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A construction of regular magic squares of odd order



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ABSTRACT

A magic square is an $n \times n$ array of numbers whose rows, columns, and the two diagonals sum to μ . A regular magic square satisfies the condition that the entries symmetrically placed with respect to the center sum to $\frac{2\mu}{n}$. Using circulant matrices we describe a construction of regular classical magic squares that are nonsingular for all odd orders. A similar construction is given that produces regular classical magic squares that are singular for odd composite orders. This paper is an extension of [3].

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1. Introduction

A *magic square* M is an $n \times n$ matrix in which entries along each row, each column, the main diagonal, and the cross diagonal add to the same value μ called the *magic sum* of M . If the entries of M are integers from 1 through n^2 where each number appears once then $\mu = \frac{n(n^2+1)}{2}$ and M is called a *classical* magic square (or *natural* magic square).

A magic square $M = [m_{i,j}]$ is said to be *regular* (also called *associated* or *symmetrical*) if the sum of the entries $m_{i,j}$ and $m_{n+1-i,n+1-j}$ that are symmetrically placed across the center of the square is equal to the number $\frac{2\mu}{n}$. In the case of classical magic square this sum is $n^2 + 1$.

Dürer’s magic square

$$\begin{bmatrix} 16 & 3 & 2 & 13 \\ 5 & 10 & 11 & 8 \\ 9 & 6 & 7 & 12 \\ 4 & 15 & 14 & 1 \end{bmatrix}$$

is an example of a regular magic square [7]. In [5] Mattingly proved that every even order regular magic square is singular (that is, determinant of the magic square is zero). In [4] Loly et al. found that not all of the 5×5 regular classical magic squares are nonsingular. In [3] an example of a 9×9 regular classical magic square that is singular is given.

As a result the question of when an odd order regular magic square is singular or nonsingular was addressed in [3]. A necessary and sufficient condition for an odd order regular magic square to be nonsingular was given. In addition a method to construct nonsingular regular classical magic squares using circulant matrices is given when the order of the magic square is an odd prime power [3].

In this paper we extend this construction method of regular classical magic squares to all odd orders. Moreover, we show that this construction method will produce a singular or nonsingular regular classical magic square based on the choice of the first row of the circulant matrix used in the construction.

2. A construction of regular magic squares

In this section we present the method of construction used in [3] to produce regular classical magic squares.

Let E denote the matrix of all 1’s for its entries and e denote the column vector of all 1’s. Since $Me = \mu e$ we observe that the magic sum μ is an eigenvalue of magic square M . The following theorem is found in [1].

Theorem 2.1. *If M is an $n \times n$ magic square and ρ is a complex number, then $M + \rho E$ has the same eigenvalues of M except that μ is replaced with $\mu + \rho n$.*

Definition 2.2. If M is a regular magic square we define

$$Z = M - \frac{\mu}{n}E$$

to be the corresponding zero regular magic square.

From [Theorem 2.1](#) it follows that zero regular magic square has the same eigenvalues as M except that μ is replaced by 0.

Let J denote the permutation matrix obtained by writing 1 in each of the cross diagonal entries and 0 elsewhere. Since multiplying a matrix on the left by J reverses the order of the rows and multiplying on the right by J reverses the order of the columns we observe that an $n \times n$ matrix M is a regular magic square if and only if $M + JMJ = \frac{2\mu}{n}E$.

Definition 2.3. An $n \times n$ matrix B with real entries is said to be *centroskew* if $JB = -B$.

It is easy to verify that the zero regular magic square Z in [Definition 2.2](#) is a centroskew matrix. The method of construction used in [\[3\]](#) uses a special type of circulant matrix which is defined below. A matrix is said to be *circulant* if each row other than the first row is obtained from the preceding row by shifting entries cyclically one column to the right.

For the rest of the paper let n denote an odd integer and S denote the set

$$S = \left\{ -\frac{n-1}{2}, \dots, -1, 0, 1, \dots, \frac{n-1}{2} \right\}. \tag{1}$$

Definition 2.4. Let $\vec{a} = (a_1, a_2, \dots, a_n)$ be a list consisting of n distinct members from S in [\(1\)](#) and $a_1 = 0$. A circulant matrix A with its first row equal \vec{a} is called an S -circulant matrix.

The following two results are from [\[3\]](#):

1. Suppose A is an S -circulant matrix. Then A is a zero magic square.
2. Suppose A is an S -circulant matrix. Then A is centroskew if and only if

$$a_{j+1} + a_{n+1-j} = 0 \quad \text{for } j = 1, \dots, n-1.$$

Example 2.5. The following is an S -circulant matrix that is centroskew.

$$\begin{bmatrix} 0 & 1 & 2 & -2 & -1 \\ -1 & 0 & 1 & 2 & -2 \\ -2 & -1 & 0 & 1 & 2 \\ 2 & -2 & -1 & 0 & 1 \\ 1 & 2 & -2 & -1 & 0 \end{bmatrix}$$

A procedure to construct a regular classical magic square

Step 1: Let A be a centroskew S -circulant matrix of odd order n . Define $Z = nA + AJ$. Then Z is a centroskew zero magic square with n^2 distinct entries from the set

$$Q = \left\{ -\frac{n^2 - 1}{2}, \dots, -1, 0, 1, \dots, \frac{n^2 - 1}{2} \right\}. \tag{2}$$

Step 2: Let $M = Z + \frac{n^2+1}{2}E$. Then M is a regular classical magic square.

Using the above procedure it is shown in [3] that

1. $\text{rank}(Z) = \text{rank}(A)$,
2. if n is an odd prime then $\text{rank}(Z) = n - 1$ and M is nonsingular,
3. if $n = p^t$ where p is an odd prime and the first row of A is $\vec{a} = (a_1, a_2, \dots, a_n)$ with $a_j = j - 1$ for $j = 1, 2, \dots, \frac{n+1}{2}$, then $\text{rank}(Z) = n - 1$ and M is nonsingular, and
4. by using other first rows for A , examples of singular M were given for $n = 9$ and $n = 15$.

The construction method makes use of the following known facts [6, p. 243], [2, p. 33, 100] about circulant matrices whose first row is given by $\vec{a} = (a_1, a_2, \dots, a_n)$. If A is a circulant matrix then $A^*A = AA^*$, so that A is normal. Hence every circulant matrix is unitarily similar to diagonal matrix. Moreover the eigenvalues of the circulant matrix A are determined by the entries of the first row and are given by

$$\left\{ \sum_{j=0}^{n-1} a_{j+1}\omega^{kj} : k = 0, 1, \dots, n - 1 \text{ and } \omega = e^{\frac{2\pi i}{n}} \right\}. \tag{3}$$

If there is only one zero eigenvalue in (3) the above construction method will produce a nonsingular regular classical magic square. If (3) has more than one zero eigenvalue then the construction method will produce a singular regular classical magic square.

In Section 3 we provide construction of nonsingular regular classical magic squares of all odd order extending the results of [3]. In Section 4 we generalize our construction to include singular regular classical magic squares of odd order. Since the construction steps are outlined above we only mention the first row $\vec{a} = (a_1, a_2, \dots, a_n)$ of the centroskew S -circulant matrix A when giving examples.

3. Nonsingular regular magic squares

We utilize the construction in previous section to create nonsingular regular magic squares for all odd n . As seen before, the designation of the first row of matrix A determines its eigenvalues by (3).

For the remainder of the paper let $\text{Re}(r)$ be the real part of complex number r and let $\text{Im}(r)$ be the imaginary part of r . With this notation $r = \text{Re}(r) + i\text{Im}(r)$.

Define the first row of matrix A by $a_j = j - 1$ for $j = 1, \dots, \frac{n+1}{2}$ and assign $a_{n-j+1} = -a_{j+1}$ for $1 \leq j \leq n - 1$. Furthermore let ω be the n th root of unity, $\omega = e^{\frac{2\pi i}{n}}$.

For simplicity, we use the notation $E_n(x)$ to denote the polynomial; $E_n(x) = \sum_{j=0}^{n-1} a_{j+1}x^j$. As an example, if $n = 5$ then the associated matrix is the matrix given in [Example 2.5](#) and the polynomial is $E_5(x) = 0 + 1x + 2x^2 - 2x^3 - 1x^4$. The polynomial notation $E_n(x)$ allows us to rewrite the set in [\(3\)](#) as

$$\{E_n(\omega^k) : k = 0, 1, \dots, n - 1\}. \tag{4}$$

With the above definition for the first row of A ,

$$E_n(x) = \sum_{j=0}^{\frac{n-1}{2}} jx^j + \sum_{j=1}^{\frac{n-1}{2}} (-j)x^{n-j}. \tag{5}$$

Lemma 3.1. *If $E_n(x)$ and ω are defined as above, then $E_n(\omega) \neq 0$.*

Proof. We examine the number $E_n(\omega) = \sum_{j=0}^{\frac{n-1}{2}} j\omega^j + \sum_{j=1}^{\frac{n-1}{2}} (-j)\omega^{n-j}$. Since $\omega^n = 1$, we have $E_n(\omega) = \sum_{j=1}^{\frac{n-1}{2}} j(\omega^j - \omega^{-j})$. Note that $(\omega^j - \omega^{-j}) = 2i \text{Im}(\omega^j)$.

Therefore, $E_n(\omega) = i \sum_{j=1}^{\frac{n-1}{2}} 2j \text{Im}(\omega^j)$. The sum $\sum_{j=1}^{\frac{n-1}{2}} 2j \text{Im}(\omega^j)$ must be positive since $\text{Im}(\omega^j) > 0$ for $j = 1, 2, \dots, \frac{n-1}{2}$. So $E_n(\omega) \neq 0$. \square

Lemma 3.2. *Let k be a divisor of n where $k \neq n$. Then $E_n(\omega^k) \neq 0$.*

Proof. If $k = 1$, then we are done by [Lemma 3.1](#). For the remainder of the proof assume $k \neq 1$. Similar computations to the proof of [Lemma 3.1](#) show that

$$E_n(\omega^k) = i \sum_{j=0}^{\frac{n-1}{2}} 2j (\text{Im}(\omega^{kj})). \tag{6}$$

Let $\frac{n}{k} = l$. Then ω^k is a primitive l th root of unity. We therefore break up $E_n(\omega^k)$ in the following way:

$$\begin{aligned} E_n(\omega^k) &= \sum_{m=0}^{\frac{k-1}{2}-1} i \left\{ \sum_{j=ml+1}^{ml+\frac{l-1}{2}} 2j \text{Im}(\omega^{kj}) + \sum_{j=ml+\frac{l-1}{2}+1}^{(m+1)l-1} 2j \text{Im}(\omega^{kj}) \right\} \\ &\quad + i \sum_{j=(\frac{k-1}{2})l+1}^{(\frac{k-1}{2})l+\frac{l-1}{2}} 2j \text{Im}(\omega^{kj}). \end{aligned} \tag{7}$$

There are $\frac{n-1}{2} + 1$ terms in (6) and $\frac{n-1}{2} = \frac{k-1}{2}l + \frac{l-1}{2}$. The first summation over m in (7) gives $\frac{k-1}{2}l + 1$ terms and the second summation over j accounts for the remaining $\frac{l-1}{2}$ terms. We now show that $\text{Im}(E_n(\omega^k)) > 0$.

Notice that $-2 \text{Im}(\omega^{(m+1)l-z}) = 2 \text{Im}(\omega^{(m+1)l+z})$ for $z = 1, 2, \dots, \frac{l-1}{2}$.

Moreover, $2 \text{Im}(\omega^{(m+1)l+z}) > 0$ for $z = 1, 2, \dots, \frac{l-1}{2}$.

So we have that $2((m + 1)l + z) \text{Im}(\omega^{(m+1)l+z}) + 2((m + 1)l - z) \text{Im}(\omega^{(m+1)l-z}) = 2((m + 1)l + z) \text{Im}(\omega^{(m+1)l+z}) - 2((m + 1)l - z) \text{Im}(\omega^{(m+1)l+z}) = 4z \text{Im}(\omega^{(m+1)l+z}) > 0$.

This shows that each pair of sums, $\sum_{ml+\frac{l-1}{2}}^{(m+1)l-1} 2j \text{Im}(\omega^{kj}) + \sum_{(m+1)l+1}^{(m+1)l+\frac{l-1}{2}} 2j \text{Im}(\omega^{kj})$ is greater than zero for $m = 0, \dots, \frac{k-1}{2} - 1$. Moreover the first sum in $E_n(\omega^k)$, which is $\sum_{j=1}^{\frac{l-1}{2}} 2j \text{Im}(\omega^{kj})$, is also greater than zero. Therefore $\text{Im}(E_n(\omega^k)) > 0$ so $E_n(\omega^k) \neq 0$. \square

Corollary 3.3. For any $k = 1, \dots, n - 1$, $E_n(\omega^k) \neq 0$.

Proof. For any $k = 1, \dots, n - 1$, ω^k is a primitive l th root of unity for some $l \in \mathbb{Z}$ which divides n . Assume $\frac{n}{l} = k'$. Note that $k = ak'$ where $\text{gcd}(a, n) = 1$. There is an isomorphism ϕ from $\mathbb{Q}[\omega^{k'}]$ to $\mathbb{Q}[\omega^k]$, given by $\phi(1) = 1$ and $\phi(\omega^{k'}) = \omega^k$ and extended to be a ring homomorphism. If we apply this isomorphism to $E_n(\omega^{k'})$, we see that $\phi(E_n(\omega^{k'})) = E_n(\omega^k)$. From Lemma 3.2, $E_n(\omega^{k'}) \neq 0$, so we must have $E_n(\omega^k) \neq 0$. \square

Theorem 3.4. Let A be the S -circulant matrix defined by $a_j = j - 1$ for $1 \leq j \leq \frac{n+1}{2}$ and $a_{n-j+1} = -a_{j+1}$ for $1 \leq j \leq n - 1$. If $Z = nA + AJ$ and E is the all ones matrix, then $M = Z + \frac{n^2+1}{2}E$ is a regular classical magic square that is nonsingular.

Proof. By Theorem 2.1 the matrix M has the same eigenvalues as Z except the eigenvalue zero is replaced with $\frac{2\mu}{n}$. From Corollary 3.3, $\text{rank}(A) = n - 1$. The matrix Z has the property that $\text{rank}(A) = \text{rank}(Z)$. Therefore, $\text{rank}(M) = n$ and M is a nonsingular classical magic square based on the construction presented in Section 2. \square

For explanation, we include the following example.

Example 3.5. If $n = 35$, the first row of A becomes

$$[0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, -17, -16, -15, -14, -13, -12, -11, -10, -9, -8, -7, -6, -5, -4, -3, -2, -1].$$

This A will create a matrix M which is a nonsingular regular classical magic square using the process outlined in Section 2.

It is obvious that there are other ways to designate the first row of A that could also produce nonsingular regular magic squares. One example would be to negate the first row which we defined.

4. Singular regular magic squares of odd composite order

In [3] it was shown that any first row containing the integers from $-\frac{n-1}{2}$ through $\frac{n-1}{2}$ and having the properties that $a_1 = 0$ and $a_{j+1} = -a_{n-j+1}$ for $j = 1, 2, \dots, n-1$ would produce a nonsingular regular magic square when n is an odd prime. However, this is not true when n is an odd composite.

Let A be an $n \times n$ centroskew S -circulant matrix with first row $\vec{b} = (b_0, b_1, \dots, b_{n-1})$. We use A to create singular regular magic squares when n is an odd composite. To do this, we need to assign integers from $-\frac{n-1}{2}$ through $\frac{n-1}{2}$ and having the property that $b_j = -b_{n-j}$ for $j = 1, 2, \dots, n$ such that $E_n(\omega^k) = 0$ for some $1 \leq k < n$ where we recall as in Section 3 that $\omega = e^{\frac{2\pi i}{n}}$ and $E_n(x) = \sum_{j=0}^{n-1} b_j x^j$.

Example 4.1. Begin with the example of $n = 35 = 5 \cdot 7$. Denote the first row of A by $[b_0, \dots, b_{34}]$. Examine $E_{35}(\omega^7)$. Notice that ω^7 is a primitive 5th root of unity.

$$E_{35}(\omega^7) = \sum_{j=0}^{34} (b_j \omega^{7j}) = \sum_{j=0}^4 \left(\sum_{i=0}^6 b_{5i+j} \right) \omega^{7j}.$$

Therefore, if each sum $\sum_{i=0}^6 (b_{5i+j})$ is zero, then the eigenvalue is zero. Due to the fact that $b_j = -b_{n-j}$, there are restrictions on what integers you can assign the b_j 's. The following two facts follow from these restrictions:

- $\sum_{i=0}^6 (b_{5i+0})$ must be zero since $b_0 = 0$ and $-b_{5i} = b_{35-5i} = b_{5(7-i)}$.
- $\sum_{i=0}^6 (b_{5i+j}) = -\sum_{i=0}^6 (b_{5i+(7-j)})$ since $-b_{5i+j} = b_{35-5i-j} = b_{5(7-i)-j}$.

For $j = 0, 1, 2, 3, 4$ denote $B_j = \{b_{5i+j} : 0 \leq i \leq 6\}$. From the previous facts, if $b \in B_j$ then $-b \in B_{5-j}$ for $j > 0$. Therefore, if we show that $\sum_{i=0}^6 (b_{5i+1}) = 0$ and $\sum_{i=0}^6 (b_{5i+2}) = 0$, then we have that $E_{35}(\omega^7) = 0$.

We simultaneously assign integers into B_1 and B_2 . We begin by placing integers in pairs which add to 1. Specifically, we place the numbers 17 and -16 in B_1 and 15 and -14 in B_2 . Then we place pairs of integers which add to -1 in each set. To do this we place -13 and 12 into B_1 and -11 and 10 into B_2 . After these elements are placed, the sum of the four elements is zero, so we must place three more numbers which add to zero in B_1 and three numbers which add to zero in B_2 .

All numbers in B_1 and B_2 have distinct absolute values. So we place their opposites in B_3 and B_4 . There are seven unassigned numbers from -17 to 17 not already assigned to B_1 through B_4 . Among these are 0 and three pairs of opposite numbers. We put the remaining seven numbers in B_0 . The complete sets B_0, B_1, B_2, B_3 , and B_4 are as follows:

- $B_0 = \{0, 3, 4, 6, -3, -4, -6\}$
- $B_1 = \{17, -16, -13, 12, 9, -8, -1\}$
- $B_2 = \{15, -14, -11, 10, 7, -5, -2\}$

- $B_3 = \{2, 5, -7, -10, 11, 14, -15\}$
- $B_4 = \{1, 8, -9, -12, 13, 16, -17\}$

Therefore, the first row of matrix A is

$$[0, 17, 15, 2, 1, 3, -16, -14, 5, 8, 4, -13, -11, -7, -9, 6, 12, 10, -10, -12, -6, 9, 7, 11, 13, -4, -8, -5, 14, 16, -3, -1, -2, -15, -17].$$

If we do this, then we have $\sum_{i=0}^6 b_{5i+j} = 0$ for each $0 \leq j \leq 4$ and at least two eigenvalues of A are zero. Since multiple eigenvalues of A are zero, the regular classical magic square M one gets by following the process in Section 2 would be singular.

We may apply a similar labeling to any composite odd number. This is done in Lemma 4.2.

Lemma 4.2. *Let $n = n_1n_2$ where n_1 and n_2 are odd integers greater than one. Then there exists an $n \times n$ centroskew S -circulant matrix A such that its first row $\vec{b} = (b_0, b_1, \dots, b_{n-1})$ gives the property that $E_n(\omega^{n_2}) = 0$.*

Proof. We have that if ω is a primitive n th root of unity then ω^{n_2} is a primitive n_1 th root of unity. We examine the eigenvalue $E_n(\omega^{n_2})$.

$$E_n(\omega^{n_2}) = \sum_{j=0}^{n-1} (b_j \omega^{jn_2}) = \sum_{j=0}^{n_1-1} \left(\sum_{i=0}^{n_2-1} b_{n_1i+j} \right) \omega^{n_2j}.$$

Let $B_j = \{b_{n_1i+j} : 0 \leq i \leq n_2 - 1\}$, the set of coefficients for ω^{n_2j} , for $0 \leq j \leq n_1 - 1$. So long as the elements of B_j add to zero for each $0 \leq j \leq n_1 - 1$, then $E_n(\omega^{n_2}) = 0$.

Note that since $b_0 = 0$ and $b_j = -b_{n-j}$ we must have two properties on the B_j 's:

- The elements of B_0 sum to zero since $b_{n_1i} = -b_{n-n_1i} = -b_{n_1(n_2-i)}$.
- The elements of B_j are the opposite of the elements in B_{n_1-j} for $1 \leq j \leq n_1 - 1$. This is due to the fact that $b_{n_1i+j} = -b_{n-n_1i-j} = -b_{n_1(n_2-i)-j}$.

To simplify the notation let $m = \frac{n-1}{2}$. Each b_j must be an integer between m and $-m$. We construct each B_j for $1 \leq j \leq \frac{n_1-1}{2}$ in the following way.

Into each B_j for $1 \leq j \leq \frac{n_1-1}{2}$ put the elements $(-1)^i(m - i(n_1 - 1) - j + 1)$ and $(-1)^{i+1}(m - i(n_1 - 1) - j)$ for $0 \leq i \leq \frac{n_2-5}{2}$. When i is even each pair of numbers sum to 1 and when i is odd the pair of numbers sum to -1 . This assignment of elements places a total of $n_2 - 3$ elements into each B_j . Moreover if $n_2 \equiv 1 \pmod 4$ the sum of all $n_2 - 3$ elements placed in B_j thus far is 1. If $n_2 \equiv 3 \pmod 4$ the sum of all $n_2 - 3$ elements placed in B_j thus far is 0. Therefore the placement of the remaining three elements depends on

the value of n_2 . We break the assignment for the last three elements based on whether $n_2 \equiv 1 \pmod 4$ or if $n_2 \equiv 3 \pmod 4$.

Case 1: $n_2 \equiv 1 \pmod 4$

Since $n_2 \equiv 1 \pmod 4$, the previous $\frac{n_2-3}{2}$ pairs of elements in B_j must sum to 1.

- Into B_1 place the elements $-(m - (\frac{n_2-3}{2})(n_1 - 1))$, $(m - (\frac{n_2-3}{2})(n_1 - 1) - 2)$, and 1
- Into B_2 place the elements $-(m - (\frac{n_2-3}{2})(n_1 - 1) - 1)$, $(m - (\frac{n_2-3}{2})(n_1 - 1) - 4)$, and 2
- Into B_3 place the elements $-(m - (\frac{n_2-3}{2})(n_1 - 1) - 3)$, $(m - (\frac{n_2-3}{2})(n_1 - 1) - 7)$, and 3
- Into B_4 place the elements $-(m - (\frac{n_2-3}{2})(n_1 - 1) - 5)$, $(m - (\frac{n_2-3}{2})(n_1 - 1) - 10)$, and 4
- \vdots
- Into $B_{\frac{n_1-1}{2}}$ place the elements $-(m - (\frac{n_2-3}{2})(n_1 - 1) - k)$, $(m - (\frac{n_2-3}{2})(n_1 - 1) - k - \frac{n_1-1}{2} - 1)$, and $\frac{n_1-1}{2}$ where k is chosen such that $|-(m - (\frac{n_2-3}{2})(n_1 - 1) - k)|$ is the highest value of any elements not previously used in a B_j .

Each triple adds to -1 so that in this case the sum of all elements in each B_j sum to zero.

Case 2: $n_2 \equiv 3 \pmod 4$

Since $n_2 \equiv 3 \pmod 4$, the previous $\frac{n_2-3}{2}$ pairs of elements in B_j must sum to 0.

- Into B_1 place the elements $(m - (\frac{n_2-3}{2})(n_1 - 1))$, $-(m - (\frac{n_2-3}{2})(n_1 - 1))$, and -1
- Into B_2 place the elements $(m - (\frac{n_2-3}{2})(n_1 - 1) - 2)$, $-(m - (\frac{n_2-3}{2})(n_1 - 1) - 4)$, and -2
- Into B_3 place the elements $(m - (\frac{n_2-3}{2})(n_1 - 1) - 3)$, $-(m - (\frac{n_2-3}{2})(n_1 - 1) - 6)$, and -3
- Into B_4 place the elements $(m - (\frac{n_2-3}{2})(n_1 - 1) - 5)$, $-(m - (\frac{n_2-3}{2})(n_1 - 1) - 9)$ and -4
- \vdots
- Into $B_{\frac{n_1-1}{2}}$ place the elements $(m - (\frac{n_2-3}{2})(n_1 - 1) - k)$, $-(m - (\frac{n_2-3}{2})(n_1 - 1) - k - \frac{n_1-1}{2})$, and $\frac{n_1-1}{2}$ where k is chosen so that $|-(m - (\frac{n_2-3}{2})(n_1 - 1) - k)|$ is the highest value of any elements not previously used in a B_j .

Each triple adds to 0 so that in this case the sum of all elements in each B_j sum to zero.

In both cases, no integers in B_1 through $B_{\frac{n_1-1}{2}}$ have the same absolute value. Therefore we may place their opposites in B_{n_1-1} through $B_{\frac{n_1+1}{2}}$. There are n_2 remaining elements not used in any B_j for $j \geq 1$. These elements consist of zero and pairs of oppo-

site numbers. Place these elements into B_0 . Therefore, the numbers in B_0 must sum to zero.

This assignment of b_i 's gives the sum of elements in each B_j is zero, namely $\sum_{i=0}^{n_2-1} b_{n_1+i+j} = 0$, therefore $E_n(\omega^{n_2}) = \sum_{j=0}^{n_1-1} (\sum_{i=0}^{n_2-1} b_{n_1+i+j}) \omega^{n_2j} = 0$. \square

Theorem 4.3. *Assume that n is a composite positive odd integer and E is the all ones matrix. Let $Z = nA + AJ$ where A is an $n \times n$ matrix as obtained in Lemma 4.2. Then $M = Z + \frac{n^2+1}{2}E$ is a regular classical magic square that is singular.*

Proof. Designating the first row of the $n \times n$ matrix A as described in Lemma 4.2 creates an S -circulant matrix A which has at least 2 eigenvalues which are zero. This means that $\text{rank}(A) = \text{rank}(Z) \leq n - 2$. If we let $M = Z + \frac{n^2+1}{2}E$ then $\text{rank}(M) \leq n - 1$ so M is singular. \square

Remark 4.4. There are other assignments to the first row of A that will produce a singular M . For example, there are other ways one may permute the elements of the B_j 's to also get zero for each summation of elements in B_j .

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