Reversible Menon-Hadamard Difference Sets in Abelian 2-groups

Jordan D. Webster
Mid Michigan Community College

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A \((v, k, \lambda)\) difference set \(D\) in a group \(G\) of order \(v\) is a \(k\)-subset of \(G\) such that each group element other than the identity appears exactly \(\lambda\) times in the multiset \(\{d_1d_2^{-1} : d_1, d_2 \in D\}\).

A Menon-Hadamard difference set has parameters \((4m^2, 2m^2 - m, m^2 - m)\) for some \(m \in \mathbb{N}\).
Example: \( \{x, x^3, y, y^3, xy, x^3y^3\} \) is a \((16, 6, 2)\) Menon-Hadamard difference set in \( C_4 \times C_4 = \langle x, y : x^4 = y^4 = [x, y] = 1 \rangle \).

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<tr>
<th></th>
<th>x</th>
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<tr>
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Reversibility

Our Example $D = \{x, x^3, y, y^3, xy, x^3y^3\}$ in $C_4 \times C_4$

Notice for each element in $D$, the inverse element is also in $D$.

A difference set $D$ is Reversible if for each $d \in D$, $d^{-1} \in D$. 
Abelian non-2 groups do not have reversible difference sets

Some Abelian 2 groups with reversible difference sets:
$C_4, (C_{2r})^{(2)}, (C_{2^{2r}})^{(3)}$

Direct products of groups that contain reversible difference sets have reversible difference sets.

Some Abelian 2 groups don’t have reversible difference sets.
$C_8 \times C_2, C_{64} \times (C_{16})^{(2)}$. 
What we don’t/didn’t know

Odd powers on groups with $C_2$:
Example $(C_{32})^3 \times C_2$

Odd powers on groups without $C_2$:
Example $(C_{128})^3 \times (C_{32})^3$

Single cyclic group in direct product with no match
Example $(C_{32})^2 \times C_{16}$
What we don’t/didn’t know

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Example $(C_{32})^3 \times C_2$

Odd powers on groups without $C_2$:
Example $(C_{128})^3 \times (C_{32})^3$

Single cyclic group in direct product with no match
Example $(C_{32})^2 \times C_{16}$
Build a difference set in $(C_8)^{(3)} \times C_2$

Convince that the pattern extends to $(C_{2^{2r+1}})^{(3)} \times C_2$
The group ring $\mathbb{C}[G]$. 

This is the ring of all formal sums of the form $\sum_{g \in G} a_g g$ where $a_g \in \mathbb{C}$. 

Addition in this ring is defined pointwise. 

$$(4g_1 + 5g_2) + (-2g_1 + 8g_2) = 2g_1 + 13g_2$$
The Group Ring

Multiplication is done with consideration of both the multiplication in $\mathbb{C}$ and by the “multiplication” of the group $G$.

$$(4g_1 + 5g_2)(2g_3) = 8g_1g_3 + 10g_2g_3$$

Multiplication distributes over addition.
Difference set in the ring is \( D = \sum_{d \in D} d \).

Also Notation of \( D^{-1} = \sum_{d \in D} d^{-1} \)

\[ DD^{-1} = (k - \lambda) + \lambda(G) \]
Difference set in the ring is \( D = \sum_{d \in D} d. \)

If \( D \) is reversible then \( D = D^{(-1)} \)

\[
DD = (k - \lambda) + \lambda(G)
\]
We wish to substitute the difference set for something else.

The Hadamard transform: \( \hat{D} = G - 2D \).

If a \((4m^2, 2m^2 - m, m^2 - m)\) difference set \( D \) exists, then
\[
\hat{D}\hat{D}(-1) = 4m^2
\]

For remainder of talk, we say that a \((4m^2, 2m^2 - m, m^2 - m)\) difference set is an element of the group ring \( \hat{D} \) with coefficients of \( \pm 1 \) and has the property that \( \hat{D}\hat{D}(-1) = 4m^2 \).
For abelian groups, a character of the group is a homomorphism into the set of complex numbers.

If $G$ is abelian, then the complete set of characters forms the dual group $G^*$. Extend each character to $\mathbb{C}$-algebra homomorphism.

$$\chi \left( \sum a_g g \right) = \sum a_g \chi(g)$$
Theorem (Turyn 1965)

\( \hat{D} \) is a Menon-Hadamard difference set in an abelian group of order \( 4m^2 \) if and only if it is an element of the group ring with \( \pm 1 \) coefficients and for each \( \chi \in G^* \), we have \( \chi(\hat{D})\chi(\hat{D}(-1)) = 4m^2 \).
The group $C_4 = \langle x : x^4 = 1 \rangle$.

Each character is homomorphism so each is defined by where it sends generator(s) of the group.

If $\chi$ is a character, then $(\chi(x))^4 = 1$

$\chi_j(x) = (e^{\frac{2\pi i}{4}})^j$ for $1 \leq j \leq 4$.

The dual group $C_4^* = \{\chi_j : 1 \leq j \leq 4\}$
We know $\chi(\hat{D})\chi(\hat{D}(-1)) = 4m^2$

When $D$ is reversible, $\hat{D}(-1) = \hat{D}$ and

$$(\chi(\hat{D}))^2 = 4m^2$$

$\chi(\hat{D}) = \pm 2m$
The ring $\mathbb{C}[G]$ has the standard basis $\{g : g \in G\}$.

Create elements for each character value.

$$e_\chi = \frac{1}{|G|} \sum_{g \in G} \chi(g)g^{-1}$$

These idempotents form basis for $\mathbb{C}[G]$. 
In the basis $\{e_\chi; \chi \in G^*\}$ we take advantage of character values.

$\chi(e_{\chi'}) = \delta_{\chi,\chi'}$

Let $Y = \sum_{\chi \in G^*} c_\chi e_\chi$

So $\chi(Y) = \chi(c_\chi)$
We write $\hat{D} = \sum_{\chi \in G^*} c_\chi e_\chi$.

So $\chi(\hat{D}) = \chi(c_\chi)$

Each $\chi(c_\chi)$ must be a complex number of modulus $2m$.

If $\hat{D}$ is reversible then $\chi(c_\chi) = \pm 2m$
We have a Menon-Hadamard Difference set \((\hat{D})\) if

- \(\pm 1\) are coefficients on each group element \(g\)
- Each coefficient, \(c_\chi\), on \(e_\chi\) has the property that \(\chi(c_\chi)\) is a complex number of modulus 2\(m\).

Reversible if \(\chi(c_\chi) = \pm 2m\)
Recall that each idempotent is \( \frac{1}{|G|} \sum_{g \in G} \chi(g) g^{-1} \).

Difference set \( \hat{D} \) has coefficients in \( \{-1, 1\} \).
Idempotents with the same kernel may be added together and we may use the same coefficient for all of them.

Sums of idempotents with the same kernel are called rational idempotents.

\([e_\chi]\) is be the rational idempotent containing \(e_\chi\) and all other \(e_{\chi'}\) such that \(\ker(\chi) = \ker(\chi')\).

The \([e_\chi]\) exist in the group ring \(\mathbb{Q}[G]\)
We have a Hadamard Difference set if

- $\pm 1$ are coefficients on each group element $g$
- Each coefficient, $c_\chi$, on $[e_\chi]$ has the property that $\chi(c_\chi)$ is a complex number of modulus $2m$. 
A tile $T$ is an element of the group ring with the following properties:

- Coefficients of $g$ are all in the set $\{1, 0, -1\}$
- For every character $\chi$ either $\chi(T) = 0$ or $\chi(T)$ is a complex number of modulus $2m$

Sums of rational idempotents with appropriate aliases create tiles.
The Goal

The Goal is to create a set of tiles so that support lines up in the correct way. We create the tiles from the rational idempotents.

- Idempotents \( \mathbb{C}[G] \)
- Rational Idempotents (with aliases) \( \mathbb{Q}[G] \)
- Tiles \( \mathbb{Z}[G] \) (Coefficients in set \( \{1, 0, -1\} \))
- Menon-Hadamard Difference Set
Goal

Goal is to do this process in \((C_8)^{(3)} \times C_2\)
In \((C_8)^{(3)} \times C_2\).

Elements of the group
\[x^8 = y^8 = z^8 = w^2 = 1\]

Size of the group is 1024.
Characters and Aliases

Group size 1024
Means $\chi(D) = \pm 32$ for all characters.

Means for each alias on an idempotent, we can set to be $32g$
where $\chi(g) = \pm 1$
For this reason, we multiply each idempotent by 32.
Rational idempotents denoted $[e_{i_1,i_2,i_3,i_4}]$

Associated with character $\chi_{i_1,i_2,i_3,i_4}$ which does

$x \rightarrow (e^{\frac{2\pi i}{8}})^{i_1}$

$y \rightarrow (e^{\frac{2\pi i}{8}})^{i_2}$

$z \rightarrow (e^{\frac{2\pi i}{8}})^{i_3}$

$w \rightarrow (-1)^{i_4}$

$0 \leq i_1 \leq 7, 0 \leq i_4 \leq 1$
The idempotents

This means there are 1024 idempotents.

Combining into rational idempotents gives 296 rational idempotents.

For convenience we will multiply rational idempotents by 32
The rational idempotents

We further break up the 296 rational idempotents by what roots of unity they send the elements of the group.

\[ \zeta_8 = e^{\frac{2\pi i}{8}} \]

There are 224 rational idempotents where an element to \( g \to \zeta_8 \)
There are 56 rational idempotents where an element to \( g \to \zeta_4 \) (no elements sent to \( \zeta_8 \))
There are 16 rational idempotents where an element \( g \to \pm 1 \)
Use the 224 rational idempotents which have \( g \rightarrow \zeta_8 \). Combine to cover the portion of the group

\[
< x > < y > < z > < w > - < x^2 > < y^2 > < z^2 > < w >
\]
The rational idempotents

Use the 72 rational idempotents $g \to \zeta_4$ or $\to \pm 1$. Combine to cover the portion of the group

$< x^2 > < y^2 > < z^2 > < w >$
Use the 224 rational idempotents $\rightarrow \zeta_8$.
Combine to cover
$\langle x \rangle \langle y \rangle \langle z \rangle \langle w \rangle - \langle x^2 \rangle \langle y^2 \rangle \langle z^2 \rangle \langle w \rangle$

How many ways to send a group element to $\zeta_8$?

Could send some power of $x$, some power of $y$, or some power of $z$ to $\zeta_8$. 
Look at rational idempotents sending $x$ to $\zeta_8$.
The elements $y$ and $z$ are not sent to $\zeta_8$.

$$32[e_{1,0,0,0}] = \frac{1}{8}(1 - x^4) < y > < z > < w >$$

Any time $i_2, i_3, i_4$ even in $32[e_{1,i_2,i_3,i_4}]$

$$32[e_{1,i_2,i_3,i_4}] = \frac{1}{8}(1 - x^4) < x^{8-i_2} y > < x^{8-i_3} z > (1 + (-1)^{i_4} w)$$
$x \rightarrow \zeta_8$

$x$ to $\zeta_8$. $y$ and $z$ not to $\zeta_8$

$32[e_{1,0,0,0}]$, $32[e_{1,4,0,0}]$, $32[e_{1,0,4,0}]$, $32[e_{1,4,4,0}]$, $32[e_{1,0,0,1}]$,
$32[e_{1,4,0,1}]$, $32[e_{1,0,4,1}]$, $32[e_{1,4,4,1}]$

$32[e_{1,2,0,0}]$, $32[e_{1,6,0,0}]$, $32[e_{1,2,4,0}]$, $32[e_{1,6,4,0}]$, $32[e_{1,2,0,1}]$,
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$x \to \zeta_8$

$x$ to $\zeta_8$. $y$ and $z$ not to $\zeta_8$

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\[ x \rightarrow \zeta_8 \]

\[ x \text{ to } \zeta_8. \ y \text{ and } z \text{ not to } \zeta_8 \]

\[ 32[e_{1,0,0,0}], 32[e_{1,4,0,0}], 32[e_{1,0,4,0}], 32[e_{1,4,4,0}], 32[e_{1,0,0,1}], \]
\[ 32[e_{1,4,0,1}], 32[e_{1,0,4,1}], 32[e_{1,4,4,1}] \]

Sum is \( A_1 = (1 - x^4) < y^2 > < z^2 > \)

Group elements that can be sent to \( \pm 1 \) by all character values:
\[ y, z, yz, yw, zw, yzw \]
\( \eta \rightarrow \zeta_8 \)

\( x \) to \( \zeta_8 \). \( y \) and \( z \) not to \( \zeta_8 \)

\[ 32[e_{1,2,0,0}], 32[e_{1,6,0,0}], 32[e_{1,2,4,0}], 32[e_{1,6,4,0}], 32[e_{1,2,0,1}], 32[e_{1,6,0,1}], 32[e_{1,2,4,1}], 32[e_{1,6,4,1}] \]

Sum is \( A_2 = (1 - x^4) < x^4y^2 > < z^2 > \)

Group elements that can be sent to \( \pm 1 \) by all character values:
\( x^2y, z, x^2yz, x^2yw, zw, x^2yzw, \)
$x \to \zeta_8$

$x$ to $\zeta_8$. $y$ and $z$ not to $\zeta_8$

$32[e_1,0,2,0], 32[e_1,4,2,0], 32[e_1,0,6,0], 32[e_1,4,6,0], 32[e_1,0,2,1], 32[e_1,4,2,1], 32[e_1,0,6,1], 32[e_1,4,6,1]$

Sum is $A_3 = (1 - x^4) < y^2 > < x^4 z^2 >$

Group elements that can be sent to $\pm 1$ by all character values:

$y, x^2 z, x^2 yz, yw, x^2 zw, x^2 yzw,$
$x \rightarrow \zeta_8$

$x$ to $\zeta_8$. $y$ and $z$ not to $\zeta_8$

$32[e_{1,2,2,0}]$, $32[e_{1,6,2,0}]$, $32[e_{1,2,6,0}]$, $32[e_{1,6,6,0}]$, $32[e_{1,2,2,1}]$, $32[e_{1,6,2,1}]$, $32[e_{1,2,6,1}]$, $32[e_{1,6,6,1}]$

Sum is $A_4 = (1 - x^4) < x^4y^2 > < x^4z^2 >$

Group elements that can be sent to $\pm 1$ by all character values:
$x^2y$, $x^2z$, $yz$, $x^2yw$, $x^2zw$, $yzw$,
Lots of ways to combine and get a few results. For instance

\[ zA_1 + zwA_2 + x^2zA_3 + x^2zwA_4 \] has support
\[ z < x^2 >> y^2 >> z^2 >> w > \]

\[ yA_1 + ywA_3 + x^2yA_2 + x^2ywA_4 \] has support
\[ y < x^2 >> y^2 >> z^2 >> w > \]

\[ yzA_1 + x^2yzwA_2 + x^2yzA_3 + yzwA_4 \] has support
\[ yz < x^2 >> y^2 >> z^2 >> w > \]
Options for the rational idempotents with $x \rightarrow \zeta_8$.

Support
$z < x^2 > < y^2 > < z^2 > < w >$
$y < x^2 > < y^2 > < z^2 > < w >$
$yz < x^2 > < y^2 > < z^2 > < w >$
We can get support of a group element multiplied by $H = \langle x^2 \rangle \langle y^2 \rangle \langle z^2 \rangle \langle w \rangle$ each time we look at sets of rational idempotents which have generators sent to primitive $8^{th}$ roots of unity.

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Gives support as
\[ < x > < y > < z > < w > - < x^2 > < y^2 > < z^2 > < w > \]

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<td>( z, y, yz )</td>
</tr>
<tr>
<td>( y )</td>
<td>( z, x, xz )</td>
</tr>
<tr>
<td>( z )</td>
<td>( x, y, yx )</td>
</tr>
<tr>
<td>( x \ y )</td>
<td>( z, xy, xyz )</td>
</tr>
<tr>
<td>( x \ z )</td>
<td>( y, xz, xyz )</td>
</tr>
<tr>
<td>( y \ z )</td>
<td>( x, yz, xyz )</td>
</tr>
<tr>
<td>( x \ y \ z )</td>
<td>( xz, xy, yz )</td>
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Similarly we can send all group elements to a power of $\zeta_4$.

$$32[e_0,0,0,*], \; 32[e_4,0,0,*], \; 32[e_0,4,0,*], \; 32[e_0,0,4,*],$$
$$32[e_4,4,0,*], \; 32[e_4,0,4,*], \; 32[e_0,4,4,*], \; 32[e_4,4,4,*]$$
$$32[e_2,0,0,*], \; 32[e_2,4,0,*], \; 32[e_2,0,4,*], \; 32[e_2,4,4,*]$$

$$32[e_0,0,2,*], \; 32[e_4,0,2,*], \; 32[e_0,4,2,*], \; 32[e_4,4,2,*],$$
$$32[e_2,0,2,*], \; 32[e_2,0,6,*], \; 32[e_2,4,2,*], \; 32[e_2,4,6,*]$$
$$32[e_0,2,0,*], \; 32[e_4,2,0,*], \; 32[e_0,2,4,*], \; 32[e_4,2,4,*],$$
$$32[e_2,2,0,*], \; 32[e_2,6,0,*], \; 32[e_2,2,4,*], \; 32[e_2,6,4,*]$$
$$32[e_0,2,2,*], \; 32[e_0,2,6,*], \; 32[e_4,2,2,*], \; 32[e_4,2,6,*],$$
$$32[e_2,2,2,*], \; 32[e_2,2,6,*], \; 32[e_2,6,2,*], \; 32[e_2,6,6,*]$$
Similarly we can send all group elements to a power of $\zeta_4$.

\[
\begin{align*}
32[e_0,0,0,*], & \quad 32[e_4,0,0,*], \quad 32[e_0,4,0,*], \quad 32[e_0,0,4,*], \\
32[e_4,4,0,*], & \quad 32[e_4,0,4,*], \quad 32[e_0,4,4,*], \quad 32[e_4,4,4,*] \\
32[e_2,0,0,*], & \quad 32[e_2,4,0,*], \quad 32[e_2,0,4,*], \quad 32[e_2,4,4,*] \\
32[e_0,0,2,*], & \quad 32[e_4,0,2,*], \quad 32[e_0,4,2,*], \quad 32[e_4,4,2,*], \\
32[e_2,0,2,*], & \quad 32[e_2,0,6,*], \quad 32[e_2,4,2,*], \quad 32[e_2,4,6,*]
\end{align*}
\]

Let sums be $Q_1$, $Q_2$
Similarly we can send all group elements to a power of $\zeta_4$.

$$32[e_{0,2,0,*}], \ 32[e_{4,2,0,*}], \ 32[e_{0,2,4,*}], \ 32[e_{4,2,4,*}],$$
$$32[e_{2,2,0,*}], \ 32[e_{2,6,0,*}], \ 32[e_{2,2,4,*}], \ 32[e_{2,6,4,*}],$$
$$32[e_{0,2,2,*}], \ 32[e_{0,2,6,*}], \ 32[e_{4,2,2,*}], \ 32[e_{4,2,6,*}],$$
$$32[e_{2,2,2,*}], \ 32[e_{2,2,6,*}], \ 32[e_{2,6,2,*}], \ 32[e_{2,6,6,*}],$$

Let sums be $Q_3$, $Q_4$
$g \rightarrow \zeta_4$

$1Q_1 + x^2 Q_2 + x^2 w Q_3 + w Q_4$ is a tile with support of

$< x^2 > < y^2 > < z^2 > < w >$
The difference set

Adding tiles we get support of $(C_8)^{(3)} \times C_2$

Since we chose aliases where characters send to $\pm 32$

We have a difference set $\hat{D}$!
$(C_8)^{(3)} \times C_2$ had a tiling structure.

The breakdown on the support was

\[
\langle x \rangle \langle y \rangle \langle z \rangle \langle w \rangle - \langle x^2 \rangle \langle y^2 \rangle \langle z^2 \rangle \langle w \rangle \\
\langle x^2 \rangle \langle y^2 \rangle \langle z^2 \rangle \langle w \rangle
\]
\((C_{32})^3 \times C_2\) has a tiling structure.

The breakdown on the support was
\[
< x > < y > < z > < w > - < x^2 > < y^2 > < z^2 > < w > \\
< x^2 > < y^2 > < z^2 > < w > - < x^4 > < y^4 > < z^4 > < w > \\
< x^4 > < y^4 > < z^4 > < w > - < x^8 > < y^8 > < z^8 > < w > \\
< x^8 > < y^8 > < z^8 > < w >
\]
Generalization

\((C_{2^{2r+1}})^{(3)} \times C_2\) has a tiling structure.

The breakdown on the support was
\[
< x > < y > < z > < w > - < x^2 > < y^2 > < z^2 > < w > \\
< x^2 > < y^2 > < z^2 > < w > - < x^4 > < y^4 > < z^4 > < w > \\
\vdots \\
< x^{2^{2r-1}} > < y^{2^{2r-1}} > < z^{2^{2r-1}} > < w >
\]
Abelian non-2 groups have no reversible difference sets

Some Abelian 2 groups with reversible difference sets:
$C_4, (C_{2r})^{(2)}, (C_{22r})^{(3)}$

Direct products of groups that contain reversible difference sets have reversible difference sets.

Some Abelian 2 groups don’t have reversible difference sets. Ex: $C_8 \times C_2, C_{64} \times (C_{16})^{(2)}$. 
What we don’t/didn’t know

Odd powers on groups with $C_2$:
Example $(C_{2^{2r+1}})^{(3)} \times C_2$

Odd powers on groups without $C_2$:
Example $(C_{2^{2r+1}})^{(3)} \times (C_{2^{2s+1}})^{(3)}$

Single cyclic group in direct product with no match
Examples $(C_{32})^{(2)} \times C_{16}$, $(C_{32})^{(3)} \times C_2$
What we don’t/didn’t know

Odd powers on groups with $C_2$:
Example $(C_{32})^3 \times C_2$

Odd powers on groups without $C_2$:
Example $(C_{128})^3 \times (C_8)^3$

Single cyclic group in direct product with no match
Examples $(C_{32})^2 \times C_{16}$, $(C_{32})^3 \times C_2$
Odd powers on groups without $C_2$:
Example $(C_{128})^{(3)} \times (C_8)^{(3)}$
Difficulties

Single cyclic group in direct product with no match
Example \((C_{32})^{(2)} \times C_{16}\)
Chipping away at Necessary and Sufficient conditions for Reversible difference sets in abelian groups.
Thank you!