Consider the *RL*-series circuit shown in the figure below, which contains a counterclockwise current $I = I(t)$, a resistance $R$, and inductance $L$, and a generator that supplies a voltage $V(t)$ when the switch is closed. Kirchhoff’s voltage law states that, “the algebraic sum of all voltage drops around an electric circuit is zero.” Applied to this *RL*-series circuit, the statement translates to the fact that the current $I = I(t)$ in the circuit satisfies the first-order linear differential equation

$$L\frac{dI}{dt} + RI = V(t),$$

where $\dot{I}$ denotes the derivative of $I$ with respect to $t$. Substituting $\frac{dI}{dt}$ for $\dot{I}$ and dividing by $L$, we obtain our usual form of a first-order nonhomogeneous equation:

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{1}{L}V(t).$$

For convenience, we present an index of units and symbols commonly used in this theory.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Unit</th>
<th>Symbol</th>
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</thead>
<tbody>
<tr>
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<td>ohm</td>
<td>$R$</td>
</tr>
<tr>
<td>Inductance</td>
<td>henry</td>
<td>$L$</td>
</tr>
<tr>
<td>Capacitance</td>
<td>farad</td>
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</tr>
<tr>
<td>Voltage</td>
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<tr>
<td>Current</td>
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<td>Charge</td>
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<td>$Q$</td>
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<tr>
<td>Time</td>
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</tbody>
</table>

*Table 1: Table of Units*
Exercises

1. Use the method of integration factors to calculate what the general solution is for this differential equation.

Solution. The integrating factor is \( \mu(t) = e^{\int \frac{R}{L} \, dt} = e^{\frac{R}{L} t} \), so we have

\[
I(t) = \frac{1}{L} e^{-\frac{R}{L} t} \int V(t) e^{\frac{R}{L} t} \, dt.
\]

In particular, if \( V(t) = V_0 \) is some constant voltage, then we may evaluate the integral on the right directly to get

\[
I(t) = \frac{1}{L} e^{-\frac{R}{L} t} \int V_0 e^{\frac{R}{L} t} \, dt = \frac{1}{L} e^{-\frac{R}{L} t} \left( \frac{L}{R} V_0 e^{\frac{R}{L} t} + c \right) = \frac{V_0}{R} + ce^{-\frac{R}{L} t}.
\]

2. In an \( RL \)-series circuit, \( L = 4 \) henries, \( R = 5 \) ohms, \( V = 8 \) volts, and \( I(0) = 0 \) amperes. Find the current at the end of 0.1 seconds. What will the current be after a very long time?

Solution. Using the formula from the last solution,

\[
I(t) = \frac{1}{L} e^{-\frac{R}{L} t} \int 8 e^{\frac{R}{L} t} \, dt = \frac{1}{L} e^{-\frac{R}{L} t} \left( \frac{32}{5} e^{\frac{R}{L} t} + c \right) = \frac{8}{5} + ce^{-\frac{R}{L} t}.
\]

Using the initial condition \( I(0) = 0 \), we get \( c = -\frac{8}{5} \), so

\[
I(t) = \frac{8}{5} \left( 1 - e^{-\frac{R}{L} t} \right).
\]

Then \( I(0.1) \approx 0.188 \) and \( I(t) \to \frac{8}{5} \) as \( t \to \infty \).

3. Assume that the voltage \( V(t) \) is given by \( V_0 \sin(\omega t) \), where \( V_0 \) and \( \omega \) are given constants. Find the solution of the differential equation above subject to the initial condition \( I(0) = I_0 \).

Solution. I’ll give the argument using the method of undetermined coefficient here, as it is different from the method done in class and it represents something new that you learned. So a solution to this DE is of the form \( y_h(t) + y_p(t) \). First,

\[
I_h(t) = e^{-\int \frac{R}{L} \, dt} = ke^{-\frac{R}{L} t}.
\]
Then, to find the particular solution $I_p(t)$, we try $I_p(t) = \alpha \cos(\omega t) + \beta \sin(\omega t)$. Substituting this into the DE, multiplying through by $L$ to get rid of the fractions, and combining like terms, we get

$$(\beta \omega L + R\alpha) \cos(\omega t) + (-\alpha \omega L + R\beta) \sin(\omega t) = V_0 \sin(\omega t).$$

Thus we have the system

$$R\alpha + \beta \omega L = 0$$
$$-\alpha \omega L + R\beta = V_0.$$

The first equation immediately gives $\alpha = -\beta \omega LR^{-1}$. Then, plugging this into the second equation gives

$$\beta \omega L \frac{\omega L + R\beta}{R} = V_0$$
$$\beta \omega L = \frac{V_0 R}{\omega^2 L^2 + R^2}.$$

Then

$$\alpha = -\beta \omega LR^{-1} \frac{\omega L}{R} = -\frac{V_0 R \omega L}{\omega^2 L^2 + R^2}.$$

Hence

$$I(t) = k e^{-\frac{\omega L}{R} t} - \frac{V_0 \omega L}{\omega^2 L^2 + R^2} \cos(\omega t) + \frac{V_0 R}{\omega^2 L^2 + R^2} \sin(\omega t).$$

It remains to solve for $k$ using the initial condition $I(0) = I_0$. We have

$$I(0) = I_0 = k - \frac{V_0 \omega L}{\omega^2 L^2 + R^2} \Rightarrow k = I_0 + \frac{V_0 \omega L}{\omega^2 L^2 + R^2}.$$

Thus

$$I(t) = I_h(t) + I_p(t) = \left(I_0 + \frac{V_0 \omega L}{\omega^2 L^2 + R^2}\right) e^{-\frac{\omega L}{R} t} - \frac{V_0 \omega L}{\omega^2 L^2 + R^2} \cos(\omega t) + \frac{V_0 R}{\omega^2 L^2 + R^2} \sin(\omega t).$$

4. Given that $L = 3$ henries, $R = 6$ ohms, $V(t) = 3 \sin(t)$, and $I(0) = 10$ amperes, compute the value of the current at any time $t$.

Solution. Using the equation from above, we have $V_0 \omega L = 3 \cdot 3 = 9$, $V_0 R = 3 \cdot 6 = 18$, and $\omega^2 L^2 + R^2 = 1^2 3^2 + 6^2 = 45$, so

$$I(t) = \frac{459}{45} e^{-\frac{\omega L}{R} t} - \frac{1}{5} \cos(t) + \frac{2}{5} \sin(t).$$

References