AN EFFECTIVE BOUND FOR REFLEXIVE SHEAVES ON CANONICALLY TRIVIAL 3-FOLDS

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ABSTRACT. We give effective bounds for the third Chern class of a semistable rank 2 reflexive sheaf on a canonically trivial threefold.

1. INTRODUCTION

We work over an algebraically closed field of characteristic 0.

In a previous work [5] we showed that if $\mathcal{F}$ is a semistable rank 2 reflexive sheaf on a smooth projective threefold X with $\text{Pic}(X) = \mathbb{Z}$, then there is an upper bound on $c_3(\mathcal{F})$ in terms of $c_i(X)$, $c_1(\mathcal{F})$ and $c_2(\mathcal{F})$ (note that $c_3(\mathcal{F}) \geq 0$). It has more recently been conjectured [2] that this remains true for any smooth, projective threefold. Here we derive explicit effective bounds in the case of a polarized smooth projective threefold X with $\omega_X = \mathcal{O}_X$, without the restriction on the Picard group.

2. STABILITY AND BOUNDEDNESS

Definition 1. Let L be a very ample line bundle on a smooth projective variety X . A rank 2 reflexive coherent sheaf $\mathcal{F}$ on X is **L -stable** (resp. **L -semistable**) if for every invertible subsheaf $\mathcal{F}'$ of $\mathcal{F}$ with $0 < \text{rank } \mathcal{F}' < \text{rank } \mathcal{F}$, we have $\mu(\mathcal{F}', L) < \mu(\mathcal{F}, L)$ (resp. $\leq$), where

$$\mu(\mathcal{F}, L) = \frac{c_1(\mathcal{F}) \cdot [L]^{\dim X - 1}}{(\text{rank } \mathcal{F}) [L]^{\dim X}}$$

□

Definition 2. A reflexive sheaf $\mathcal{F}$ is **normalized with respect to L** if $-1 < \mu(\mathcal{F}, L) \leq 0$. As L is typically fixed, we usually say simply that $\mathcal{F}$ is **normalized**. Note that as $\mu(\mathcal{F} \otimes L, L) = \mu(\mathcal{F}, L) + 1$, there exists, for any fixed $\mathcal{F}$, a unique $k \in \mathbb{Z}$ such that $\mathcal{F} \otimes L^k$ is normalized with respect to L . □

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For a fixed smooth, canonically trivial $X \subset \mathbb{P}^n$, our goal is to give a bound on $c_3(\mathcal{F})$ in terms of $c_1(\mathcal{F})$ and $c_2(\mathcal{F})$.

Lemma 3. *Let X be a smooth projective canonically trivial threefold, L any very ample line bundle on X , $\mathcal{F}$ a rank two reflexive sheaf. If $\mathcal{F}$ admits a section $s \in \Gamma(\mathcal{F})$ whose zero locus is a curve Y , then*

$$c_3(\mathcal{F}) \leq d^2 - 3d - c_1(\mathcal{F})c_2(\mathcal{F})$$

where $d = c_1(L)c_2(\mathcal{F})$ is the degree of Y in the embedding induced by L .

PROOF: This is directly related to the Hartshorne-Serre correspondence [3, Theorem 4.1]. It is shown there that any such Y is locally Cohen-Macaulay, and it can be immediately deduced [5, Theorem 1] that $c_3(\mathcal{F}) = 2p_a(Y) - 2 - c_2(\mathcal{F})c_1(\omega_X) - c_1(\mathcal{F})c_2(\mathcal{F})$. In the canonically trivial case $c_2(\mathcal{F})c_1(\omega_X) = 0$, and in any case the degree of the curve section Y in the embedding given by L is $d = c_1(L)c_2(\mathcal{F})$. The fact that $2p_a(Y) - 2 \leq d^2 - 3d$ is the bound coming from the degree of a plane curve. $\square$

In some common situations, we can do better:

Proposition 4. *Let X be a smooth projective canonically trivial threefold, L a very ample line bundle, $\mathcal{F}$ a stable (resp. semistable) rank two reflexive sheaf. Suppose that $\mu(\mathcal{F}, L) \geq 1$ (resp. $\mu(\mathcal{F}, L) > 1$) and that $H^1(X, \det \mathcal{F}^* \otimes L) = 0$. If $s \in \Gamma(\mathcal{F})$ is a section whose zero locus is a smooth curve Y , then*

$$c_3(\mathcal{F}) \leq m(m-1)(h^0(X, L) - 2) + 2m\epsilon - 2 - c_1(\mathcal{F})c_2(\mathcal{F})$$

where

$$m = \left\lfloor \frac{c_1(L)c_2(\mathcal{F}) - 1}{h^0(X, L) - 2} \right\rfloor$$

$$\epsilon = c_1(L)c_2(\mathcal{F}) - 1 - m(h^0(X, L) - 2)$$

PROOF: The section gives the sequence

$$0 \rightarrow \det \mathcal{F}^* \otimes L \rightarrow \mathcal{F}^* \otimes L \rightarrow \mathcal{I}_Y \otimes L \rightarrow 0$$

By hypothesis, $\mu(\mathcal{F}^* \otimes L, L) \leq 0$ when $\mathcal{F}$ is stable and $\mu(\mathcal{F}^* \otimes L, L) < 0$ when $\mathcal{F}$ is semistable. In either case $H^0(X, \mathcal{F}^* \otimes L) = 0$, and so $H^0(X, \mathcal{I}_Y \otimes L) = 0$. This implies that Y is a non-degenerate curve in the embedding induced by L , hence we may apply Castelnuovo's Bound [1, p. 116]. $\square$

The idea now is: given a very ample line bundle L , bound the twist of $\mathcal{F}$ by L^r needed to produce a section, and then use the bound in

Lemma 3. We do this by first finding a bound for the vanishing of $h^2(\mathcal{F} \otimes L^r)$ and then by making the Euler characteristic positive.

The next two results (Proposition 5 and Corollary 7) follow directly from more general results in [5]. The first proceeds by showing that a minimal m satisfying the given inequality but contradicting the claim must be positive; the second proceeds from the first by Serre Duality on the smooth surface D , and then the stated inequality forces the Euler characteristic to be positive via Riemann-Roch, hence the relevant h^0 must be strictly greater than h^1 , and in particular must be positive.

Proposition 5. [5, Proposition 9] *Let X be a smooth, projective canonically trivial threefold with very ample line bundle L , $\mathcal{F}$ a normalized L -semistable rank 2 reflexive sheaf. Let D be a general member of the linear system $|L|$. Suppose $m < 0$ is an integer satisfying*

$$m + \mu(\mathcal{F}, L) + 1 < 0$$

Then $H^0(D, \mathcal{F}_D(mD)) = 0$. $\square$

Remark 6 Note that the reference contains the hypothesis $\text{Pic}(X) = \mathbb{Z}$, but it is not relevant for this particular proposition. $\square$

Corollary 7. [5, Corollary 11] *With notation and hypotheses as in Proposition 5, if*

$$r > 2 + \mu(\mathcal{F}, L)$$

then for the general member D , $H^2(D, \mathcal{F}_D^(rD)) = 0$. If, furthermore, r is such that:*

$$(6r^2 - 6r + 2) - 2(6r - 3)\mu(\mathcal{F}, L) \geq \frac{(6c_2(\mathcal{F}) - 3c_1(\mathcal{F})^2 - c_2(X))[L]}{[L]^3}$$

then $H^0(D, \mathcal{F}_D^(rD)) \neq 0$.* $\square$

Corollary 8. *With notation and hypotheses as in Proposition 5 there exists a constant ρ_1 depending on $L, c_1(\mathcal{F}), c_2(\mathcal{F})$ and $c_2(X)$ such that if $r \geq \rho_1$ then $H^1(D, \mathcal{F}_D^*(rD)) = 0$.*

PROOF:

By the previous Corollary, there is a constant depending on the above parameters such that if k is larger than that constant, then $\mathcal{F}_D^*(kD)$ has a section. Choosing the *smallest* such integer k we have a sequence:

$$0 \rightarrow \mathcal{O}_D \rightarrow \mathcal{F}_D^*(kD) \rightarrow \mathcal{I}_Z(2kD) \otimes \det \mathcal{F}^* \rightarrow 0$$

where $Z \subset D$ is zero dimensional of length

$$\ell = c_1(L)c_2(\mathcal{F}) - kc_1^2(L)c_1(\mathcal{F}) + k^2c_1^3(L)$$

Because $K_D = L \otimes \mathcal{O}_D$ is (very) ample, $H^1(D, \mathcal{O}(D)) = H^2(X, \mathcal{O}_X)$ and $H^1(D, \mathcal{O}(2D)) = 0$.

Let $\alpha \in \mathbb{Z}$ be such that $|\det \mathcal{F}^* \otimes L^\alpha \otimes \mathcal{O}_D|$ contains a curve that misses Z (e.g. if the system is very ample). Because $\mathcal{O}_D(D)$ is very ample, we have $H^1(D, \mathcal{I}_Z((2k+t)D)) = 0$ when $2k+t \geq \ell-1$ by a uniform regularity result of Mumford [4, p. 103].

Therefore, by choosing a curve that misses Z in the system $|\det \mathcal{F}^* \otimes L^\alpha \otimes \mathcal{O}_D|$, we find $H^1(D, \mathcal{I}_Z((2k+t+\alpha)D) \otimes \det \mathcal{F}^*) = 0$ when $2k+t+\alpha \geq \ell-1$ and $2k+t \geq 2$.

Consequently, $H^1(D, \mathcal{F}_D^*(rD)) = 0$ for

$$r \geq \max\{k+2, \ell-1-k, 2-k+\alpha\}$$

□

Corollary 9. *With notation and hypotheses as in Proposition 5, there exists an integer ρ_2 depending on $L, c_1(\mathcal{F}), c_2(\mathcal{F})$ and $c_2(X)$ such that if $r \geq \rho_2$ then $H^0(X, \mathcal{F}^* \otimes L^r) \neq 0$.*

PROOF: The vanishing of H^1 and H^2 on D described in the above Corollaries gives $H^2(X, \mathcal{F}^* \otimes L^r) = 0$. The result now follows by an Euler characteristic argument. □

Theorem 10. *Let X be a smooth, projective canonically trivial threefold with very ample line bundle L , and $\mathcal{F}$ an L -semistable rank 2 reflexive sheaf. Then there exists an integer C depending on $L, c_1(\mathcal{F}), c_2(\mathcal{F})$ and $c_2(X)$ such that $C \geq c_3(\mathcal{F})$.*

PROOF: As $c_3(\mathcal{F})$ is unaffected by twisting by a line bundle, we may assume $\mathcal{F}$ is normalized. Then the above results apply and we can take a section of $\mathcal{F}^* \otimes L^k$ for some k , bounded as in Corollary 9. We then have an exact sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{F}^* \otimes L^k \rightarrow \mathcal{I}_Y \otimes L^{2k} \otimes \det \mathcal{F}^* \rightarrow 0$$

where $Y \subset X$ is a curve. Computing Euler characteristics gives:

$$2p_a(Y) - 2 = d_1 d_2 + c_3(\mathcal{F})$$

where

$$d_1 = c_1(\mathcal{F}^* \otimes L^k) = -c_1(\mathcal{F}) + 2kc_1(L)$$

and

$$\begin{aligned} d_2 &= c_2(\mathcal{F}^* \otimes L^k) \\ &= c_2(\mathcal{F}) - kc_1(\mathcal{F})c_1(L) + k^2 c_1^2(L) \end{aligned}$$

Note that in the embedding determined by L , the degree of the curve Y is precisely $d_2c_1(L)$. This implies $d_2c_1(L)(d_2c_1(L) - 3) \geq 2p_a(Y) - 2$ and so

$$d_2c_1(L)(d_2c_1(L) - 3) - d_1d_2 \geq c_3(\mathcal{F})$$

□

The following is a straightforward calculation as in [5]. For notational convenience we let $d = c_1(L)c_2(\mathcal{F})$.

Corollary 11. *Let $X \subset \mathbb{P}^4$ be a smooth quintic hypersurface. Let $\mathcal{F}$ be a rank two semistable reflexive sheaf with $\mu(\mathcal{F}, \mathcal{O}_X(1)) = 0$. Then for $d \leq 40$ we have*

$$c_3(\mathcal{F}) \leq (5d^2 + 136d + 910)(5d^2 + 134d + 880).$$

PROOF: We let $L = \mathcal{O}_X(1)$ and note that $[L]^3 = 5$ and $c_2(X) = 10c_1(L)^2$. The first bound in Corollary 7 is $r \geq 3$ and the second becomes $6r^2 - 6r + 2 \geq \frac{6d-50}{5}$, or

$$r \geq \frac{5 + \sqrt{20d - 175}}{10}.$$

As long as $d \leq 40$, the second inequality is weaker than the first, hence we need only take $r \geq 3$. In Corollary 8, we have $\alpha = 2$ and $\ell = d - 5r + 5r^2$, and hence the bound there becomes $\rho_1 \geq \max\{r + 2, 5r^2 - 6r + d - 1, 4 - r\}$, but the second terms is strongest and we have $\rho_1 \geq d + 14$.

Turning to Corollary 9, we need $\chi(\mathcal{F}^* \otimes L^{\rho_2}) > 0$, but Riemann-Roch gives

$$\chi(\mathcal{F}^* \otimes L^{\rho_2}) - \frac{1}{2}c_3(\mathcal{F}) = \frac{\rho_2}{3}(5\rho_2^2 + 25 - 3d)$$

and this is clearly positive for $\rho_2 \geq d + 14$.

Finally, from Theorem 10 we have $d_1 = (2\rho_2 - 1)c_1(L)$ and $d_2 = c_2(\mathcal{F}) + (\rho_2^2 - \rho_2)c_1^2(L)$, and hence the bound for c_3 becomes

$$\begin{aligned} c_3(\mathcal{F}) &\leq d_2c_1(L)(d_2c_1(L) - 3) - d_1d_2 \\ &= (d + 5(\rho_2^2 - \rho_2))(d + 5(\rho_2^2 - \rho_2) - 3) - (2\rho_2 - 1)(d + 5(\rho_2^2 - \rho_2)) \end{aligned}$$

Substituting $\rho_2 = d + 14$ yields $c_3(\mathcal{F}) \leq (5d^2 + 136d + 910)(5d^2 + 134d + 880)$. □

Remark 12 Note that this is somewhat better than the bound found in [5], though that bound was also quartic in d . Note that for $\mathbb{P}^3$, $c_3(\mathcal{F})$ is bounded by a quadratic in d [3, 8.2]. □

For the next result, recall [6, Notation 6] that $\Delta(\mathcal{F}) = c_2(\mathcal{F} \otimes \mathcal{F}^*) = 4c_2(\mathcal{F}) - c_1^2(\mathcal{F})$.

Proposition 13. *Let X be a smooth projective Fano threefold. If $\mathcal{F}$ is a rank 2 semistable reflexive sheaf, then*

$$(1) \quad c_1(X)\Delta(\mathcal{F}) \geq \frac{c_1(X)c_2(X)}{3} - 2.$$

PROOF: [6, Corollary 12] gives

$$c_1(X)\Delta(\mathcal{F}) = 2\operatorname{ext}_{\mathcal{O}_X}^1(\mathcal{F}, \mathcal{F}) - 2\operatorname{ext}_{\mathcal{O}_X}^2(\mathcal{F}, \mathcal{F}) + \frac{c_1(X)c_2(X)}{3} - 2$$

Further, [6, Corollary 15] states that the relevant moduli space of sheaves admits a perfect tangent-obstruction complex and has a virtual cycle of the expected dimension $\operatorname{ext}_{\mathcal{O}_X}^1(\mathcal{F}, \mathcal{F}) - \operatorname{ext}_{\mathcal{O}_X}^2(\mathcal{F}, \mathcal{F})$. In particular, this expression is non-negative. $\square$

As a corollary, we have:

Corollary 14. *Let $X \subset \mathbb{P}^4$ be a smooth Fano hypersurface of degree r , $\mathcal{F}$ a rank 2 semistable reflexive sheaf with $\mu(\mathcal{F}, \mathcal{O}_X(1)) = 0$.*

- (1) *If $r = 1, 2, 3$ then $c_1(L)c_2(\mathcal{F}) \geq 1$.*
- (2) *If $r = 4$ then $c_1(L)c_2(\mathcal{F}) \geq 0$.*

$\square$

PROOF: These follow from (1) by direct computation. $\square$

Remark 15 Note that the case $r = 1$ is related to well-known bound on $\mathbb{P}^3$ [3, 3.3]; we do not know if these bounds are the best possible. $\square$

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