes corresponding to the equation in $H^1_0 \times L^2$ and $D(A) \times H^1_0$.

1. Introduction.

We consider the following weakly damped non-autonomous hyperbolic equation

$$\frac{\partial^2 u}{\partial t^2} + \alpha \frac{\partial u}{\partial t} - \Delta u + f(u) = g(x,t), \qquad u|_{\partial\Omega} = 0, \quad x \in \Omega, \tag{1.1}$$

where Ω is an open bounded subset of $\mathbb{R}^n$ with sufficiently smooth boundary, α is a positive constant, and f is a C^1 function from $\mathbb{R}$ to $\mathbb{R}$ satisfying the following conditions:

$$\liminf_{|s|\to\infty} \frac{F(s)}{s^2} \geqslant 0; \tag{1.2}$$

$$\limsup_{|s|\to\infty} \frac{f'(s)}{|s|^\gamma} = 0, \tag{1.3}$$

with $0 \leqslant \gamma < \infty$ if $n = 1,2$ and $0 \leqslant \gamma \leqslant \frac{2}{n-2}$ if $n \geqslant 3$. Here, F is the primitive function of f:

$$F(s) = \int_0^s f(r)dr.$$

Furthermore, there exists a $C_1(>0)$ such that

$$\liminf_{|s|\to\infty} \frac{sf(s) - C_1 F(s)}{s^2} \geqslant 0. \tag{1.4}$$

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