Strongly Regular Cayley Graphs in Rank Two Abelian Groups

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Strongly regular graphs play a central role in algebraic combinatorics, providing a graph-theoretic foundation for a variety of incidence structures including generalized quadrangles and quasi-symmetric designs.

In this talk we explore a question raised by Jim Davis and John Polhill, examining parameters of strongly regular graphs which can be described as Cayley graphs in $C_{p^n} \times C_{p^n}$.

The work begins with a close look at characters of abelian groups and their group rings. These algebraic tools are then translated into efficient algorithms for finding Cayley graphs with prescribed parameter sets.
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Strongly regular graphs “stand on the cusp between the random and the highly structured”, says combinatorialist Peter Cameron.

Constructions of strongly regular graphs often use finite fields & so many strongly regular graphs can be defined as a Cayley graph in which the underlying group is elementary abelian.

Here we are interested in abelian groups with high exponent, in particular, $G = C_{p^n} \times C_{p^n}$.

The study of Cayley graphs uses algebraic techniques such as group representations and group homomorphisms. If the underlying group is abelian, the characters form an isomorphic group and one may then construct a dual graph.

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A graph is **regular** of degree $k$ if the degree of every vertex is $k$.

A regular graph with $v$ vertices and degree $k$ is **strongly regular** with parameters $(v, k, \lambda, \mu)$ if the number of paths of length two between two vertices $x$ and $y$ is dependent only on whether $x$ is adjacent to $y$.

The number of paths of length two from $x$ to $y$ is

$$\lambda \text{ if } x \sim y$$

and

$$\mu \text{ if } x \not\sim y$$
Regular graphs and strongly regular graphs

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$$\lambda \text{ if } x \sim y$$

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If $x$ and $y$ are adjacent, $\lambda$ counts the number of vertices $z$ in this configuration:

$$
\begin{array}{c}
\text{x} \\
\text{z} \\
\text{y}
\end{array}
$$

while if $x$ and $y$ are not adjacent, $\mu$ counts the number of vertices $z$ in this configuration:

$$
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We give the parameters as $(v, k, \lambda, \mu)$ where $v$ is the total number of vertices and $k$ is the degree of each vertex.
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![Diagram 1](image1)

while if $x$ and $y$ are not adjacent, $\mu$ counts the number of vertices $z$ in this configuration:

![Diagram 2](image2)

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SRG $\langle 9, 4, 1, 2 \rangle$
The Shrikhande graph...
and the $4 \times 4$ Rook graph (lattice graph):
Fix a vertex $x$ and count ordered pairs $(y, z)$ such that $y \sim z$ and $x \sim y$ but $x \not\sim z$, so that the vertices $y$ and $z$ induce the subgraph below.

$$k(k - \lambda - 1) = (v - k - 1)\mu.$$  \hspace{1cm} (1)

This is our first feasibility condition for the strongly regular graph parameters.
The first feasibility condition for strongly regular graphs

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(1)

This is our **first feasibility condition** for the strongly regular graph parameters.
Strongly regular graphs have a particularly interesting spectrum!

If $A$ is the adjacency matrix of a $(v, k, \lambda, \mu)$ strongly regular graph then

$$A^2 = kI + \lambda A + \mu(J - I - A).$$

Rewrite this:

$$A^2 - (\lambda - \mu)A - (k - \mu)I = \mu J.$$

If $\vec{v}$ is an eigenvector with eigenvalue $\theta$ different from $k$ then

$$\theta^2 - (\lambda - \mu)\theta - (k - \mu) = 0.$$
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\[
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\]

1. (9, 4, 1, 2) has eigenvalue \( k = 4 \) and the others are roots of

\[
\theta^2 + \theta - 2 = 0.
\]

So \( \theta = 1, -2 \)

2. (16, 5, 0, 2) has eigenvalue \( k = 5 \) and the others are roots of

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The eigenvalues of the (16, 6, 2, 2) Lattice Graph
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$$A = \begin{pmatrix} J - I & I & I & I \\ I & J - I & I & I \\ I & I & J - I & I \\ I & I & I & J - I \end{pmatrix}$$

has eigenvalues $k = 6$ and the roots of $x^2 - 4$:

$$6, 2, 2, 2, 2, 2, 2, -2, -2, -2, -2, -2, -2, -2, -2, -2.$$
The eigenvalues of the \((16, 6, 2, 2)\) Lattice Graph

\[ A = \begin{pmatrix} 
J - I & I & I & I \\
I & J - I & I & I \\
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The $(16, 6, 2, 2)$ Lattice Graph as a Cayley Graph
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The \( (16, 6, 2, 2) \) Lattice Graph as a Cayley Graph

\[
G = \mathbb{Z}_4 \oplus \mathbb{Z}_4, \quad S = \{10, 20, 30, 01, 02, 03\}
\]
The \((16, 6, 2, 2)\) Lattice Graph as a Cayley Graph

\[ G = C_4 \times C_4, \quad S = x + x^2 + x^3 + y + y^2 + y^3 \]
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Recall

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The general problem

We seek strongly regular Cayley graphs (partial difference sets) with $m^2$ vertices and degree $k = r(m - 1)$ or $k = r(m + 1)$ (for various values of $r$) in abelian groups with exponent higher than that given by structures in the additive group of a field.

In particular, we are interested in groups of order $p^{2n}$ with exponent higher than $p$. The case $p = 2$ seems to be distinct from the odd prime cases.

Jim Davis (U. Richmond), John Polhill (Bloomsburg U.) and others have constructed PDS in direct products of cyclic groups of order 4.

The highest that the exponent could be is $2^n$.

(Polhill & Davis) Can we construct PDS in the group $G = C_{2^n} \times C_{2^n}$?
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The characters of $C_4 \times C_4$ and the Lattice Graph

The 16 elements of $C_4 \times C_4$ can be mapped into $\mathbb{C}$ in 16 different ways!

<table>
<thead>
<tr>
<th>$\chi$</th>
<th>$x$</th>
<th>$y$</th>
<th>( S = (x + x^2 + x^3) + (y + y^2 + y^3) )</th>
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<tbody>
<tr>
<td>$\chi_{0,0}$</td>
<td>1</td>
<td>1</td>
<td>$3 + 3 = 6$</td>
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<tr>
<td>$\chi_{1,0}$</td>
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Let's focus on the characters which map $S$ to 2.

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<td>$3 + 3 = 6$</td>
</tr>
<tr>
<td>$\chi_{1,0}$</td>
<td>$i$</td>
<td>1</td>
<td>$-1 + 3 = 2$</td>
</tr>
<tr>
<td>$\chi_{2,0}$</td>
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<td>$i$</td>
<td>$3 - 1 = 2$</td>
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<td>$i$</td>
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</table>
The dual Cayley graph

\[ S^* = (\chi_{1,0} + \chi_{2,0} + \chi_{3,0}) + (\chi_{0,1} + \chi_{0,2} + \chi_{0,3}) \text{ in } G^*. \]

<table>
<thead>
<tr>
<th>\chi</th>
<th>x</th>
<th>y</th>
<th>[ S = (x + x^2 + x^3) + (y + y^2 + y^3) ]</th>
</tr>
</thead>
<tbody>
<tr>
<td>\chi_{0,0}</td>
<td>1</td>
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</tr>
<tr>
<td>\chi_{1,0}</td>
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<tr>
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<td>-\text{i}</td>
<td>[ -1 - 1 = -2 ]</td>
</tr>
</tbody>
</table>
Here are the values of the characters of $C_4 \times C_4$ on the Shrikhande Cayley Graph.

<table>
<thead>
<tr>
<th>$\chi$</th>
<th>$x$</th>
<th>$y$</th>
<th>$S = (x + x^3) + (y + y^3) + (xy + x^3y^3)$</th>
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<tbody>
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The dual of the Shrikhande Graph

\[ S^* = (\chi_{1,0} + \chi_{3,0}) + (\chi_{0,1} + \chi_{0,3}) + (\chi_{1,1} + \chi_{3,3}) \]

<table>
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<tr>
<th>( \chi )</th>
<th>( x )</th>
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The group $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ and its dual
The rook graph
The Shrikhande graph
We define an equivalence relationship on $\sim$ on $G$ by $g \sim h$ if there exists $s \in \mathbb{Z}, \gcd(s, |G|) = 1$ such that $h = g^s$.

The dual group $G^*$ of characters of $G$ is isomorphic to $G$.

On $G^*$ we define an equivalence relation by $\chi \sim \psi$ if $\ker(\chi) = \ker(\psi)$.

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$$e_\chi := \frac{1}{|G|} \sum_{g \in G} \chi(g) \ g.$$  \hspace{1cm} (2)

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Given a character $\chi \in G^*$, defined the rational idempotent corresponding to $\chi$ as

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Each element of $G^*/\sim$ provides us with a rational idempotent; the rational idempotents are in a natural one-to-one correspondence with $G^*/\sim$. 

The rational idempotents graph
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Smith

Cayley SRGs

2015

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The rational idempotents of $C_4 \times C_4$ may be represented as $4 \times 4$ arrays.

For example, the idempotent corresponding to the trivial representation is

$$\frac{1}{16}\langle x, y \rangle = \frac{1}{16} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

while the character which maps $x$ to $-1$ and $y$ to $1$ has idempotent

$$\frac{1}{16}(1 - x)\langle x^2, y \rangle = \frac{1}{16} \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$
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An example

The character which maps \( x \) to \( i \) and \( y \) to \( i \) is equivalent to the character which maps both \( x \) and \( y \) to \( -i \).

This pair of characters give rational idempotent

\[
\frac{1}{8} (1 - x) \langle x^2, y \rangle = \frac{1}{16} \begin{bmatrix}
2 & 0 & -2 & 0 \\
0 & -2 & 0 & 2 \\
-2 & 0 & 2 & 0 \\
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\end{pmatrix}$$
The rational idempotents graph

Suppose $S^* = \{\chi_{1,0}, \chi_{3,0}, \chi_{1,1}, \chi_{3,3}, \chi_{0,1}\chi_{0,3}\}$, so that the characters in $S^*$ map $S$ to 2 while all the other nontrivial characters map $S$ to $-2$. Then

$$S = 6[e_{0,0}] - 2(\overline{S^*}) + 2S^* = -2 + 8[e_{0,0}] + 4([e_{1,0}] + [e_{1,1}] + [e_{0,1}]).$$

$$= \frac{1}{16} \left( \begin{bmatrix} -32 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 8 & 8 & 8 & 8 \\ 8 & 8 & 8 & 8 \\ 8 & 8 & 8 & 8 \\ 8 & 8 & 8 & 8 \end{bmatrix} + \begin{bmatrix} 24 & 8 & -8 & 8 \\ 8 & 8 & -8 & -8 \\ -8 & -8 & -8 & -8 \\ 8 & 8 & -8 & -8 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} = x + x^3 + y + y^3 + xy + x^3 y^3.$$
Suppose $S^* = \{\chi_{1,0}, \chi_{3,0}, \chi_{1,1}, \chi_{3,3}, \chi_{0,1}\}$, so that the characters in $S^*$ map $S$ to 2 while all the other nontrivial characters map $S$ to $-2$.

Then

$$S = 6[e_{0,0}] - 2(S^*) + 2S^* = -2 + 8[e_{0,0}] + 4([e_{1,0}] + [e_{1,1}] + [e_{0,1}]).$$

$$= \frac{1}{16} \left( \begin{array}{cccc}
-32 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array} \right) + \begin{array}{cccc}
8 & 8 & 8 & 8 \\
8 & 8 & 8 & 8 \\
8 & 8 & 8 & 8 \\
8 & 8 & 8 & 8
\end{array} + \begin{array}{cccc}
24 & 8 & -8 & 8 \\
8 & 8 & -8 & -8 \\
-8 & -8 & -8 & -8 \\
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\end{array} \right)$$

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The Shrikhande graph ...
... and its dual
The Clebsch graph and its dual
... and its dual
The group $\mathbb{Z}_8 \times \mathbb{Z}_8$ and its dual
The unique \((64, 18, 2, 6)\) Cayley Graph in \(\mathbb{Z}_8 \times \mathbb{Z}_8\)

\[ S = \{10, 30, 50, 70\} \cup \{11, 33, 55, 77\} \cup \{01, 03, 05, 07\} \cup \{24, 64\} \cup \{26, 62\} \cup \{42, 46\} \]
The group ring element of a rational idempotent

(16 by 16, order 16 characters)
$G = C_{16} \times C_{16}, \ k = 190, \ \theta_i = 4, -12$
Extending this to larger groups

Strongly regular Cayley graphs in $C_{16} \times C_{16}$ generated by Marty Malandro.

- $k=120 \ 8^{120} \ -8^{135}$:
- $k=119 \ 7^{136} \ -9^{119}$:
- $k=90 \ 10^{90} \ -6^{165}$:
- $k=119 \ 7^{136} \ -9^{119}$:
- $k=75$
Extending this to larger groups

Strongly regular Cayley graphs in $C_{16} \times C_{16}$ generated by Marty Malandro.
\[ G = C_{32} \times C_{32}, \quad k = 372, \quad \theta_i = 20, -12 \]
The rational idempotents graph of $C_{27} \times C_{27}$

Suppose $G = C_{p^m} \times C_{p^m}$. We define a directed graph on the elements of $G/\sim$ by mapping $(x/\sim) \mapsto (y/\sim)$ if $x^p \sim y$.
The rational idempotents graph of $C_{27} \times C_{27}$

Suppose $G = C_{p^m} \times C_{p^m}$. We define a directed graph on the elements of $G/\sim$ by mapping $(x/\sim) \mapsto (y/\sim)$ if $x^p \sim y$.

Equivalence classes of $C_{27} \times C_{27}$
We have worked through the groups $C_{2^n} \times C_{2^n}$ up to $n = 6$.

The size of the automorphism group of the tree grows exponentially, from $3 \cdot 2^{22}$ for $C_{16} \times C_{16}$ to $3 \cdot 2^{46}$ for $C_{32} \times C_{32}$ and then $3 \cdot 2^{94}$ for $C_{64} \times C_{64}$.

GAP is quite happy to create those groups, as long as we do not have to look too closely at the elements.

But my program hangs up immediately on the second entry of $C_{32} \times C_{32}$ and eventually quits due to memory allocation.

Some more theory is needed. If there is some way for me to use the cosets of the automorphism group of the group within the automorphism group of the tree, I can reduce $3 \cdot 2^{46}$ by $3 \cdot 2^{13}$, so that might search space is $2^{33} = 8$ billion which is feasible, maybe...
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In $C_4 \times C_4$ there are, of course, the known PDS:

1. (16, 6, 2, 2): 2 inequivalent PDS (Rook graph & Shrikhande graphs), each with a different idempotent tree.
2. (16, 5, 0, 2): 1. (The Clebsch graph is unique.)
Results in $C_4 \times C_4$

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There are 11 inequivalent PDS represented by 8 different idempotent trees.

1. \((64, 14, 6, 2) : 1\)
2. \((64, 18, 2, 6) : 1\)
3. \((64, 21, 8, 6) : 3\) inequivalent PDS in 2 idempotent trees,
4. Hadamard: \((64, 27, 10, 12) : 3\) inequivalent PDS in 2 idempotent trees,
5. Hadamard: \((64, 28, 12, 12) : 3\) inequivalent PDS in 2 idempotent trees.
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Results in $C_8 \times C_8$

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Results in $C_{16} \times C_{16}$

There are 268 inequivalent PDS in 13 different idempotent trees.

1. $(256, 30, 14, 2)$: 1 PDS, 1 tree,
2. $(256, 45, 16, 6)$: 1 PDS, 1 tree,
3. $(256, 75, 26, 20)$: 6 PDS in 1 idempotent tree,
4. $(256, 90, 34, 30)$: 36 PDS in 2 idempotent trees,
5. $(256, 105, 44, 42)$: 36 PDS in 1 idempotent tree,
   Hadamard:
6. $(256, 119, 54, 56)$: 86 PDS in 3 idempotent trees,
7. $(256, 120, 56, 56)$: 102 PDS in 4 idempotent trees.
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Results in $C_{32} \times C_{32}$

There are 12 different ways to fill in the idempotent tree so as to get a PDS.

In addition to the latin square PDS in $C_{32} \times C_{32}$ with $k = 2(31)$ and $k = 3(31)$ (which are unique), there are the following:

1. $(1024, k = 372 = 12(31), 140, 132)$ more than $43,690$ inequivalent PDS
2. $(1024, k = 465 = 15(31), 212, 210)$ at least $2^{19}$ inequivalent PDS

Hadamard:

3. $(1024, 495, 238, 240)$ at least 1.3 million
4. $(1024, 496, 240, 240)$ at least 1.4 million
Results in $C_{32} \times C_{32}$

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Results in $C_{64} \times C_{64}$

There are 19 different ways to fill in the idempotent tree so as to get a PDS.

In addition to the unique latin square PDS in $C_{64} \times C_{64}$ with $k = 2(63)$ and $k = 3(63)$, there are the following:

1. $k = 1890 = 30(63)$: there are at least $(2/3) \times 2^{49} = 3.8 \times 10^{14}$ of these.
2. $k = 1953 = 31(63)$: there are at least $2^{49} = 5.6 \times 10^{14}$ of these.
3. $k = 2015 = 31(65)$, Hadamard: there are at least $21 \times 2^{46} = 1.48 \times 10^{15}$ of these.
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Results

We have worked through the groups $C_{3^n} \times C_{3^n}$ up to $n = 4$ and, more generally, found some nested families of PDS in $C_{p^n} \times C_{p^n}$ for odd primes $p$.

There are infinite families of PDS with the Paley parameters,

$$(p^{2n}, \frac{1}{2}(p^{2n} - 1), \frac{1}{4}(p^{2n} - 1) - 1, \frac{1}{4}(p^{2n} - 1))$$

begun by choosing half of the nodes of smallest order ($p$) and then continuing the choices according to the pattern below.

\[ (p = 3) \]
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![Diagram](attachment:image.png)

$(p = 3)$
Further Questions

1. Are there other nested PDS families \((p \text{ odd})\)?
2. Are there nested Hadamard difference set families? \((p = 2)\)
3. Is there a general theory for the Hadamard difference sets? \((p = 2)\)
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(END)
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