A COUNTER EXAMPLE TO FULTON'S CONJECTURE ON $\overline{M}_{0,n}$

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Abstract. We describe an elementary counter example to a conjecture of Fulton on the set of effective divisors on the space $\overline{M}_{0,n}$ of marked rational curves.

1. Introduction

We work over an algebraically closed field of characteristic 0.

Consider the following conjecture of Fulton

Conjecture 1.1. [KM] *Every effective k-cycle on $\overline{M}_{0,n}$ is an effective sum of vital k-cycles; where a codimension r vital cycle is an irreducible component parameterizing curves with r singular points.* □

Following [GKM] we denote this conjecture by $F_k(0,n)$. For $k = 1$, the same question can be posed for $\overline{M}_{g,n}$; we denote analogous conjecture $F_1(g,n)$. We denote by $F^k(0,n)$ the conjecture stated for codimension k cycles. Denoting by $\widetilde{M}_{0,n}$ the quotient of $\overline{M}_{0,n}$ by the natural action of S_n, we can make the analogous conjecture on $\widetilde{M}_{0,n}$, denoted $\widetilde{F}_1(0,n)$.

Two main results are

Theorem 1.2. [GKM] *If $F_1(0,n+g)$ holds then $F_1(g,n)$ holds. Further, $F_1(g,n)$ is equivalent to $\widetilde{F}_1(0,g+n)$.* □

Theorem 1.3. *Notation as above;*

1. $F_1(0,n)$ *holds for* $n \leq 7$ [KM]
2. $\widetilde{F}_1(0,n)$ *holds for* $n \leq 11$ [KM]
3. $\widetilde{F}_1(0,n)$ *holds for* $n \leq 13$ [FG] □

We also recall the following description of $\overline{M}_{0,n}$ due to Kapranov:

Let $X \subset \mathbb{P}^n$ be a set of $n+2$ reduced points in linear general position. Construct the sequence of blow-ups: $f^0 : B^0 \to \mathbb{P}^n$ is the blow up of $\mathbb{P}^n$ along X

$f^1 : B^1 \to B^0$ is the blow up of B^0 along the proper transforms of the $\binom{n+2}{2}$ chords through the points

$f^2 : B^2 \to B^1$ is the blow up of B^1 along the proper transforms of the $\binom{n+2}{3}$ 2-planes through three points

$\vdots$

$f^{n-2} : B^{n-2} \to B^{n-3}$ is the blow up of B^{n-3} along the proper transforms of the $\binom{n+2}{n-1}$ $(n-2)$-planes though $(n-1)$ points

We then have:

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Theorem 1.4. [K] *With notation as above,* $B^{n-2} \cong \overline{M}_{0,n+3}$ □

We understand this identification as follows: If you fix $n+2$ general points in $\mathbb{P}^n$, then through any other point in general position there is a unique rational normal curve, which now has $n+3$ markings. When this extra point moves out of general position, we are forced to blow up various linear spans of the original $n+2$ points. It is, furthermore, worth noting that to make the identification in Theorem 1.4 explicit, we must choose one of the marked points to be "special". For example, $\overline{M}_{0,5}$ is isomorphic to $\mathbb{P}^2$ with four general points blown up. There are, however, five different maps $\overline{M}_{0,5} \to \mathbb{P}^2$ that fit this description, each corresponding to a case where four points are considered fixed and the fifth is allowed to move (it is also worth noting that these maps are induced by the five ψ_i classes).

Our purpose here is to discuss $F^1(0,n)$ from the point of view of Theorem 1.4. For notational convenience, we denote the collection of $(i+1)$-secant i-planes by Σ_i.

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2. Effective Divisors

In the Spring of 2000, Keel announced a collection of counterexamples to $F^1(0,n)$, $n \geq 6$. Here we describe a set of counterexamples which are elementary from the point of view of Theorem 1.4.

We begin with $\overline{M}_{0,6}$. Divisors on $\overline{M}_{0,6}$ are denoted D_I, $I \subseteq \{1,2,3,4,5,6\}$, where D_I is the divisor parameterizing curves with the elements of I marked on one component and the complementary indices marked on the other. In particular, for example, $D_{123} = D_{456}$. On the blow up of $\mathbb{P}^3$, divisors are denoted H, E_i, E_{ij} where H is the proper transform of a hyperplane, E_i is the exceptional divisor coming from the blow up of the point p_i and E_{ij} is the exceptional divisor coming from the blow up of the line through p_i and p_j.

The divisor dictionary is then as follows (note that we take 6 to be the special point):

$\overline{M}_{0,6}$	B^1	$\overline{M}_{0,6}$	B^1
D_{12}	$H - E_3 - E_4 - E_5 - E_{34} - E_{35} - E_{45}$	D_{46}	E_4
D_{13}	$H - E_2 - E_4 - E_5 - E_{24} - E_{25} - E_{45}$	D_{56}	E_5
D_{14}	$H - E_2 - E_3 - E_5 - E_{23} - E_{25} - E_{35}$	D_{123}	E_{45}
D_{15}	$H - E_2 - E_3 - E_4 - E_{23} - E_{24} - E_{34}$	D_{124}	E_{35}
D_{16}	E_1	D_{125}	E_{34}
D_{23}	$H - E_1 - E_4 - E_5 - E_{14} - E_{15} - E_{45}$	D_{126}	E_{12}
D_{24}	$H - E_1 - E_3 - E_5 - E_{13} - E_{15} - E_{35}$	D_{134}	E_{25}
D_{25}	$H - E_1 - E_3 - E_4 - E_{13} - E_{14} - E_{34}$	D_{135}	E_{24}
D_{26}	E_2	D_{136}	E_{13}
D_{34}	$H - E_1 - E_2 - E_5 - E_{12} - E_{15} - E_{25}$	D_{145}	E_{23}
D_{35}	$H - E_1 - E_2 - E_4 - E_{12} - E_{14} - E_{24}$	D_{146}	E_{14}
D_{36}	E_3	D_{156}	E_{15}
D_{45}	$H - E_1 - E_2 - E_3 - E_{12} - E_{13} - E_{23}$		

As the points are in linear general position, we may as well assume they are $p_1 = [1,0,0,0]$, $p_2 = [0,1,0,0]$, $p_3 = [0,0,1,0]$, $p_4 = [0,0,0,1]$, and $p_5 = [1,1,1,1]$. We take our variables to be x_0, x_1, x_2, x_3.

Lemma 2.1. *The divisor*

$$Q = 2H - E_1 - E_2 - E_3 - E_4 - E_5 - E_{13} - E_{14} - E_{23} - E_{24}$$

has a section.

PROOF: It is easy to verify that the proper transform of $x_0x_1 - x_2x_3$ is a section. In fact, it is not hard to show that $h^0(B^1, \mathcal{O}(Q)) = 1$. $\square$

Lemma 2.2. *Q is not an effective sum of boundary divisors.*

PROOF: Suppose it is the effective sum of boundary divisors. Then by the dictionary on the previous page, we must have precisely two of the D_{ij}, $j \neq 6$. We attempt to construct them.

At least one of them must contain $-E_5$, and so we begin with

$$H-?$$
$$H - E_5 -?$$

Now, note that the second divisor will contain two terms of the form $-E_{i5}$, neither of which help to build Q. This leaves four other $-E_{ij}$ to be determined. *However*, note that the three divisors of the form $-E_{ij}$ in the first expression will form a cycle. As Q has no such cycle, we waste one of our four choices and so cannot build Q. $\square$

Translating back, we see

$$\begin{aligned} Q &= D_{25} + D_{36} + D_{46} + 2D_{125} + D_{15} - D_{56} \\ &= D_{1346} + D_{1245} + D_{1235} + 2D_{125} + D_{15} - D_{1234} \end{aligned}$$

The same argument can now be used to show that

$$Q + \sum_{i=1}^{4} a_i D_{i5}$$

is also effective and is not an effective sum of boundary divisors (for $a_i \in \mathbb{N}$).

Remark 2.3 It should be noted that Q is not nef, consistent with Faber's proof [F] that in $\overline{M}_{0,6}$ any nef divisor is an effective sum of boundary divisors. In particular, if C is the proper transform of the line $\{[a,b,a,b]\}$ then $Q.C = -1$. It should also be noted here that $C = D_{245} \cap D_{135}$, hence is contained in the boundary. $\square$

Example 2.4 One can just as easily construct fourteen more counter examples in essentially the same way. The divisor Q depends on the choice of a point, in this case p_5, and then the choice of a line not to be included, in this case the line between p_1 and p_2 or equivalently the line between p_3 and p_4.

For example, the divisor

$$2H - E_1 - E_2 - E_3 - E_4 - E_5 - E_{12} - E_{13} - E_{24} - E_{34}$$

is also effective and is not an effective sum of boundary divisors. □

These examples can now be lifted to $\overline{M}_{0,n}$ for any $n \geq 7$. Explicitly for $n = 7$, consider the rational map $\mathbb{P}^4 \dashrightarrow \mathbb{P}^3$ given by projection from the point $[0,0,0,0,1]$. It is not hard to see how to resolve this to a morphism $\overline{M}_{0,7} \to \overline{M}_{0,6}$, and simply by pulling the divisors described above back to $\overline{M}_{0,7}$ we obtain counter examples to $F^1(0,7)$.

3. Keel's Example

A counter example to $F^1(0,6)$ was constructed by S. Keel as follows: Take the involution of $\overline{M}_{0,6}$ determined by the element $(12)(34)(56) \in S_6$. The fixed locus has a non-boundary divisorial component which is the counter example. We show here that viewed appropriately, this can be made to yield precisely the fifteen examples described in Example 2.4. We do not, however, show that these divisors are counter examples, except after identification with the examples above.

A point not in the boundary of $\overline{M}_{0,6}$ is a smooth rational curve C with six distinct marked points which we can take to be $[1,0],[0,1],[1,1],[1,a],[1,b],[1,c]$. For this point to be fixed we must have an automorphism of C exchanging pairs of points as described above; it is easy to see that this is equivalent to the condition $a = bc$.

To identify this locus as the proper transform of a hypersurface in $\mathbb{P}^3$, we note that this curve corresponds to the image of $[1,c]$ under a map from C to $\mathbb{P}^3$ identifying the other five markings of C with the five points $p_1,\ldots,p_5$ in $\mathbb{P}^3$. We choose the map sending $[1,0] \mapsto p_1$, $[0,1] \mapsto p_2$, $[1,1] \mapsto p_3$, $[1,a] \mapsto p_4$, and $[1,b] \mapsto p_5$. The point $[1,c]$ is then sent to

$$[(1-c)(1-a^{-1}c),(1-c)(1-a^{-1}c)(b^{-1}c),c(1-a^{-1}c)(b^{-1}-1),c(1-c)(b^{-1}-a^{-1})]$$

Substituting $c = ab^{-1}$, the locus of points is seen to be given by $x_0x_1 - x_2x_3$, whose proper transform is a section of

$$2H - E_1 - E_2 - E_3 - E_4 - E_5 - E_{13} - E_{14} - E_{23} - E_{24}$$

exactly as in Lemma 2.1.

More generally, if we send $[1,0] \mapsto p_i$, $[0,1] \mapsto p_j$, $[1,1] \mapsto p_k$, $[1,a] \mapsto p_\ell$, and $[1,b] \mapsto p_m$, then the proper transform of the locus of points fixed by the involution $(ij)(k\ell)(m6) \in S_6$ is a section of

$$2H - E_1 - E_2 - E_3 - E_4 - E_5 - E_{ik} - E_{i\ell} - E_{jk} - E_{j\ell}$$

This gives precisely the fifteen examples described in Example 2.4.

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