Photonic Crystal Fiber Analysis Using Cylindrical FDTD with Bloch Boundary Conditions

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Outline

- Introduction to FDTD methods for rotationally invariant geometries
- Spatially looped boundary conditions for structures with discrete rotational invariance
- Demonstration and application of the FDTD method to photonic crystal fiber
- Future work on grid density
Finite-difference time-domain method

Source-free Maxwell's curl equations

\[ \frac{\partial \vec{D}}{\partial t} = \nabla \times \vec{H} \]

\[ \frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E} \]

- Explicit spatial and temporal discretization
- Staggered spatial arrangement
- Second order accuracy
- Linear scaling in problem size
- Easy to implement and easily parallelizable
- General approach: applies to arbitrary geometry
- Can incorporate nonlinear effects
FDTD in cylindrical coordinates

For structures with continuous azimuthal invariance:

\[ \epsilon(r, \phi + \Lambda, z) = \epsilon(r, \phi, z) \]

Assume functional form \( F(r, \phi, z) = f_m(r, z)e^{im\phi} \)

for 6 field components with integer \( m \) specified by user.

Derivatives with respect to \( \phi \) can be evaluated analytically:

\[ \frac{\partial}{\partial \phi} F(r, \phi, z) = imf_m(r, z)e^{im\phi} \]

FDTD analysis is required only for the two dimensional field quantity: \( f_m(r, z) \)

FDTD in cylindrical coordinates

For structures with azimuthal invariance:

Assume functional form \( F(r, \phi, z) = f_m(r, z)e^{im\phi} \)

for 6 field components with integer \( m \) specified by user

\[
\begin{align*}
\frac{\partial D_r^{i,j+1/2}}{\partial t} & = \frac{1}{\Delta z} (H_{\phi}^{i,j} - H_{\phi}^{i,j+1}) - \frac{m}{r_0} \frac{H_z^{i,j+1/2}}{r} \\
\frac{\partial B_r^{i,j+1/2}}{\partial t} & = \frac{1}{\Delta z} (E_{\phi}^{i,j+1} - E_{\phi}^{i,j}) - \frac{m}{r_1} \frac{E_z^{i,j+1/2}}{r} \\
\frac{\partial D_{\phi}^{i+1/2,j+1/2}}{\partial t} & = \frac{1}{\Delta r} (H_z^{i+1/2} - H_z^{i+1,j+1/2}) + \frac{1}{\Delta z} (H_{r}^{i+1/2,j+1} - H_{r}^{i+1/2,j}) \\
\frac{\partial B_{\phi}^{i+1/2,j+1/2}}{\partial t} & = \frac{1}{\Delta r} (E_z^{i+1,j+1/2} - E_z^{i+1/2,j+1/2}) + \frac{1}{\Delta z} (E_{r}^{i+1/2,j} - E_{r}^{i+1/2,j+1}) \\
\frac{\partial D_z^{i+1/2,j}}{\partial t} & = \frac{2r_2}{r_2^2 - r_1^2} H_{\phi}^{i+1,j} - \frac{2r_1}{r_2^2 - r_1^2} H_{\phi}^{i,j} + \frac{2m\Delta r}{r_2^2 - r_1^2} H_{r}^{i+1/2,j} \\
\frac{\partial B_z^{i+1/2,j}}{\partial t} & = \frac{2r_2}{r_2^2 - r_1^2} E_{\phi}^{i,j} - \frac{2r_1}{r_2^2 - r_1^2} E_{\phi}^{i+1,j} + \frac{2m\Delta r}{r_2^2 - r_1^2} E_{r}^{i+1/2,j}
\end{align*}
\]

Cylindrical coordinates: discrete rotational invariance

Russell, JLT, 24 4729 2006

Fujita and Baba, APL, 80 2051 2002

Tarwara, et al., Optics Express, 16 5199 2008

Chen and Towe, JSTQE, 12 117 2006
Cylindrical coordinates: discrete rotational invariance

Can we apply similar techniques to reduce the computational domain size for structures with discrete rotational invariance?

Photonic crystal fiber

Air holes along the length of the fiber

Applications include:
- sensing
- nonlinear optics
- high power beam delivery
- supercontinuum generation

User can control modal properties by using different cross-sectional geometries

Confinement due to total internal reflection and Bragg reflection: Numerical method is necessary for accurate analysis.
Spatially looped boundary conditions

The strategy will be to apply spatially looped boundary conditions.

\[ e^{\beta a} \]

\[ \begin{align*}
\epsilon(x, y, z + a) &= \epsilon(x, y, z) \\
F(x, y, z + a) &= e^{i\beta a} F(x, y, z)
\end{align*} \]

\[ \epsilon(r, \phi + \Lambda, z) = \epsilon(r, \phi, z) \]

\[ \beta \text{ values span 0 to } \pi \]

Consider the functional form used for structures with continuous rotational invariance

\[ F_m(r, \phi, z) = f_m(r, z)e^{im\phi} \]

Define an operator \( \Phi(\Lambda) \) that rotates a function about the \( z \) axis by an angle \( \Lambda \).

\[ \Phi(\Lambda)e(r, \phi, z) = e(r, \phi + \Lambda, z) = e(r, \phi, z) \]

Therefore, we expect the solutions to Maxwell's equations to be eigenfunctions of the rotation operator \( \Phi(\Lambda) \)

\[ \Phi(\Lambda)F_m(r, \phi, z) = F_m(r, \phi + \Lambda, z) = f_m(r, z)e^{im(\phi + \Lambda)} = f_m(r, z)e^{im\Lambda}e^{im\phi} = e^{im\Lambda}F_m(r, \phi, z) \]
Therefore, $F_m$ is an eigenfunction of $\Phi(\Lambda)$ with eigenvalue $e^{im\Lambda}$

$$\Phi(\Lambda)F_m(r, \phi, z) = e^{im\Lambda}F_m(r, \phi, z)$$

Substituting $m + n2\pi/\Lambda$ for $m$ above yields an eigenvalue problem yielding the same eigenvalue $e^{im\Lambda}$

Therefore, we construct superpositions of these degenerate eigenfunctions by summing over $n$

$$\tilde{F}_m(r, \phi, z) = \sum_n F_{m+n2\pi/\Lambda}(r, \phi, z)$$

$$= \sum_n f_{m+n2\pi/\Lambda}(r, z)e^{i(m+n\frac{2\pi}{\Lambda})\phi}$$

**Fourier series representation of a function periodic in $\phi$ with period $\Lambda$**

$$= e^{im\phi}\sum_n f_{m+n2\pi/\Lambda}(r, z)e^{in\frac{2\pi}{\Lambda}\phi}$$
TM Field solutions in spherical coordinates

\[ \tilde{F}_m(r, \phi, z) = e^{im\phi} \sum_{n} f_{m+n2\pi/\Lambda}(r, z)e^{in\frac{2\pi}{\Lambda} \phi} \]

\[ \tilde{F}_m(r, \phi, z) = e^{im\phi}u(r, \phi, z) \quad \text{with} \quad u(r, \phi + \Lambda, z) = u(r, \phi, z) \]

These results are similar to Bloch's theorem for electrons in a periodic potential.

Bloch's theorem applies to structures with angular periodicity.

\[ \tilde{F}_m(r, \phi, z) = e^{im\phi}u(r, \phi, z) \]

\[ u(r, \phi + \Lambda, z) = u(r, \phi, z) \]

\[ \tilde{F}_m(r, \phi + \Lambda, z) = e^{im\Lambda} \tilde{F}_m(r, \phi, z) \]

\( m \): user specified integer
Photonic crystal fiber analysis

For analysis of waveguide structures uniform along the propagation direction, “compact” FDTD can be used

\[ \tilde{F}_m(r, \phi, z) = f_m(r, \phi)e^{j\beta z} \]

\[ = e^{im\phi}u(r, \phi)e^{j\beta z} \]

With compact FDTD, a 3D problem is solved using 2D computational arrays.

With the cylindrical Bloch form derived here, the 2D computational arrays are reduced by a factor

\[ n = \frac{2\pi}{\Lambda} = \frac{2\pi}{(\pi/3)} \]

\[ = 6 \text{ in this case} \]

Qiu, Microwave and Optical Technology Letters, 30 327 2001
Chen and Mittra, Microwave and Optical Technology Letters, 15 201 1997
Photonic crystal fiber analysis

Compact FDTD in cylindrical coordinates with spatially looped boundary conditions. User specifies $\beta$ and $m$ for each program run.

For a lattice constant ($a$) near 0.5 $\mu$m, the operating wavelength corresponding to $\beta a = 3.0$ is near $\lambda = 1.5$ $\mu$m which is the low loss fiber transmission wavelength.
FDTD in cylindrical coordinates: advantages compared to cartesian coordinates

“Staircase error” is reduced as cylindrical discretization conforms naturally to structures with cylindrical surfaces.
FDTD in cylindrical coordinates: advantages compared to cartesian coordinates

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Computational domain in corner regions is wasted in cartesian coordinates as minimum domain size is defined by distance between device and the adjacent computational boundary.
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Spatially looped boundary conditions can be easily applied.
FDTD in cylindrical coordinates: disadvantages compared to cartesian coordinates

Nonuniform grid spacing:
uniform along $r$: $\Delta r$
nonuniform along $\phi$: $\Delta s = i\Delta r\Delta \phi$

The grid spacing increases as radius increases:
- high resolution at the origin
- low origin at larger radii
FDTD in cylindrical coordinates: disadvantages compared to cartesian coordinates

Maximum stable time step is limited by smallest grid spacing:

$$\Delta t < \frac{1}{c \sqrt{\left(\frac{1}{\Delta r}\right)^2 + \left(\frac{2}{\Delta r \Delta \phi}\right)^2 + \left(\frac{\beta}{2}\right)^2}}$$

dominant term

$$\Delta t < \frac{\Delta r \Delta \phi}{c \sqrt{2}}$$

cylindrical FDTD

$$\Delta t < \frac{\Delta x}{c \sqrt{3}}$$

cartesian FDTD

For $\Delta \phi = \pi/100$, time step for cylindrical FDTD is 1/30 that of cartesian FDTD

Dib et al. IEEE Trans. MTT 47 509 1999
FDTD in cylindrical coordinates: disadvantages compared to cartesian coordinates

Maximum stable time step is limited by smallest grid spacing:

\[ \Delta t < \frac{1}{c \sqrt{\left(\frac{1}{\Delta r}\right)^2 + \left(\frac{2}{\Delta r \Delta \phi}\right)^2 + \left(\frac{\beta}{2}\right)^2}} \]

Dominant term

(Dib et al. IEEE Trans. MTT 47 509 1999)

Proposed solution:

- compensate lower resolution with more grid points
- choose a course $\Delta \phi$ and then increase grid density as radius increases

(Yu and Mittra. IEEE Trans. MTT 47 353 1999)
Conclusions

- Derived spatially looped boundary condition applicable to structures with discrete rotational invariance.

- Fully vectorial 3D analysis of photonic crystal fibers using 2D grid and a factor of 6 reduction in cross section size.

- Primary drawbacks: decreasing resolution at large radius and small time step – use additional grid points to overcome.

Russell, JLT, 24 4729 2006