The Life and Mathematics of Ramanujan

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Graduate Student Seminar

Early life

Ramanujan was born in 1887 in a Brahmin family near Kumbakonam.

He entered high school at the age of seven. The sixth-form boys were delighted to find a youngster who was ready to do all of their hard sums for them. By the time he was twelve or thirteen he was recognized as quite an abnormal boy.

Soon after beginning the study of trigonometry, he discovered Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

He was very disappointed when he read in Loney's *Trigonometry* that it was known already."

Until the age of 16, Ramanujan had no access to mathematical books of higher class.

"Whittaker's *Modern Analysis* had not yet spread so far, and Bromwich's *Infinite Series* did not exist. Either of these books would have made a tremendous difference to him."

(*This and subsequent quotes taken from the book Ramanujan by G. H. Hardy.*)

When he was 16, Ramanujan obtained the book *A Synopsis of Elementary Results in Pure and Applied Mathematics* by George Carr.

Carr was a private tutor in London, and came to Cambridge when he was nearly 40.

"Carr is now completely forgotten, even in his own college, except in so far as Ramanujan has kept his name alive."

"The book contains enunciations of 6165 theorems, systematically and quite scientifically arranged. ... Proofs are often little more than cross-references and are the least interesting parts of the book. ... All this is exaggerated in Ramanujan's famous notebooks, and any student of the notebooks can see that Ramanujan's ideal of presentation had been copied from Carr's."

Comment: Bruce Berndt has pointed out that numbering in Carr's book contains many gaps. According to Berndt, it has only 4417 theorems, not 6165.

Ramanujan's academic career in India was undistinguished.

- In December 1903, he sat for the matriculation exam of Madras Univeristy. He obtained a Second Class place and entered the Government College of Kumbakonam with a scholarship.
- However, by this time he was totally immersed in mathematics and would not study any other subject. He failed his exams (except for mathematics) and lost his scholarship.

- In 1907, Ramanujan appeared for the First Arts (FA) Examination at Pachayiappa College after private study. He took exams in English, Sanskrit, and Mathematics. He scored 85 out of 150 on Mathematics, but he failed the other exams.
- In Fall 2002, C.A. Reddi found copies of the 1903 and 1907 exams that Ramanujan took. He and Bruce Berndt (UIUC) published them, along with commentary, in the *American Mathematical Monthly.*

- About 1910 he found influential Indian friends, who tried to find a position for him and failed.
- He began working as a clerk in Madras at a salary of 30 pounds per year.
- His first substantial paper was published in 1911. By 1912, his exceptional powers began to be understood.
- Sir Francis Spring and Sir Gilbert Walker obtained a scholarship of 60 pounds per year for Ramanujan.

- In 1913, Ramanujan wrote to G.H. Hardy at Cambridge.
- The letter contained 120 mathematical formulas and theorems from Ramanujan's notebooks.
- At first Hardy thought it might be a hoax–the results must be known theorems artfully disguised.

- Some of the results in the letter were familiar to Hardy.
- Some of the were not familiar, but Hardy was able to prove them "but not without more trouble than expected."
- Some of them he could not prove. "They defeated me completely; I had never seen the like of them before. A single look at them is enough to show that could only be written down by a mathematician of the highest class. They must be true, because, if they were not true, no one would have the imagination to invent them."

- E. H. Neville visited Madras in 1914, met Ramanujan, and saw one of his notebooks.
- He invited Ramanujan to Cambridge. Ramanujan's parents initially objected, but they withdrew their opposition after his mother had a dream.
- She had seen him surrounded by Europeans and heard the goddess Namagiri commanding her to no longer stand between her son and the fulfillment of his life's purpose.

- Ramanujan had three years of uninterrupted activity.
- He was elected Fellow of the Royal Society in 1918 and a Fellow of Trinity College later the same year.
- He fell ill in 1917, and never completely recovered.
- But he continued working at a high level until his death in 1920 at the age of 32.

- Ramanujan's published works fill a volume of over 400 pages.
- His unpublished notebooks were studied for many years.
- Bruce Berndt has published five volumes on Ramanujan's notebooks.

Partitions

A *partition* of n is division of representation of n as a sum of positive integers. Therefore

$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1$$

has 5 partitions. The number of partitions of n is denoted $p(n)$.

A Simple Partition Problem

Let $r_2(n)$ be the number of partitions of n where the summands are all 1 or 2. In other words, $r_2(n)$ is number of ways of writing $n = a + 2b$ where a and b are non-negative integers. For example,

$$5 = 1 \cdot 5 + 2 \cdot 0 = 1 \cdot 3 + 2 \cdot 1 = 1 \cdot 1 + 2 \cdot 2,$$

so $r_2(5) = 3$.

Observe that

$$\sum_{n=0}^{\infty} r_2(n) z^n = (1+z+z^2+\ldots)(1+z^2+z^4+\ldots) = \frac{1}{(1-z)} \frac{1}{(1-z^2)}$$

By partial fractions, we also have

$$\frac{1}{(1-z)(1-z^2)} = \frac{1}{4(1-z)} + \frac{1}{4(1+z)} + \frac{1}{2(1-z)^2}.$$

Therefore

$$r_2(n) = \frac{2n+3+(-1)^n}{4} = \left[\frac{n}{2}\right] + 1.$$

A Slightly More Complicated Partition Problem

Let $r_3(n)$ be the number of ways of writing n as a sum of $1s$, $2s$, and $3s$. Observe that

$$\sum_{n=0}^{\infty} r_3(n) z^n = \frac{1}{(1-z)} \frac{1}{(1-z^2)} \frac{1}{(1-z^3)}.$$

By partial fractions, we also have

$$\begin{aligned}\frac{1}{(1-z)} \frac{1}{(1-z^2)} \frac{1}{(1-z^3)} &= \frac{1}{6(1-z)^3} + \frac{1}{4(1-z)^2} + \frac{17}{72(1-z)} \\ &\quad + \frac{1}{8(1+z)} + \frac{1}{9(1-\omega z)} + \frac{1}{9(1-\omega^2 z)},\end{aligned}$$

where $\omega = e^{2\pi i/3}$ is a primitive cube root of unity.

Formula for $r_3(n)$

With a little bit of algebra, one sees that

$$r_3(n) = \frac{(n+3)^2}{12} - \frac{7}{72} + \frac{(-1)^n}{8} + \frac{\omega^n}{9} + \frac{\omega^{2n}}{9},$$

The first two terms come from the pole at $z = 1$. The other terms come from poles at

$$z = -1, z = \omega = e^{2\pi i/3}, z = \omega^2 = e^{4\pi i/3}$$

respectively. In other words, the poles come from roots of 1 of degree of at most 3.

Further examination shows that

$$r_3(n) = \frac{(n+3)^2}{12} + E(n),$$

where $|E(n)| \leq 1/2$. Therefore $r_3(n)$ is the closest integer to $(n+3)^2/12$.

Hardy and Ramanujan work on $p(n)$

The generating function for $p(n)$ is

$$F(z) = \sum_{n=0}^{\infty} p(n) z^n = \frac{1}{(1-z)(1-z^2)(1-z^3)\ldots}.$$

Let C be a circle centered at 0 with radius < 1. By Cauchy's Theorem

$$p(n) = \frac{1}{2\pi i} \int_C \frac{F(z)}{z^{n+1}} dz.$$

First Hardy-Ramanujan asymptotic for $p(n)$

The idea is to compute the contribution of a small arc near $z = 1$ and estimate rest of integral crudely. Hardy and Ramanujan obtained

$$p(n) = \frac{1}{2\pi\sqrt{2}} \frac{d}{dn} \frac{e^{K\lambda_n}}{\lambda_n} + O(e^{Hn^{1/2}}),$$

where $K = \pi\sqrt{2/3}, \lambda_n = \sqrt{n - 1/24}$ and $H < K$.

Hardy-Ramanujan Formula with tiny error term

$F(z)$ has singularities at $z = e^{2\pi i p/q}$ for all rational numbers p/q. By taking into account the effect of these other singularities, Hardy and Ramanujan were led to conjecture that

$$p(n) \sim \sum_{q=1}^{Q} L_q(n)\phi_q(n),$$

where Q is some function of n,

$$\begin{aligned} \phi_q(n) &= \frac{q^{1/2}}{2\pi\sqrt{2}} \frac{d}{dn}\left(\frac{e^{K\lambda_n/q}}{\lambda_n}\right), \\ L_q(n) &= \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \omega_{a,q} e^{-2\pi i a/q}, \end{aligned}$$

and $\omega_{a,q}$ is a certain 24th root of 1.

Major MacMahon used a recursive formula for $p(n)$ to compute

$$p(200) = 3972999029388.$$

This agreed with the first 8 terms of the Hardy-Ramanujan sum to .004. This motivated Hardy and Ramanujan to prove that

$$p(n) = \sum_{q<\alpha n^{1/2}} L_q(n)\phi_q(n) + O(n^{-1/4}),$$

which is an unusually good error term for a result in analytic number theory.

Rademacher Exact Formula

H. and R. actually did not start with

$$\phi_q(n) = \frac{q^{1/2}}{2\pi\sqrt{2}} \frac{d}{dn} \left(\frac{e^{K\lambda_n/q}}{\lambda_n} \right),$$

but with the "nearly equivalent"

$$\frac{q^{1/2}}{\pi\sqrt{2}} \frac{d}{dn} \left(\frac{\cosh(K\lambda_n/q) - 1}{\lambda_n} \right),$$

In the 1930s, Rademacher tried to simplify their proof by using

$$\psi_q(n) = \frac{q^{1/2}}{\pi\sqrt{2}} \frac{d}{dn} \left(\frac{\sinh(K\lambda_n/q)}{\lambda_n} \right),$$

To his surprise, this lead to an exact formula for $p(n)$:

$$p(n) = \sum_{q=1}^{\infty} L_q(n)\psi_q(n).$$

Selberg independently discovered the same result at about the same time. He said later that "I am inclined to believe that Rademacher and I are the only ones to have studied this paper thoroughly since the time it was written."

In the 1920s, Hardy and Littlewood realized that the "circle method" could be used as a tool for attacking other additive problems. They applied it to numerous such problems, including

- Waring's problem,
- Goldbach's conjecture, and
- Small gaps between primes,

The circle method is still an important tool in analytic number theory.