CLASSES OF STABLE COMPLEX MATRICES DEFINED VIA THE THEOREMS OF GERŠGORIN AND LYAPUNOV

BRYAN CAIN, TERRY D. LENKER, SIVARAM K. NARAYAN*, AND PETER VERMEIRE

ABSTRACT. We consider (and characterize) mainly classes of (positively) stable complex matrices defined via methods of Geršgorin and Lyapunov. Although the real matrices in most of these classes have already been studied, we sometimes improve upon (and even correct) what has been previously published. Many of the classes turn out quite naturally to be products of common sets of matrices. A Venn diagram shows how the classes are related.

1. INTRODUCTION AND PRELIMINARIES

We consider subsets of the set of $n \times n$ (positively) stable complex matrices

$$\mathcal{S} = \{A \in M_n(\mathbb{C}) | \operatorname{Re}(\lambda) > 0 \text{ for all eigenvalues } \lambda \text{ of } A\}.$$

The study of various kinds of real and complex stable matrices has a long history, cf. e.g. [2], [4], [6], [8], [13]. Our work generalizes parts of [2] by allowing the matrices to be complex instead of just real, by including new results, and by making some corrections. We include a Venn diagram which summarizes how our results fit in with those of others.

Here are two well-known methods of establishing that an $n \times n$ complex matrix $A = (a_{ij})$ is stable. We use them to define other sets of stable matrices.

1. Via Geršgorin's Theorem: Let the i th coordinate of the *Geršgorin vector* $g(A)$ of A be

$$g(A)_i = \operatorname{Re}(a_{ii}) - \sum_{\substack{j=1 \\ j \neq i}}^n |a_{ij}|.$$

If $g(A)_i > 0$ for all i then A is stable.

2. Via Lyapunov's Theorem: A is stable if and only if there is an $H > 0$ such that $HA + A^*H > 0$ (throughout the paper, the condition that a matrix B is Hermitian positive definite is denoted $B > 0$).

Many of our results come from a few simple observations.

Observation 1.1. Let $A \in GL_n(\mathbb{C})$, $H > 0$, and U be unitary. Set $B = U^*AU$ and $G = U^*HU$. Then

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* Corresponding Author. Email: sivaram.narayan@cmich.edu, Phone: 989-774-3566, Fax: 989-774-2414

- (1) $(A^{-1})^*(HA + A^*H)A^{-1} = HA^{-1} + (A^{-1})^*H.$
- (2) $(H^{-1})^*(HA + A^*H)H^{-1} = H^{-1}A^* + AH^{-1}.$
- (3) $U^*(HA + A^*H)U = GB + B^*G.$

Let S and T be subsets of some ring R and suppose the elements of S are invertible.

- (4) $\{a \in R | \exists s \in S \text{ such that } sa \in T\} = S^{-1}T = \{s^{-1}t | s \in S \text{ and } t \in T\}.$

If $S^{-1} = S$ then

- (5) $\{a \in R | \exists s \in S \text{ such that } sa \in T\} = ST.$
- (6) $\{a \in R | \exists s \in S \text{ such that } as \in T\} = TS.$

□

Note that if $\mathcal{S}_1, \mathcal{S}_2$ are subsets of $M_n(\mathbb{C})$, we denote the collection of products of elements by juxtaposition, i.e. $\mathcal{S}_1\mathcal{S}_2 = \{S_1S_2 \in M_n(\mathbb{C}) | S_1 \in \mathcal{S}_1, S_2 \in \mathcal{S}_2\}.$

Notation 1.2. For $A \in M_n(\mathbb{C})$

- (1) $\mathcal{H}_{pd} = \{A | A > 0\}$ is the set of Hermitian positive definite matrices.
- (2) $\mathcal{R}_{pd} = \{A | 2\text{Re}(A) = A + A^* > 0\}.$
- (3) $\mathcal{D} = \{A | A = \text{diag}(d), d \in \mathbb{C}^n\}$ is the set of diagonal matrices.
- (4) $\mathcal{S}_{\mathcal{D}} = \mathcal{D} \cap \mathcal{S}.$
- (5) $\mathcal{D}^+ = \mathcal{D} \cap \mathcal{H}_{pd} = \{A | A = \text{diag}(d), d_i > 0\}.$
- (6) $\mathbb{D}_{stab} = \{A | A\mathcal{D}^+ \subseteq \mathcal{S}\}.$
- (7) $\mathbb{H}_{stab} = \{A | A\mathcal{H}_{pd} \subseteq \mathcal{S}\}.$
- (8) $\mathcal{G}_+ = \{A | g(A) > 0\}.$

$\mathbb{D}_{stab}$ is the set of **D-stable** matrices and $\mathcal{G}_+$ is the set of **row diagonally dominant** matrices. For a class $\mathcal{M}$ of matrices, we define

$$\mathcal{L}(\mathcal{M}) = \{A \in M_n(\mathbb{C}) | \exists M \in \mathcal{M} \text{ such that } MA + A^*M > 0\}.$$

$\mathcal{L}(\mathcal{D}^+)$ is the set of **Lyapunov diagonally stable** matrices. We let $\mathcal{M}(\mathbb{R})$ denote the real matrices in $\mathcal{M}.$

Proposition 1.3. If $\mathcal{F} \subseteq \mathcal{H}_{pd}$ then $\mathcal{L}(\mathcal{F}) = \mathcal{F}^{-1}\mathcal{R}_{pd}$ and $\mathcal{L}(\mathcal{F}^{-1}) = \mathcal{L}(\mathcal{F})^*.$ In particular, if $\mathcal{F} = \mathcal{F}^{-1}$ then $\mathcal{L}(\mathcal{F}) = \mathcal{R}_{pd}\mathcal{F} = \mathcal{F}\mathcal{R}_{pd}$ and $\mathcal{L}(\mathcal{F}) = \mathcal{L}(\mathcal{F})^*.$

PROOF: The first equality follows from Observation 1.1.4 since $\mathcal{L}(\mathcal{F}) = \{A | \exists S \in \mathcal{F} \text{ such that } SA \in \mathcal{R}_{pd}\}$; the second follows from Observation 1.1.2 after writing $H^{-1} = \mathcal{F}.$

If $\mathcal{F} = \mathcal{F}^{-1}$ then $\mathcal{F}\mathcal{R}_{pd} = \mathcal{L}(\mathcal{F}) = \mathcal{L}(\mathcal{F})^* = (\mathcal{F}\mathcal{R}_{pd})^* = \mathcal{R}_{pd}\mathcal{F}.$

□

Corollary 1.4. (cf. [2, 3.1.I.2]) $\mathcal{S} = \mathcal{L}(\mathcal{H}_{pd}) = \mathcal{R}_{pd}\mathcal{H}_{pd} = \mathcal{H}_{pd}\mathcal{R}_{pd}.$

□

Corollary 1.5. (cf. [2, 3.1.I.3]) $\mathcal{L}(\mathcal{D}^+) = \mathcal{D}^+\mathcal{R}_{pd} = \mathcal{R}_{pd}\mathcal{D}^+.$

□

Theorem 1.6.

- (1) $\mathcal{R}_{pd} \subseteq \mathbb{H}_{stab}.$
- (2) $\mathcal{D}^+ \subseteq \mathcal{H}_{pd} \subseteq \mathcal{R}_{pd} \subseteq \mathcal{L}(\mathcal{D}^+) \subseteq \mathbb{D}_{stab} \subseteq \mathcal{S}.$

PROOF: The first part is in [4]. We show $\mathcal{L}(\mathcal{D}^+) \subset \mathbb{D}_{stab}$: if $E \in \mathcal{D}^+$ and $EA + A^*E > 0$, then for every $D \in \mathcal{D}^+$ we have

$$(ED)(AD) + (AD)^*(ED) = D(EA + A^*E)D > 0$$

and so AD is stable by Lyapunov's Theorem. This is presented in [1] for real matrices.

The rest of the relations are immediate from the definitions.

□

2. LYAPUNOV'S CONDITION AND THE STABLE DIAGONAL MATRICES

H. Wimmer [13] has characterized three classes of stable matrices defined via Lyapunov's condition.

Theorem 2.1. *Let $A \in M_n(\mathbb{C})$.*

- (1) $DA + A^*D > 0$ for every $D \in \mathcal{D}^+$ if and only if $A \in \mathcal{S}_{\mathcal{D}}$.
- (2) $PA + A^*P > 0$ for all $P \in \mathcal{H}_{pd} \cap \mathcal{G}_+$ if and only if A is a stable scalar matrix.
- (3) $PA + A^*P > 0$ for all $P \in \mathcal{H}_{pd}$ if and only if A is a stable scalar matrix.

PROOF: Part (3) is [13, Theorem 8]; the proof there establishes parts (1) and (2) also. $\square$

Theorem 2.2. *The three sets defined in (1), (2), and (3) below are equal.*

- (1) $\{A \in M_n(\mathbb{C}) \mid \exists W \in \mathcal{H}_{pd} \text{ such that } \forall D \in \mathcal{D}^+, W(AD) + (AD)^*W > 0\}$
- (2) $\mathcal{H}_{pd}\mathcal{S}_{\mathcal{D}}$
- (3) $\mathcal{S}_{\mathcal{D}}\mathcal{H}_{pd}$

PROOF: Rewriting gives $D(WA)^* + (WA)D > 0$ for all $D \in \mathcal{D}^+$, part (1) of Theorem 2.1 yields $WA \in \mathcal{S}_{\mathcal{D}}$, and Observation 1.1.5 gives equality of (1) and (2). To see (2) and (3) are equal, let $P \in \mathcal{H}_{pd}$ and $D \in \mathcal{S}_{\mathcal{D}}$. Then $PD = (D^*)^{-1}(D^*PD) \in \mathcal{S}_{\mathcal{D}}\mathcal{H}_{pd}$ and $DP = (DPD^*)(D^*)^{-1} \in \mathcal{H}_{pd}\mathcal{S}_{\mathcal{D}}$. $\square$

Remark 2.3. If in Theorem 2.2 (1) we require $A \in M_n(\mathbb{R})$ but permit W to be complex, we obtain the set $\mathcal{H}_{pd}(\mathbb{R})\mathcal{D}^+$. To see this, if $A = PD$ with $P \in \mathcal{H}_{pd}$ and $D \in \mathcal{S}_{\mathcal{D}}$ then D must be real because A is real and the diagonal of P is real; therefore $D > 0$ and P is real. $\square$

Theorem 2.4. *Let $M = \begin{bmatrix} a + iA & b + iB \\ c + iC & d + iD \end{bmatrix}$ with $a, b, c, d, A, B, C, D \in \mathbb{R}$. Set $E = b^2 + B^2$, $F = 4ad - 2bc + 2BC$, and $G = c^2 + C^2$. Then $M \in \mathcal{L}(\mathcal{D}^+)$ if and only if $a > 0$, $d > 0$, $F > 0$, and $F^2 - 4EG > 0$.*

PROOF:

$M \in \mathcal{L}(\mathcal{D}^+)$ if and only if there is an $x > 0$ such that $Q = \text{Diag}(x, 1) \cdot M + M^* \cdot \text{Diag}(x, 1) > 0$; i.e. $q_{11} = 2ax > 0$, $q_{22} = 2d > 0$, and $\det(Q) = -Ex^2 + Fx - G > 0$.

If $E = 0$, then clearly such an $x > 0$ exists if and only if a, d, F , and $F^2 - 4EG$ are positive. We therefore assume $E > 0$. The maximum value of $\det(Q)$ is $\frac{F^2 - 4EG}{4E}$

which is achieved at $x_0 = \frac{F}{2E}$. If a, d, F and $F^2 - 4EG$ are positive, then letting $x = x_0$ suffices and $M \in \mathcal{L}(\mathcal{D}^+)$.

Conversely, suppose $M \in \mathcal{L}(\mathcal{D}^+)$. Then a and d are positive as observed above. Since $\det(Q) > 0$ for some $x > 0$ and $\det(Q) = -G < 0$ when $x = 0$, the maximum value $\frac{F^2 - 4EG}{4E} > 0$ and is achieved at $\frac{F}{2E} = x_0 > 0$; hence $F^2 - 4EG > 0$ and $F > 0$. $\square$

The next example shows that $\mathcal{H}_{pd}\mathcal{S}_{\mathcal{D}} \not\subseteq \mathcal{L}(\mathcal{D}^+)$.

Example 2.5. Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. Then $A \cdot \text{Diag}(1, 1 + 3i) \in \mathcal{H}_{pd}\mathcal{L}_{\mathcal{D}}$, but it is not in $\mathcal{L}(\mathcal{D}^+)$ by Theorem 2.4.

By Corollary 1.5, there are no such examples among real matrices since we have $\mathcal{H}_{pd}(\mathbb{R})\mathcal{L}_{\mathcal{D}}(\mathbb{R}) \subseteq \mathcal{R}_{pd}(\mathbb{R})\mathcal{D}^+ = \mathcal{L}(\mathcal{D}^+)(\mathbb{R})$. [2, 4.2] would correctly say that $\mathcal{H}_{pd}(\mathbb{R})\mathcal{L}_{\mathcal{D}}(\mathbb{R})$ is a proper subset of $\mathcal{L}(\mathcal{D}^+)(\mathbb{R})$ if the subscript *sym* were dropped; unfortunately the proof given there fails when the first row of W is $\begin{bmatrix} 1 & -3 \end{bmatrix}$. $\square$

Our next result describes all 2×2 real matrices which are in $\mathcal{L}(\mathcal{D}^+)$ but do not belong to $\mathcal{H}_{pd}(\mathbb{R})\mathcal{D}^+$. One example is $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ which is in $\mathcal{L}(\mathcal{D}^+)$ by Theorem 6.1.5.

An $n \times n$ real matrix $A = (a_{ij})$ is said to be **strongly sign symmetric** if for all $i, j \leq n$ either $a_{ij}a_{ji} > 0$ or $a_{ij} = a_{ji} = 0$.

Corollary 2.6. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{R})$. Then $A \in \mathcal{H}_{pd}\mathcal{D}^+$ if and only if $A \in \mathcal{L}(\mathcal{D}^+)$ and is strongly sign symmetric.

PROOF: Suppose $A \in \mathcal{H}_{pd}\mathcal{D}^+$. We know that $\mathcal{H}_{pd}\mathcal{D}^+ \subseteq \mathcal{L}(\mathcal{D}^+)$ and clearly any such A is strongly sign symmetric.

Conversely, Theorem 6.1.5 says that a, d , and $ad - bc$ are positive. If $b = c = 0$ then $A = AI$ with $A \in \mathcal{H}_{pd}$; if $bc > 0$ then

$$A = \begin{bmatrix} \frac{ab}{c} & b \\ b & d \end{bmatrix} \begin{bmatrix} \frac{c}{b} & 0 \\ 0 & 1 \end{bmatrix} \in \mathcal{H}_{pd}\mathcal{D}^+.$$

$\square$

Note that if $A \in M_n(\mathbb{R})$ then $A \in \mathcal{H}_{pd}(\mathbb{R})\mathcal{D}^+$ if and only if there exists $D \in \mathcal{D}^+$ such that $AD \in \mathcal{H}_{pd}(\mathbb{R})$. In this case, M -matrix facts give us more relevant information. $A = (a_{ij})$ is an M -matrix if and only if A is stable and its off diagonal entries are non-positive.

We borrow the following ideas from [7]. If $A \in M_n(\mathbb{R})$ we say that A is $\mathcal{D}_{SYM}^+$ if there exists $E \in \mathcal{D}^+$ such that AE is symmetric. If A is strongly sign symmetric then A is $\mathcal{D}_{SYM}^+$ if and only if the **comparison matrix** $C(A) = (c_{ij})$ defined by

$$c_{ij} = \begin{cases} \text{lla}_{ii} & i = j \\ -|a_{ij}| & i \neq j \end{cases}$$

is also $\mathcal{D}_{SYM}^+$.

Since the diagonal of A is immaterial as to whether or not $A \in \mathcal{D}_{SYM}^+$, we may assume that the diagonal is sufficiently large so that $C(A)$ is an M -matrix. Denote the minimum real eigenvalue of an M -matrix B by $q(B)$.

Definition 2.7. The **Hadamard product** of two matrices $A = (a_{ij})$ and $B = (b_{ij})$ of the same dimensions is $A \circ B = (a_{ij}b_{ij})$. $\square$

Theorem 2.8. [7] Let A be an M -matrix. Then $A^{-1} \circ A$ is an M -matrix and $q(A^{-1} \circ A) \leq 1$. If A is irreducible, then equality holds if and only if A is $\mathcal{D}_{SYM}^+$.

$\square$

Theorem 2.9. *Let $A \in M_n(\mathbb{R})$ be strongly sign symmetric, irreducible, and have all its leading principal minors positive. Choose $D \in \mathcal{D}^+$ such that $B = C(A) + D$ is an M -matrix. Then $A \in \mathcal{H}_{pd}(\mathbb{R})\mathcal{D}^+$ if and only if $q(B^{-1} \circ B) = 1$.*

PROOF: By Theorem 2.8, B is $\mathcal{D}_{SYM}^+$ if and only if $q(B^{-1} \circ B) = 1$, which holds if and only if $A \in \mathcal{D}_{SYM}^+$ by previous remarks. Hence there exists $E \in \mathcal{D}^+$ such that AE is symmetric. Since all the leading principal minors of A are positive if and only if all the leading principal minors of AE are positive, we see that $AE \in \mathcal{H}_{pd}(\mathbb{R})$ if and only if $A \in \mathcal{H}_{pd}(\mathbb{R})\mathcal{D}^+$. $\square$

If A is reducible, apply Theorem 2.9 to each irreducible direct summand of A .

If A has non-negative entries the following theorem of Johnson and Dias da Silva [10] tells us when $A \in \mathcal{H}_{pd}(\mathbb{R})\mathcal{D}^+$.

We say that A is **cycle balanced** if for any sequence $\{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$ we have $a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1} = a_{i_1 i_k} a_{i_k i_{k-1}} \cdots a_{i_2 i_1}$.

Theorem 2.10. [10] *Let $A \in M_n(\mathbb{R})$ be a non-negative irreducible strongly sign symmetric matrix. Then $A \in \mathcal{H}_{pd}(\mathbb{R})\mathcal{D}^+$ if and only if*

- (1) *the leading principal minors of A are positive, and*
- (2) *A is cycle balanced.*

$\square$

3. VIA GERŠGORIN'S THEOREM

By Geršgorin's Theorem, $\mathcal{G}_+ = \{A \in M_n(\mathbb{C}) | g(A) > 0\} \subseteq \mathcal{S}$. By Observation 1.1.6, the set of **strictly row-diagonally quasi-dominant** matrices

$$\begin{aligned} QD_s &= \{A \in M_n(\mathbb{C}) | \exists D \in \mathcal{D}^+ \text{ such that } g(AD) > 0\} \\ &= \mathcal{G}_+ \mathcal{D}^+ \end{aligned}$$

is also contained in $\mathcal{S}$ because $g(D^{-1}AD) = D^{-1}g(AD) > 0$ tells us $D^{-1}AD$, and hence A , is stable.

Since $g(DA) = Dg(A)$ when $D > 0$ is diagonal, we have $\mathcal{D}^+ \mathcal{G}_+ = \mathcal{G}_+$ and so

$$\mathcal{D}^+ QD_s = QD_s = \mathcal{G}_+ \mathcal{D}^+ = \mathcal{G}_+ \mathcal{D}^+ \mathcal{D}^+ = QD_s \mathcal{D}^+ = \mathcal{D}^+ QD_s \mathcal{D}^+.$$

Note that [2, 3.1.I.1], whose proof is easily corrected, is the corresponding result for $\{A \in M_n(\mathbb{R}) | g(A) \geq 0\}$ and real matrices with every diagonal entry positive.

Remark 3.1. In [2] there is confusion between $g(X) > 0$ and $g(X) \geq 0$. For example, $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is a counterexample to [2, 1.1.10] and [2, 2.2], as well as the relations [2, 3.1.II.2] and [2, 3.1.II.3]. $\square$

Here is a connection between M -matrices and some of the types of stable matrices we are discussing.

Theorem 3.2. [9] *If every diagonal entry of A is positive and every off-diagonal entry is non-positive, then the following are equivalent:*

- (1) $A \in \mathcal{S}$.
- (2) $A \in QD_s$.
- (3) $A \in \mathcal{L}(\mathcal{D}^+)$.
- (4) *There exist $D, E \in \mathcal{D}^+$ such that $g(DAE) > 0$ and $g(EA^T D) > 0$; i.e. DAE is strictly row- and column-diagonally dominant.*

$\square$

The next theorem is well-known for real matrices cf. [11].

Theorem 3.3. $QD_s \subseteq \mathcal{L}(\mathcal{D}^+)$.

PROOF: First, note that $A = (a_{ij}) \in QD_s$ if and only if $B = (b_{ij})$ defined by $b_{ii} = \operatorname{Re}(a_{ii})$, $b_{ij} = a_{ij}$, $i \neq j$, is in QD_s , and that $B \in QD_s$ if and only if its comparison matrix $C(B) \in QD_s$. It follows that there exist $D, E \in \mathcal{D}^+$ such that $g(DBE) > 0$ and $g(EB^*D) > 0$ by Theorem 3.2.4. Therefore $g(DBE + EB^*D) > 0$ and so $M = DBE + EB^*D$ is a stable Hermitian matrix by Geršgorin's Theorem. Next, $M > 0$ if and only if $(E^{-1}D)B + B^*(E^{-1}D) > 0$ by Observation 1.1.2. Finally, note that $B \in \mathcal{L}(\mathcal{D}^+)$ if and only if $A \in \mathcal{L}(\mathcal{D}^+)$. $\square$

Theorem 3.4. Let $M = \begin{bmatrix} a + iA & \beta \\ \gamma & d + iD \end{bmatrix}$ with $a, A, d, D \in \mathbb{R}$ and $\beta, \gamma \in \mathbb{C}$. Then $M \in QD_s = \mathcal{G}_+\mathcal{D}^+$ if and only if $a > 0$, $d > 0$, and $ad - |\beta\gamma| > 0$.

PROOF: $M \in QD_s$ if and only if there exist $p, q > 0$ such that $ap > |\beta|q$ and $dq > |\gamma|p$. Clearly then, $M \in QD_s$ implies the stated inequalities. Conversely, the point $R = \left(\frac{|\beta|}{a}, \frac{|\gamma|}{d}\right)$ lies in the first quadrant below the curve $xy = 1$. However, there is a point (p, p^{-1}) above and to the right of R if and only if $ap > |\beta|$ and $d > |\gamma|p$. Setting $q = 1$, we see $M \in QD_s$. $\square$

4. GERŠGORIN AND LYAPUNOV UNITE

We define the Geršgorin-Lyapunov stable matrices:

$$\begin{aligned} \mathcal{GL}_s &= \{A \in M_n(\mathbb{C}) \mid \exists P \in \mathcal{H}_{pd} \cap \mathcal{G}_+ \text{ such that } PA \in \mathfrak{R}_{pd}\} \\ &= (\mathcal{H}_{pd} \cap \mathcal{G}_+)^{-1} \mathfrak{R}_{pd} \subseteq \mathcal{S}. \end{aligned}$$

Remark 4.1. Suppose $P \in \mathcal{H}_{pd}$ with $g(P) \geq 0$ and $PA \in \mathfrak{R}_{pd}$. Set $Q = P + rI$ where $r > 0$. Then $Q \in \mathcal{H}_{pd}$ with $g(Q) > 0$, and $QA \in \mathfrak{R}_{pd}$ if $r > 0$ is small enough since $QA + A^*Q = PA + A^*P + r(A + A^*)$. So the more relaxed requirement $g(P) \geq 0$ would have resulted in the same set $\mathcal{GL}_s$, and hence what [2] calls $\mathcal{M}$ is what we would have obtained by restricting A and P in our definition of $\mathcal{GL}_s$ to be real. $\square$

Lemma 4.2. For $A \in M_n(\mathbb{C})$ the following are equivalent:

- (1) $A \in \mathcal{GL}_s$
- (2) $A^{-1} \in \mathcal{GL}_s$
- (3) $W^T A W \in \mathcal{GL}_s$ if W is a permutation matrix.

PROOF: The equivalence of the first two is Observation 1.1.1. The equivalence of the first and third is Observation 1.1.3 and the invariance of $\mathcal{H}_{pd} \cap \mathcal{G}_+$ under permutation similarity. $\square$

Remark 4.3. The invariance under permutation displayed in Lemma 4.2 must be present in any condition which characterizes $\mathcal{GL}_s$, but it is absent from [2, 4.1(5)]. The following theorem displays that invariance and contradicts [2, 4.1]. The proof in [2, 4.1] seems to establish the contrapositive when it should have proved the converse. $\square$

We close this section by characterizing $\mathcal{GL}_s \cap M_2(\mathbb{R})$.

Theorem 4.4. *Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{R})$. Then the following are equivalent:*

- (1) $A \in \mathcal{GL}_s$.
- (2) *There is a real $P \in \mathcal{H}_{pd} \cap \mathcal{G}_+$ such that $F = PA + A^*P > 0$.*
- (3) *A is stable, $a + |b| > 0$, $a + |c| > 0$, $d + |b| > 0$, and $d + |c| > 0$.*

PROOF:

Throughout we let $e = \det(A)$. (2) clearly implies (1).

(1) $\Rightarrow$ (3): If $A \in \mathcal{GL}_s$ then A is stable by Lyapunov's Theorem, hence $e > 0$.

Let $P = \begin{bmatrix} x & z \\ \bar{z} & y \end{bmatrix} \in \mathcal{H}_{pd} \cap \mathcal{G}_+$ be such that

$$F = \begin{bmatrix} 2ax + 2cRe(z) & bx + cy + (a+d)z \\ bx + cy + (a+d)\bar{z} & 2bRe(z) + 2dy \end{bmatrix} > 0.$$

Since $bRe(z) + dy > 0$ and since $y > |z|$ we have

$$0 < \frac{bRe(z) + dy}{y} \leq \frac{|b||Re(z)|}{y} + d < |b| + d.$$

Now, by Lemma 4.2.2 we know $A^{-1} = \frac{1}{e} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \in \mathcal{GL}_s$; arguing as above

we see $|b| + a > 0$. By Lemma 4.2.3, if $W = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ then $W^T A W = \begin{bmatrix} d & c \\ b & a \end{bmatrix}$ belongs to $\mathcal{GL}_s$ which gives $|c| + a > 0$. Finally, examining the inverse of $W^T A W \in \mathcal{GL}_s$ gives $|c| + d > 0$.

(3) $\Rightarrow$ (2): Suppose A is stable and satisfies the given inequalities in (3). It suffices to find $P = \begin{bmatrix} x & z \\ z & y \end{bmatrix} \in \mathcal{H}_{pd}(\mathbb{R}) \cap \mathcal{G}_+$ such that $2(bz + dy) > 0$ and

$$\det(F) = 4(ax + cz)(bz + dy) - (az + bx + cy + dz)^2 > 0.$$

Since A is stable, $e > 0$ and $a + d > 0$, hence either a or d must be positive. Using Lemma 4.2.3 we may, and do, assume $a > 0$.

If $d > 0$ there exists $P \in \mathcal{D}^+$ such that $F > 0$ by Theorem 6.1.5.

When $d \leq 0$ we have $bc = ad - e < 0$ and so we may set $z = \pm 1$ so that $bz = |b|$ and $cz = -|c|$.

Suppose $d = 0$. Then $bz + dy = |b| > 0$; since $bc < 0$ the line $bx + cy = 0$ will pass through points with both $x, y > 1$ and sufficiently large to make $\det(F) = 4|b|(ax - |c|) - a^2 > 0$. In this case $P = \begin{bmatrix} x & z \\ z & y \end{bmatrix} \in \mathcal{H}_{pd} \cap \mathcal{G}_+$.

Finally, assume $d < 0$ and set $f(x, y) = \det(F)$. The discriminant of $f(x, y)$ is $16ade < 0$, hence $f(x, y) = 0$ is an ellipse. Since $f(x, 0) = 4(ax + cz)bz - (az + bx + dz)^2 < 0$ for $x \gg 0$, $f(x, y) > 0$ at all points inside the ellipse $f(x, y) = 0$. The ellipse passes through the points

$$\begin{aligned} s &= \left(\frac{cz}{d}, \frac{z(e + d^2)}{cd} \right) \\ t &= \left(-\frac{z(e + d^2)}{bd}, -\frac{bz}{d} \right). \end{aligned}$$

Note that $s = t$ implies that $a = -d$ contradicting $\text{tr}(A) > 0$. Now, s and t lie in the convex region strictly above the curve $xy = 1$ when $x > 0$ (note that for either point the product of the coordinates is $\frac{e}{d^2} + 1$; note also that the choice of $z = \pm 1$ above puts both points in the right half-plane). Let $\bar{k}$ be the open line segment connecting s and t . Then $f(x, y) > 0$ on $\bar{k}$. By (3), $s \in \{(x, y) | x > 1\}$ and $t \in \{(x, y) | y > 1\}$. Both lie above the curve $xy = 1$ so $\bar{k}$ contains points in $\{(x, y) | x, y > 1\}$. At every such point (x, y) we have $P = \begin{bmatrix} x & z \\ z & y \end{bmatrix} \in \mathcal{H}_{pd} \cap \mathcal{G}_+$. To show $g(x, y) = bz + dy = |b| + dy > 0$ on $\bar{k}$, note that $g(t) = 0$ and $g(s) = \frac{zd}{c}(a + d) > 0$. $\square$

Remark 4.5. We have been informed that this and certain other Lyapunov-condition based theorems can be proven via the following Lemma 4.6.

Lemma 4.6. *Suppose A, R , and S are real matrices and let $H^* = H = R + iS$. If $HA + A^*H > 0$ then $RA + A^*R > 0$.*

PROOF: Conjugating gives $\overline{H}A + A^*\overline{H} > 0$; adding this to the assumed inequality and dividing by 2 produces the result. $\square$

5. GERŠGORIN AGAIN

The set of matrices

$$\begin{aligned} \mathcal{W}_{sdom} &= \mathcal{H}_{pd}\mathcal{G}_+\mathcal{D}^+ \\ &= \{A \in M_n(\mathbb{C}) | \exists W \in \mathcal{H}_{pd} \text{ such that } WA \in \mathcal{G}_+\mathcal{D}^+\} \end{aligned}$$

by Observation 1.1.5. Restricting both A and W to be real, we obtain the set $\mathcal{H}_{pd}(\mathbb{R})\mathcal{G}_+(\mathbb{R})\mathcal{D}^+$ which was introduced in [12].

Contrary to [2, 3.1.II.3], $\mathcal{W}_{sdom}(\mathbb{R}) \not\subseteq \mathcal{S}$:

Example 5.1. Let a, b, c, d be positive and let $ab > 1$ and $cd > 1$. Let

$$A = \begin{bmatrix} a & -1 \\ -1 & b \end{bmatrix} \begin{bmatrix} c & r \\ \frac{1}{r} & d \end{bmatrix}$$

Then $A \in \mathcal{H}_{pd}(\mathbb{R})\mathcal{G}_+(\mathbb{R})\mathcal{D}^+$ by Theorem 3.4, but A is not always stable since $\text{tr}(A) = ac + bd - (r + r^{-1}) < 0$ for $r > 0$ sufficiently small. $\square$

Theorem 5.2. $\mathcal{S}\mathcal{D}^+ = \mathcal{H}_{pd}\mathcal{L}(\mathcal{D}^+)$.

PROOF: Let D and P be Hermitian and set

$$L = P(AD) + (AD)^*P = (PA)D + D(PA)^*.$$

If $A \in \mathcal{S}\mathcal{D}^+$ then there exists $D \in \mathcal{D}^+$ such that AD is stable, so there exists $P \in \mathcal{H}_{pd}$ such that $L > 0$. Hence $(PA)^* \in \mathcal{L}(\mathcal{D}^+)$, and by Proposition 1.3 so is PA . Thus $A \in \mathcal{H}_{pd}\mathcal{L}(\mathcal{D}^+)$.

If $A \in \mathcal{H}_{pd}\mathcal{L}(\mathcal{D}^+)$, there is a $P \in \mathcal{H}_{pd}$ such that $PA \in \mathcal{L}(\mathcal{D}^+)$. So by Proposition 1.3, $(PA)^* \in \mathcal{L}(\mathcal{D}^+)$. Then there is a $D \in \mathcal{D}^+$ such that $L > 0$, which shows that AD is stable and hence $A \in \mathcal{S}\mathcal{D}^+$. $\square$

In [2, 2.3], this result is stated for real matrices but only the containment $\mathcal{S}(\mathbb{R})\mathcal{D}^+ \subseteq \mathcal{H}_{pd}(\mathbb{R})\mathcal{L}(\mathcal{D}^+)(\mathbb{R})$ is proven there.

Theorems 3.3 and 5.2 together yield

Corollary 5.3. $\mathcal{H}_{pd}QD_s \subseteq \mathcal{H}_{pd}\mathcal{L}(\mathcal{D}^+) = \mathcal{S}\mathcal{D}^+.$ $\square$

Note that for each $A \in \mathcal{H}_{pd}QD_s$ there exists a $D \in \mathcal{D}^+$ such that $AD \in \mathcal{S}$. Here is another class of matrices which shares this property:

Theorem 5.4. [8] *Let $A \in M_n(\mathbb{C})$ have all principal minors real and all leading principal minors positive. Then there exists $D \in \mathcal{D}^+$ such that AD has simple positive eigenvalues.* $\square$

The following example shows that $\mathcal{H}_{pd}QD_s$ and the class of matrices in Theorem 5.4 are not the same:

Remark 5.5.

$$A = \begin{bmatrix} 5 & -1 & 3 \\ -1 & 2 & -2 \\ 3 & -2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 5 \end{bmatrix} \in \mathcal{H}_{pd}(\mathbb{R})QD_s(\mathbb{R})$$

but $a_{11} < 0.$ $\square$

Theorems 5.2 and 5.4 combine to give

Theorem 5.6. *Let $A \in M_n(\mathbb{C})$ have all principal minors real and all leading principal minors positive. Then $A \in \mathcal{H}_{pd}\mathcal{L}(\mathcal{D}^+).$* $\square$

6. SET INCLUSIONS

The following Venn diagram displays inclusion relations between sets of real stable matrices. Each of the sets depicted is the intersection of $M_n(\mathbb{R})$ with a set of complex stable matrices studied above. Our diagram generalizes Venn diagrams in [2] and [4], and corrects [2, Fig. 2]. Several of the examples we give originally appeared in [2] and [4]. The nine containments in the diagram can be justified as follows, starting from the largest set and working down:

$\mathbb{D}_{stab} \subseteq \mathcal{S}$: from the definition with $D = I$.

$\mathcal{GL}_s \subseteq \mathcal{S}$: from Lyapunov's Theorem.

$\mathbb{H}_{stab} \subseteq \mathbb{D}_{stab}$: from the definition.

$\mathcal{L}(\mathcal{D}^+) \subseteq \mathbb{D}_{stab}$: from [4, 2.3].

$\mathcal{L}(\mathcal{D}^+) \subseteq \mathcal{GL}_s$: from the definition.

$\mathcal{R}_{pd} \subseteq \mathbb{H}_{stab}$: from [4, 2.3].

$\mathcal{R}_{pd} \subseteq \mathcal{L}(\mathcal{D}^+)$: from [4, 2.3].

$\mathcal{G}_+(\mathbb{R})\mathcal{D}^+ \subseteq \mathcal{L}(\mathcal{D}^+)(\mathbb{R})$: from [11, 3.4].

$\mathcal{H}_{pd}(\mathbb{R})\mathcal{D}^+ \subseteq \mathcal{L}(\mathcal{D}^+)(\mathbb{R})$: from the sentence following Example 2.5.

When $n = 1$ these sets coincide with the positive real numbers. When $n > 1$, we show that some of the regions in the diagram are non-empty by placing a letter in the region denoting a matrix (defined below) which is located there. Forming the direct sum of a given example and an identity matrix produces an example of larger dimension. To justify the locations of the 2×2 examples use the following theorem which also shows that there are no 2×2 real examples in the regions where F, H, M , and O appear.

Theorem 6.1. *Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{R}).$*

- (1) $A \in \mathcal{S}$ if and only if $\text{tr}(A) > 0$ and $\det(A) > 0$.
- (2) $A \in \mathcal{GL}_s$ if and only if $A \in \mathcal{S}$, $a + |b| > 0$, $a + |c| > 0$, $d + |b| > 0$, and $d + |c| > 0$.
- (3) $A \in \mathbb{D}_{stab}$ if and only if $A \in \mathcal{S}$ and $a, d \geq 0$.
- (4) $A \in \mathbb{H}_{stab}$ if and only if $A \in \mathbb{D}_{stab}$ and $4ad - (b + c)^2 \geq 0$.
- (5) $A \in \mathcal{L}(\mathcal{D}^+)$ if and only if $a, d > 0$ and $\det(A) > 0$.
- (6) $A \in QD_s$ if and only if $a, d > 0$ and $ad > |bc|$.
- (7) $A \in \mathcal{H}_{pd}\mathcal{L}_{\mathcal{D}}$ if and only if $A \in \mathcal{L}(\mathcal{D}^+)$ and A is strongly sign symmetric.
- (8) $A \in \mathbb{R}_{pd}$ if and only if $a > 0$ and $4ad - (b + c)^2 > 0$.

PROOF: (1) is a well-known characterization of stability; (2) is Theorem 4.4; (6) is Theorem 3.4; (7) is Corollary 2.6. The rest can be found in [4]. $\square$

Figure 1: For $n \times n$ real matrices with $n > 1$. Dotted lines are conjectural as we know of no examples for H or for M .

Example 6.2. *Examples of matrices in various sets are given:*

$$\begin{aligned}
A &= \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} & H &= \text{unknown} \\
B &= \begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix} & J &= \begin{bmatrix} 4 & 10 \\ -2 & -3 \end{bmatrix} \\
C &= \begin{bmatrix} 1 & -\frac{1}{2} \\ -3 & 2 \end{bmatrix} & K &= \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix} \\
D &= \begin{bmatrix} 1 & -\frac{1}{8} \\ 15 & 2 \end{bmatrix} & L &= \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \\
E &= \begin{bmatrix} 1 & 2\sqrt{2}-3 \\ 3 & 2 \end{bmatrix} & M &= \text{unknown} \\
F &= \begin{bmatrix} 1 & -4 & 5 \\ \frac{1}{5} & 1 & -\frac{1}{2} \\ -\frac{1}{4} & 2 & 1 \end{bmatrix} & N &= \frac{1}{8} \begin{bmatrix} -2 & 8 \\ -7 & 4 \end{bmatrix} \\
G &= \begin{bmatrix} 0 & \frac{1}{2} \\ -1 & 1 \end{bmatrix} & O &= \begin{bmatrix} 5 & -1 & 3 \\ -1 & 2 & -2 \\ 3 & -2 & 3 \end{bmatrix}
\end{aligned}$$

Example 6.3. $F = \begin{bmatrix} 1 & -4 & 5 \\ \frac{1}{5} & 1 & -\frac{1}{2} \\ -\frac{1}{4} & 2 & 1 \end{bmatrix} \in \mathcal{L}(\mathcal{D}^+)$ by [3, p.77]. $F \notin \mathbb{H}_{stab}$ since $F + F^*$ is not positive semidefinite [5].

If F were in QD_s , there would exist $D = \text{Diag}(d_1, d_2, d_3) \in \mathcal{D}^+$ such that $g(FD) > 0$; this holds if and only if d_1, d_2, d_3 satisfy the system of inequalities

$$\begin{aligned}
d_1 - 4d_2 - 5d_3 &> 0 \\
-0.2d_1 + d_2 - 0.5d_3 &> 0 \\
-0.25d_1 - 2d_2 + d_3 &> 0.
\end{aligned}$$

Since

$$\begin{aligned}
0 &< (d_1 - 4d_2 - 5d_3) + 10(-0.2d_1 + d_2 - 0.5d_3) + 4(-0.25d_1 - 2d_2 + d_3) \\
&= -2d_1 - 2d_2 - 6d_3 \\
&< 0,
\end{aligned}$$

$F \notin QD_s$.

Finally, since F is not strongly sign-symmetric, $F \notin \mathcal{H}_{pd}\mathcal{S}_{\mathcal{D}}$.

Note that in the real 2×2 case, $QD_s = \mathcal{L}(\mathcal{D}^+)$ by [6]. □

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MATHEMATICS DEPARTMENT, IOWA STATE UNIVERSITY, AMES IA 50011

E-mail address: `bcain@iastate.edu`

DEPARTMENT OF MATHEMATICS, 214 PEARCE, CENTRAL MICHIGAN UNIVERSITY, MOUNT PLEASANT MI 48859

E-mail address: `lenke1td@cmich.edu`

DEPARTMENT OF MATHEMATICS, 214 PEARCE, CENTRAL MICHIGAN UNIVERSITY, MOUNT PLEASANT MI 48859

E-mail address: `naray1sk@cmich.edu`

DEPARTMENT OF MATHEMATICS, 214 PEARCE, CENTRAL MICHIGAN UNIVERSITY, MOUNT PLEASANT MI 48859

E-mail address: `verme1pj@cmich.edu`