

Secant varieties and birational geometry

Peter Vermeire

Department of Mathematics, Oklahoma State University, Stillwater, OK 74078, USA
(e-mail: petvermi@math.okstate.edu)

Received: 29 November 1999; in final form: 4 September 2000 /

Published online: 23 July 2001 – © Springer-Verlag 2001

Abstract. We show how to use information about the equations defining secant varieties to smooth projective varieties in order to construct a natural collection of birational transformations. These were first constructed as flips in the case of curves by M. Thaddeus via Geometric Invariant Theory, and the first flip in the sequence was constructed by the author for varieties of arbitrary dimension in an earlier paper. We expose the finer structure of a second flip; again for varieties of arbitrary dimension. We also prove a result on the cubic generation of the secant variety and give some conjectures on the behavior of equations defining the higher secant varieties.

Mathematics Subject Classification (1991): 14E05

1. Introduction

In this paper we continue the geometric construction of a sequence of flips associated to an embedded projective variety begun in [V2]. We give hypotheses under which this sequence of flips exists, and state some conjectures on how positive a line bundle on a curve must be to satisfy these hypotheses. These conjectures deal with the degrees of forms defining various secant varieties to curves and seem interesting outside of the context of the flip construction.

As motivation, we have the work of A. Bertram and M. Thaddeus. In [T1] this sequence of flips is constructed in the case of smooth curves via GIT, in the context of the moduli space of rank two vector bundles on a smooth curve. An understanding of this as a sequence of log flips is given in [B3], and further examples of sequences of flips of this type, again constructed

via GIT, are given in [T2],[T3]. Our construction, however, does not use the tools of Geometric Invariant Theory and is closer in spirit to [B1],[B2].

In Sect. 2, we review the constructions in [B1] and [T1] and describe the relevant results from [V2]. In Sect. 3 we discuss the generation of $\text{Sec}X$ by cubics. In particular, we show (Theorem 3.2) that large embeddings of varieties have secant varieties that are at least set theoretically defined by cubics. We also offer some general conjectures and suggestions in this direction for the generation of higher secant varieties.

The construction of the new flips is somewhat more involved than that of the first in [V2]. We give a general construction of a sequence of birational transformations in Sect. 4, and we describe in detail the second flip in Sect. 5.

We mention that some of the consequences of these constructions and this point of view are worked out in [V3].

Notation. We will decorate a projective variety X as follows: X^d is the d^{th} cartesian product of X ; $S^d X$ is $\text{Sym}^d X = X^d / S_d$, the d^{th} symmetric product of X ; and $\mathcal{H}^d X$ is $\text{Hilb}^d(X)$, the Hilbert Scheme of zero dimensional subschemes of X of length d . Recall (Cf. [Go]) that if X is a smooth projective variety then $\mathcal{H}^d X$ is also projective, and is smooth if and only if either $\dim X \leq 2$ or $d \leq 3$.

Write $\text{Sec}_k^\ell X$ for the (complete) variety of k -secant ℓ -planes to X . As this notation can become cluttered, we simply write $\text{Sec}^\ell X$ for $\text{Sec}_{\ell+1}^\ell X$ and $\text{Sec}X$ for $\text{Sec}_2^1 X$. Note also the convention $\text{Sec}^0 X = X$. If V is a k -vector space, we denote by $\mathbb{P}(V)$ the space of 1-dimensional quotients of V . Unless otherwise stated, we work throughout over the field $k = \mathbb{C}$ of complex numbers. We use the terms locally free sheaf (resp. invertible sheaf) and vector bundle (resp. line bundle) interchangeably. Recall that a line bundle \mathcal{L} on X is *nef* if $\mathcal{L} \cdot C \geq 0$ for every irreducible curve $C \subset X$. A line bundle \mathcal{L} is *big* if $\mathcal{L}^{\otimes n}$ induces a birational map for all $n \gg 0$.

Acknowledgements. I would like to thank Aaron Bertram, Sheldon Katz, Zhenbo Qin, and Jonathan Wahl for their helpful conversations and communications.

2. Overview of stable pairs and the geometry of $\text{Sec}X$

Fix a line bundle Λ on a fixed smooth curve X , and denote by $M(2, \Lambda)$ the moduli space of semi-stable rank two vector bundles E with $\wedge^2 E = \Lambda$. There is a natural rational map, the *Serre Correspondence*

$$\Phi : \mathbb{P}(\Gamma(X, K_X \otimes \Lambda)^*) \dashrightarrow M(2, \Lambda)$$

given by the duality $\text{Ext}^1(\Lambda, \mathcal{O}) \cong H^1(X, \Lambda^{-1}) \cong H^0(X, K_X \otimes \Lambda)^*$, taking an extension class $0 \rightarrow \mathcal{O} \rightarrow E \rightarrow \Lambda \rightarrow 0$ to E . One has an

embedding $X \hookrightarrow \mathbb{P}(\Gamma(X, K_X \otimes \Lambda)^*)$ (at least in the case $d = c_1(\Lambda) \geq 3$) and Φ , defined only for semi-stable E , is a morphism off $\text{Sec}^k X$ where $k = [\frac{d-1}{2}]$ [B2]. This map is resolved in [B1] by first blowing up along X , then along the proper transform of $\text{Sec} X$, then along the transform of $\text{Sec}^2 X$ and so on until we have a morphism to $M(2, \Lambda)$.

A different approach is taken in [T1]. There, for a fixed smooth curve X of genus at least 2 and a fixed line bundle Λ , the moduli problem of semi-stable pairs (E, s) consisting of a rank two bundle E with $\wedge^2 E = \Lambda$, and a section $s \in \Gamma(X, E) - \{0\}$, is considered. This, in turn, is interpreted as a GIT problem, and by varying the linearization of the group action, a collection of (smooth) moduli spaces M_1, M_2, \dots, M_k (k as above) is constructed. As stability is an open condition, these spaces are birational. In fact, they are isomorphic in codimension one, and may be linked via a diagram

$$\begin{array}{ccccc} & \widetilde{M}_2 & & \widetilde{M}_3 & & \widetilde{M}_k \\ & \searrow & & \searrow & & \searrow \\ M_1 & & M_2 & & \dots & & M_k \end{array}$$

where there is a morphism $M_k \rightarrow M(2, \Lambda)$. The relevant observations are first that this is a diagram of flips (in fact it is shown in [B3] that it is a sequence of log flips) where the ample cone of each M_i is known. Second, M_1 is the blow up of $\mathbb{P}(\Gamma(X, K_X \otimes \Lambda)^*)$ along X , \widetilde{M}_2 is the blow up of M_1 along the proper transform of the secant variety, and all of the flips can be seen as blowing up and down various higher secant varieties. Finally, the M_i are isomorphic off loci which are projective bundles over appropriate symmetric products of X .

Our approach is as follows: The sequence of flips in Thaddeus' construction can be realized as a sequence of geometric constructions depending only on the embedding of $X \subset \mathbb{P}^n$. An advantage of this approach is that the smooth curve X can be replaced by any smooth variety. Even in the curve case, our approach applies to situations where Thaddeus' construction does not hold (e.g. for canonical curves with Cliff $X > 2$). In [V2], we show how to construct the first flip using only information about the syzygies among the equations defining the variety $X \subset \mathbb{P}^n$. We summarize this construction here.

Definition 2.1. Let X be a subscheme of \mathbb{P}^n . The pair (X, F_i) satisfies condition (K_d) if X is scheme theoretically cut out by forms F_0, \dots, F_s of degree d such that the trivial (or Koszul) relations among the F_i are generated by linear syzygies.

We say (X, V) satisfies (K_d) for $V \subseteq H^0(\mathbb{P}^n, \mathcal{O}(d))$ if V is spanned by forms F_i satisfying the above condition. We say simply X satisfies (K_d) if

there exists a set $\{F_i\}$ such that (X, F_i) satisfies (K_d) , and if the discussion depends only on the existence of such a set, not on the choice of a particular set.

As (K_2) is a weakening of Green's property (N_2) [G], examples of varieties satisfying (K_2) include smooth curves embedded by complete linear systems of degree at least $2g + 3$, canonical curves with Cliff $X \geq 3$, and sufficiently large embeddings of arbitrary projective varieties.

To any projective variety $X \subset \mathbb{P}^{s_0}$ defined (as a scheme) by forms F_0, \dots, F_{s_1} of degree d , there is an associated rational map $\varphi : \mathbb{P}^{s_0} \dashrightarrow \mathbb{P}^{s_1}$ defined off the common zero locus of the F_i , i.e. off X . This map may be resolved to a morphism $\tilde{\varphi} : \widetilde{\mathbb{P}^{s_0}} \rightarrow \mathbb{P}^{s_1}$ by blowing up \mathbb{P}^n along X , or equivalently by projecting from the closure of the graph $\overline{I_\varphi} \subset \mathbb{P}^{s_0} \times \mathbb{P}^{s_1}$. We have the following results on the structure of $\tilde{\varphi}$:

Theorem 2.2. [V2, 2.4-2.10] Let (X, F_i) be a pair that satisfies (K_d) . Then:

1. $\varphi : \mathbb{P}^{s_0} \setminus X \rightarrow \mathbb{P}^{s_1}$ is an embedding off of $\text{Sec}_d^1 X$, the variety of d -secant lines.
2. The projection of a positive dimensional fiber of $\tilde{\varphi}$ to \mathbb{P}^{s_0} is either contained in a linear subspace of X or is a linear space intersecting X in a d -tic hypersurface.

If, furthermore, X does not contain a line then $\tilde{\varphi}$ is an embedding off the proper transform of $\text{Sec}_d^1 X$. \square

Theorem 2.3. [V2, 3.8] Let (X, V) satisfy (K_2) and assume $X \subset \mathbb{P}^{s_0}$ is smooth, irreducible, contains no lines and contains no quadrics. Then:

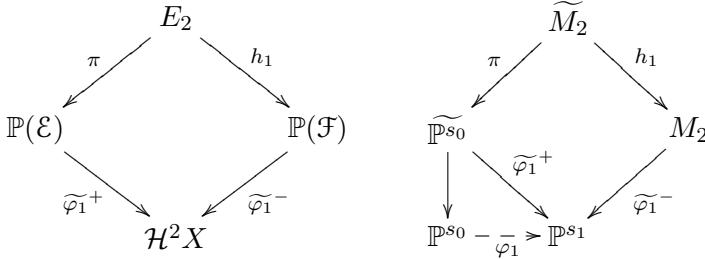
1. The image of $\widetilde{\text{Sec}X} = \widetilde{\text{Sec}_2^1 X}$ under $\tilde{\varphi}$ is $\mathcal{H}^2 X$.
2. $\mathcal{E} = \tilde{\varphi}_*(\mathcal{O}_{\widetilde{\text{Sec}X}}(H))$ is a rank two vector bundle on $\mathcal{H}^2 X$, where $\mathcal{O}_{\widetilde{\mathbb{P}^{s_0}}}(H)$ is the proper transform of the hyperplane section on \mathbb{P}^{s_0} .
3. $\tilde{\varphi} : \widetilde{\text{Sec}X} \rightarrow \mathcal{H}^2 X$ is the \mathbb{P}^1 -bundle $\mathbb{P}_{\mathcal{H}^2 X} \mathcal{E} \rightarrow \mathcal{H}^2 X$. \square

This implies $\widetilde{\text{Sec}X}$, and hence $\widetilde{M}_2 = \text{Bl}_{\widetilde{\text{Sec}X}}(\widetilde{\mathbb{P}^{s_0}})$, are smooth. To complete the flip, we construct a base point free linear system on \widetilde{M}_2 , and take M_2 to be the image of the associated morphism. Denoting $\widetilde{\text{Sec}X} = \mathbb{P}(\mathcal{E})$, the sheaf $\mathcal{F} = \tilde{\varphi}_*(N_{\mathbb{P}(\mathcal{E})/\widetilde{\mathbb{P}^{s_0}}}^* \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1))$ is locally free of rank $n - 2 \dim X - 1$ on $\mathcal{H}^2 X$. Write $\mathbb{P}(\mathcal{F}) = \mathbb{P}_{\mathcal{H}^2 X}(\mathcal{F})$ and rename $\tilde{\varphi}$ as $\tilde{\varphi}_1^+$:

Theorem 2.4. [V2, 4.13] Let (X, V) satisfy (K_2) and assume $X \subset \mathbb{P}^{s_0}$ is smooth, irreducible, contains no lines and contains no plane quadrics. Then there is a flip as pictured below with:

1. $\widetilde{\mathbb{P}^{s_0}}$, \widetilde{M}_2 , and M_2 smooth

2. $\widetilde{\mathbb{P}^{s_0}} \setminus \mathbb{P}(\mathcal{E}) \cong M_2 \setminus \mathbb{P}(\mathcal{F})$, hence if $\text{codim}(\mathbb{P}(\mathcal{E}), \widetilde{\mathbb{P}^{s_0}}) \geq 2$ then $\text{Pic } \widetilde{\mathbb{P}^{s_0}} \cong \text{Pic } M_2$
3. h_1 is the blow up of M_2 along $\mathbb{P}(\mathcal{F})$
4. π is the blow up of $\widetilde{\mathbb{P}^{s_0}}$ along $\mathbb{P}(\mathcal{E})$
5. $\widetilde{\varphi_1^-}$, induced by $\mathcal{O}_{M_2}(2H - E)$, is an embedding off of $\mathbb{P}(\mathcal{F})$, and the restriction of $\widetilde{\varphi_1^-}$ is the projection $\mathbb{P}(\mathcal{F}) \rightarrow \mathcal{H}^2 X$
6. $\widetilde{\varphi_1^+}$, induced by $\mathcal{O}_{\widetilde{\mathbb{P}^{s_0}}}(2H - E)$, is an embedding off of $\mathbb{P}(\mathcal{E})$, and the restriction of $\widetilde{\varphi_1^+}$ is the projection $\mathbb{P}(\mathcal{E}) \rightarrow \mathcal{H}^2 X$



To continue this process following Thaddeus, we need to construct a birational morphism $\widetilde{\varphi}_2^+ : M_2 \rightarrow \mathbb{P}^{s_2}$ which contracts the transforms of 3-secant 2-planes to points, and is an embedding off their union. The natural candidate is the map induced by the linear system $\mathcal{O}_{M_2}(3H - 2E)$. We discuss two different reasons for this choice that will guide the construction of the entire sequence of flips. Sect. 3 addresses the question of when this system is globally generated. Note that we abuse notation throughout and identify line bundles via the isomorphism $\text{Pic } \widetilde{\mathbb{P}^{s_0}} \cong \text{Pic } M_k$.

The first reason is quite naive: Just as quadrics collapse secant lines because their restriction to such a line is a quadric hypersurface, so too do cubics vanishing twice on a variety collapse every 3-secant \mathbb{P}^2 because they vanish on a cubic hypersurface in such a plane. Similarly, to collapse the transform of each $k + 1$ -secant \mathbb{P}^k via a morphism $\widetilde{\varphi}_k^+ : M_k \rightarrow \mathbb{P}^{s_k}$, the natural system is $\mathcal{O}_{M_k}((k + 1)H - kE)$.

Another reason is found by studying the ample cones of the M_i . Note that the ample cone on $\widetilde{\mathbb{P}^{s_0}} (= M_1)$ is bounded by the line bundles $\mathcal{O}_{\widetilde{\mathbb{P}^{s_0}}}(H)$ and $\mathcal{O}_{\widetilde{\mathbb{P}^{s_0}}}(2H - E)$. Both of these bundles are globally generated, and by Theorems 2.2 and 2.3, they each give birational morphisms whose exceptional loci are projective bundles over Hilbert schemes of points of X ($\mathcal{H}^1 X \cong X$ and $\mathcal{H}^2 X$ respectively).

On M_2 , the ample cone is bounded on one side by $\mathcal{O}_{M_2}(2H - E)$. This gives the map $\widetilde{\varphi}_1^- : M_2 \rightarrow \mathbb{P}^{s_1}$ mentioned in Theorem 2.4; in particular it is globally generated, the induced morphism is birational, and its exceptional locus is a projective bundle over $\mathcal{H}^2 X$. On the other side, the ample cone contains a line bundle of the form $\mathcal{O}_{M_2}((2m - 1)H - mE)$ ([V2, 4.9]). In

fact, if X is a smooth curve embedded by a line bundle of degree at least $2g+5$, it is shown in [T1] that the case $m = 2$ suffices, i.e. that the ample cone is bounded by $\mathcal{O}_{M_2}(2H - E)$ and $\mathcal{O}_{M_2}(3H - 2E)$. Therefore, it is natural to look for conditions under which $\mathcal{O}_{M_2}(3H - 2E)$ is globally generated. Thaddeus further shows that under similar positivity conditions, the ample cone of M_k is bounded by $\mathcal{O}_{M_k}(kH - (k-1)E)$ and $\mathcal{O}_{M_k}((k+1)H - kE)$.

Noting the fact that $h_1^*\mathcal{O}_{M_2}(3H - 2E) = \widetilde{\mathcal{O}_{M_2}}(3H - 2E_1 - E_2)$, it is not difficult to see (using Zariski's Main Theorem) that this system will be globally generated if $\text{Sec}X \subset \mathbb{P}^{s_0}$ is scheme theoretically defined by cubics, because a cubic vanishing twice on a variety must also vanish on its secant variety. Unfortunately, there are no general theorems on the cubic generation of secant varieties analogous to quadric generation of varieties. We address this question in the next section.

3. Cubic generation of secant varieties

Example 3.1. Some examples of varieties whose secant varieties are *ideal* theoretically defined by cubics include:

1. X is any Veronese embedding of \mathbb{P}^n [Ka]
2. X is the Plücker embedding of the Grassmannian $\mathbb{G}(1, n)$ for any n [H, 9.20].
3. X is the Segre embedding of $\mathbb{P}^n \times \mathbb{P}^m$ [H, 9.2]. □

We prove a general result:

Theorem 3.2. *Let $X \subset \mathbb{P}^{s_0}$ satisfy condition (K_2) . Then $\text{Sec}(v_d(X))$ is set theoretically defined by cubics for $d \geq 2$.*

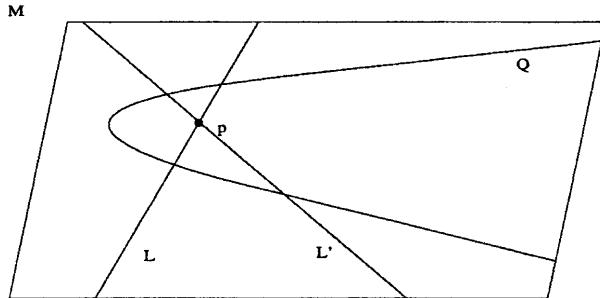
Proof. We begin with the case $d = 2$, the higher embeddings being more elementary.

Let $Y = v_2(X)$, $V = v_2(\mathbb{P}^{s_0}) \subset \mathbb{P}^N$, and H the linear subspace of \mathbb{P}^N defined by the hyperplanes corresponding to all the quadrics in \mathbb{P}^{s_0} vanishing on X . Then $Y = V \cap H$ as schemes and we show, noting that $\text{Sec}V$ is ideal theoretically defined by cubics, that $\text{Sec}Y = \text{Sec}V \cap H$ as sets.

Note that the map $\varphi_1 : \mathbb{P}^{s_0} \dashrightarrow \mathbb{P}^{s_1}$ can be viewed as the composition of the embedding $v_2 : \mathbb{P}^{s_0} \hookrightarrow \mathbb{P}^N$ with the projection from H , $\mathbb{P}^N \dashrightarrow \mathbb{P}^{s_1}$.

Let $p \in \text{Sec}V \cap H$. If $p \in V$, then $p \in Y = V \cap H$ hence $p \in \text{Sec}Y$.

Otherwise, any secant line L to V through p intersects V in a length two subscheme Z . Z considered in \mathbb{P}^{s_0} determines a unique line in \mathbb{P}^{s_0} whose image in \mathbb{P}^N is a plane quadric $Q \subset V$ spanning a plane M . If $H \cap Q = Z' \subset Y$ is non-empty then $Z' \cup \{p\} \subset H \cap M$, hence either H intersects M in a line L' through p or $M \subset H$. In the first case L' is a secant line to Y , in the second $Q \subset Y$. In either situation $p \in \text{Sec}Y$.



All that remains is the case $H \cap M = \{p\}$ and $H \cap Q$ is empty. However in this case the line L , and hence the scheme $Z = L \cap Q$ is collapsed to a point by the projection. As the rational map φ is an embedding off $\text{Sec}X$, this implies Z lies on the image of a secant line to $X \subset \mathbb{P}^{s_0}$. As a length two subscheme of \mathbb{P}^{s_0} determines a unique line, Q must be the image of a secant line to $X \subset \mathbb{P}^{s_0}$ contradicting the assumption that $H \cap Q$ is empty.

For $d > 2$, note that the projection from H is an embedding off $V \cap H$ (this can be derived directly from Theorem 2.2 or see [V1, 3.3.1]). Therefore, if H intersects a secant line, the line lies in H , hence is a secant line to Y . \square

Example 3.3. As Green's (N_2) implies (K_2) , this shows that the secant varieties to the following varieties are set theoretically defined by cubics:

1. X a smooth curve embedded by a line bundle of degree $4g + 6 + 2r$, $r \geq 0$.
2. X a smooth curve with Cliff $X > 2$, embedded by $K_X^{\otimes r}$, $r \geq 2$.
3. X a smooth variety embedded by $(K_X \otimes L^{\otimes(\dim X+3+\alpha)})^{\otimes r}$, $\alpha \geq 0$, $r \geq 2$, L very ample.
4. X a smooth variety embedded by $L^{\otimes 2r}$ for all $r \gg 0$, L ample. \square

Remark 3.4. Notice that in the case $d = 2$ of Proposition 3.2, the cubics that at least set theoretically define the secant variety satisfy (K_3) . This is because:

1. The ideal of the secant variety of $v_2(\mathbb{P}^{s_0})$ is generated by cubics, and the module of syzygies is generated by linear relations [JPW, 3.19]. Hence $\text{Sec}(v_2(\mathbb{P}^{s_0}))$ satisfies (K_3) .
2. It is clear from the definition that if $X \subset \mathbb{P}^n$ satisfies (K_d) , then any linear section does as well. \square

Example 3.5. If $X \subset \mathbb{P}^n$ is a smooth quadric hypersurface, then $v_2(X)$ is given by the intersection of $v_2(\mathbb{P}^n)$ with a hyperplane H . Furthermore, the

intersection of $\text{Sec}(v_2(\mathbb{P}^n))$ with H is a scheme S with $S_{\text{red}} \cong \text{Sec}(v_2(X))$. Therefore, a general smooth quadric hypersurface has $\text{Sec}(v_2(X)) \cong \text{Sec}(v_2(\mathbb{P}^n)) \cap H$ as schemes, hence $\text{Sec}(v_2(X))$ satisfies (K_3) . \square

We record here a related conjecture of Eisenbud, Koh, and Stillman as well as a partial answer proven by M.S. Ravi:

Conjecture 3.6. [EKS] Let L be a very ample line bundle that embeds a smooth curve X . For each k there is a bound on the degree of L such that $\text{Sec}^k X$ is ideal theoretically defined by the $(k+2) \times (k+2)$ minors of a matrix of linear forms.

Theorem 3.7. [R] If $\deg L \geq 4g + 2k + 3$, then $\text{Sec}^k X$ is set theoretically defined by the $(k+2) \times (k+2)$ minors of a matrix of linear forms.

These statements provide enough evidence to make the following basic:

Conjecture 3.8. Let L be an ample line bundle on a smooth variety X , $k \geq 1$ fixed. Then for all $n \gg 0$, L^n embeds X so that $\text{Sec}^k X$ is ideal theoretically defined by forms of degree $k+2$, and furthermore satisfies condition (K_{k+2}) .

Remark 3.9. If X is a curve with a 5-secant 3-plane, then any cubic vanishing on $\text{Sec}X$ must vanish on that 3-plane. Hence $\text{Sec}X$ cannot be set theoretically defined by cubics. This should be compared to the fact that if X has a trisecant line, then X cannot be defined by quadrics. In particular, this shows that Green's condition (N_2) is not even sufficient to guarantee that there exists a cubic vanishing on $\text{Sec}X$. For example, if X is an elliptic curve embedded in \mathbb{P}^4 by a line bundle of degree 5, then $\text{Sec}X$ is a quintic hypersurface. Therefore, any uniform bound on the degree of a linear system that would guarantee $\text{Sec}X$ is even set theoretically defined by cubics must be at least $2g + 4$. \square

We can use earlier work to give a more geometric necessary condition for $\text{Sec}X$ to be defined as a scheme by cubics. Specifically, in [V2, 3.7] it is shown that the intersection of $\widetilde{\text{Sec}X}$ with the exceptional divisor E of the blow up of \mathbb{P}^{s_0} along X is isomorphic to $\text{Bl}_\Delta(X \times X)$. This implies that if $\pi : \widetilde{\text{Sec}X} \rightarrow \text{Sec}X$ is the blow up along X , then $\pi^{-1}(p) \cong \text{Bl}_p(X)$, $p \in X$. In fact, it is easy to verify that if X is embedded by a line bundle L , then $\pi^{-1}(p) \cong \text{Bl}_p(X) \subset \mathbb{P}\Gamma(X, L \otimes \mathcal{I}_p^2)$ where $\mathbb{P}\Gamma(X, L \otimes \mathcal{I}_p^2)$ is identified with the fiber over p of the projectivized conormal bundle of $X \subset \mathbb{P}^{s_0}$. Now, if $\text{Sec}X$ is defined as a scheme by cubics, then the base scheme of $\mathcal{O}_{\widetilde{\text{Sec}X}}(3H - 2E)$ is precisely $\widetilde{\text{Sec}X}$. The restriction of this series to $\mathbb{P}\Gamma(X, L \otimes \mathcal{I}_p^2)$ is thus a system of quadrics whose base scheme is $\text{Bl}_p(X)$.

In other words, if X is a smooth variety embedded by a line bundle L that satisfies (K_2) and if $\text{Sec}X$ is scheme theoretically defined by cubics, then for every $p \in X$ the line bundle $L \otimes \mathcal{O}(-2E_p)$ is very ample on $\text{Bl}_p(X)$ and $\text{Bl}_p(X) \subset \mathbb{P}\Gamma(\text{Bl}_p(X), L \otimes \mathcal{O}(-2E_p))$ is scheme theoretically defined by quadrics.

In the case X is a curve, this implies that a uniform bound on $\deg L$ that would imply $\text{Sec}X$ is defined by cubics must be at least $2g + 4$, the same bound encountered in Remark 3.9. The construction in [B1] shows similarly that any uniform bound that would imply $\text{Sec}^k X$ is defined by $(k+2)$ -tics must be at least $2g + 2 + 2k$. We combine these observations with the degree bounds encountered in the constructions of [T1] and [B1] to form the following:

Conjecture 3.10. Let X be a smooth curve embedded by a line bundle L . If $\deg(L) \geq 2g + 2k$ then $\text{Sec}^{k-1} X$ is defined as a scheme by forms of degree $k+1$. If $\deg(L) \geq 2g + 2k + 1$ then $\text{Sec}^{k-1} X$ satisfies condition (K_{k+1}) . \square

4. The general birational construction

Suppose that X satisfies (K_2) , is smooth, and contains no lines and no plane quadrics. Suppose further that $\text{Sec}X$ is scheme theoretically defined by cubics C_0, \dots, C_{s_2} , and that $\text{Sec}X$ satisfies (K_3) . Under these hypotheses, we construct a second flip as follows: We know that $\mathcal{O}_{M_2}(3H - 2E)$ is globally generated by the discussion above; hence this induces a morphism $\widetilde{\varphi}_2^+ : M_2 \rightarrow \mathbb{P}^{s_2}$ which agrees with the map given by the cubics $\varphi_2 : \mathbb{P}^{s_0} \dashrightarrow \mathbb{P}^{s_2}$ on the locus where M_2 and \mathbb{P}^{s_0} are isomorphic. By Theorem 2.2, $\widetilde{\varphi}_2^+$ is a birational morphism. We wish first to identify the exceptional locus of $\widetilde{\varphi}_2^+$. It is clear that $\widetilde{\varphi}_2^+$ will collapse the image of a 3-secant 2-plane to a point, hence the exceptional locus must contain the transform of $\text{Sec}^2 X$. However by Theorem 2.2, we know that the rational map φ_2 is an embedding off $\text{Sec}_3^1(\text{Sec}X)$, the trisecant variety to the secant variety. This motivates the following

Lemma 4.1. *Let $X \subset \mathbb{P}^n$ be an irreducible variety. Assume either of the following:*

1. $\text{Sec}^k X$ is defined as a scheme by forms of degree $\leq 2k + 1$.
2. X is a smooth curve embedded by a line bundle of degree at least $2g + 2k + 1$.

Then $\text{Sec}^k X = \text{Sec}_{k+1}^1(\text{Sec}^{k-1} X)$ as schemes.

Proof. First, choose a $(k+1)$ -secant k -plane M . M then intersects $\text{Sec}^{k-1} X$ in a hypersurface of degree $k+1$, hence every line in M lies in

$\text{Sec}_{k+1}^1(\text{Sec}^{k-1}X)$. As $\text{Sec}^k X$ is reduced and irreducible, $\text{Sec}^k X \subseteq \text{Sec}_{k+1}^1(\text{Sec}^{k-1}X)$ as schemes.

For the converse, assume the first condition is satisfied. Choose a line L that intersects $\text{Sec}^{k-1}X$ in a scheme of length at least $k+1$. It is easy to verify that $\text{Sec}^k X$ is singular along $\text{Sec}^{k-1}X$, hence every form that vanishes on $\text{Sec}^k X$ must vanish $2k+2$ times on L . By hypothesis, however, $\text{Sec}^k X$ is scheme theoretically defined by forms of degree $\leq 2k+1$, hence each of these forms must vanish on L .

The sufficiency of the second condition follows from Thaddeus' construction and [B3, Sect. 2,(i)]. \square

This implies that if $\text{Sec}^k X$ satisfies (K_{k+2}) and if $\text{Sec}^k X = \text{Sec}_{k+1}^1(\text{Sec}^{k-1}X)$, then the map $\varphi_{k+1} : \mathbb{P}^{s_0} \dashrightarrow \mathbb{P}^{s_{k+1}}$ given by the forms defining $\text{Sec}^k X$ is an embedding off of $\text{Sec}^{k+1} X$. We use Theorem 2.2 to understand the structure of these maps via the following two lemmas:

Lemma 4.2. *If the embedding of a projective variety $X \subset \mathbb{P}^n$ is $(2k+4)$ -very ample, then the intersection of two $(k+2)$ -secant $(k+1)$ -planes, if nonempty, must lie in $\text{Sec}^k X$ (in fact, it must be an $\ell+1$ secant \mathbb{P}^ℓ for some $\ell \leq k$). In particular, $\text{Sec}^{k+1} X$ has dimension $(k+2) \dim X + k + 1$.*

Proof. The first statement is elementary: Assume two $(k+2)$ -secant $(k+1)$ -planes intersect at a single point. If the point is not on X , then there are $2k+4$ points of X that span a $(2k+2)$ -plane, which is impossible by hypothesis. Hence the intersection lies in $\text{Sec}^0 X = X$. A simple repetition of this argument for larger dimensional intersections gives the desired result. The statement of the dimension follows immediately; or see [H, 11.24]. \square

Lemma 4.3. *Let $X \subset \mathbb{P}^{s_0}$ be an irreducible variety whose embedding is $(2k+4)$ -very ample. Assume that $\text{Sec}^k X$ satisfies (K_{k+2}) , and that $\text{Sec}^{k+1} X = \text{Sec}_{k+2}^1(\text{Sec}^k X)$ as schemes. Let Γ be the closure of the graph of φ_{k+1} with projection $\pi : \Gamma \rightarrow \mathbb{P}^{s_0}$. If a is a point in the closure of the image of φ_{k+1} and $F_a \subset \Gamma$ is the fiber over a then $\pi(F_a)$ is one of the following:*

1. a reduced point in $\mathbb{P}^{s_0} \setminus \text{Sec}^{k+1} X$
2. a $(k+2)$ -secant $(k+1)$ -plane
3. contained in a linear subspace of $\text{Sec}^k X$

Proof. The first and third possibilities follow directly from Theorem 2.2.

For the second, note that a priori $\pi(F_a)$ could be any linear space intersecting $\text{Sec}^k X$ in a hypersurface of degree $k+2$. However, Lemma 4.2 and the hypothesis that $\text{Sec}^{k+1} X = \text{Sec}_{k+2}^1(\text{Sec}^k X)$ immediately imply that any such linear space must be $k+1$ dimensional; hence a $(k+2)$ -secant $(k+1)$ -plane. \square

With these results in hand we present the general construction.

Let Y_0 be an irreducible projective variety and suppose $\beta_i : Y_0 \dashrightarrow Y_i$, $1 \leq i \leq j$, is a collection of dominant, birational maps. Define the *dominating variety* of the collection, denoted $\mathcal{B}_{(0,1,\dots,j)}$, to be the closure of the graph of

$$(\beta_1, \beta_2, \dots, \beta_j) : Y_0 \dashrightarrow Y_1 \times Y_2 \times \cdots \times Y_j$$

Denote by $\mathcal{B}_{(a_1,a_2,\dots,a_r)}$ the projection of $\mathcal{B}_{(0,1,\dots,j)}$ to $Y_{a_1} \times Y_{a_2} \times \cdots \times Y_{a_r}$. Note that $\mathcal{B}_{(a_1,a_2,\dots,a_r)}$ is birationally isomorphic to $\mathcal{B}_{(b_1,b_2,\dots,b_k)}$ for all $0 \leq a_r, b_k \leq j$. Note further that if the β_i are all morphisms then $\mathcal{B}_{(0,1,\dots,j)} \cong Y_0$, in other words only rational maps contribute to the structure of the dominating variety.

Definition/Notation 4.4 We say $X \subset \mathbb{P}^{s_0}$ satisfies condition (K_2^j) if $\text{Sec}^i X$ satisfies condition (K_{2+i}) for $0 \leq i \leq j$; hence X satisfies (K_2^0) if and only if X satisfies (K_2) , X satisfies (K_2^1) if and only if X satisfies (K_2) and $\text{Sec} X$ satisfies (K_3) , etc. \square

If $X \subset \mathbb{P}^{s_0}$ satisfies condition (K_2^j) , then each rational map $\varphi_i : \mathbb{P}^{s_0} \dashrightarrow \mathbb{P}^{s_i}$ is birational onto its image for $1 \leq i \leq j+1$, and assuming the conclusion of Lemma 4.1 each φ_i is an embedding off $\text{Sec}^i X$. Therefore $\mathcal{B}_{(i)}$ is the closure of the image of φ_i , $\mathcal{B}_{(0,i)}$ is the closure of the graph of φ_i , and in the notation of Theorem 2.4 $\mathcal{B}_{(0,1,2)} \cong \widetilde{M}_2$ and $\mathcal{B}_{(1,2)} \cong M_2$. Note $\mathcal{B}_{(0)} = \mathbb{P}^{s_0}$.

Lemma 4.5. $\mathcal{B}_{(0,1,2,\dots,i)}$ is the blow up of $\mathcal{B}_{(0,1,2,\dots,i-1)}$ along the proper transform of $\text{Sec}^{i-1} X$, $1 \leq i \leq j+1$.

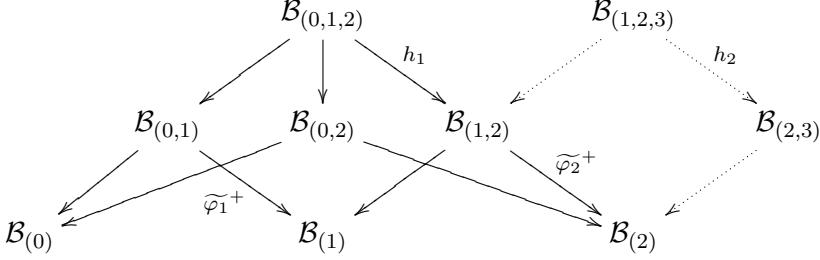
Proof. This is immediate from the definition (or see [V1, 3.1.1]). \square

Remark 4.6. The spaces constructed in [B1] are of the type $\mathcal{B}_{(0,1,2,\dots,k)}$. The spaces \widetilde{M}_k and M_k constructed in [T1] are $\widetilde{M}_k \cong \mathcal{B}_{(k-2,k-1,k)}$ and $M_k \cong \mathcal{B}_{(k-1,k)}$. \square

Our goal is to understand explicitly the geometry of this web of varieties generalizing Theorem 2.4. In the next section we describe in detail the structure of the second flip. As each subsequent flip requires the understanding of $\mathcal{H}^k X$ for larger k , it is not clear that the process will continue nicely beyond the second flip (at least for varieties of arbitrary dimension).

5. Construction of the second flip

Let $X \subset \mathbb{P}^{s_0}$ be a smooth, irreducible variety that satisfies (K_2^1) . The diagram of varieties we study in this section is:



where $\mathcal{B}_{(0,1,2)}$ is the dominating variety of the pair of birational maps $\varphi_1 : \mathbb{P}^{s_0} \dashrightarrow \mathbb{P}^{s_1}$ and $\varphi_2 : \mathbb{P}^{s_0} \dashrightarrow \mathbb{P}^{s_2}$; and where we have yet to construct the two rightmost varieties. We write $\text{Pic } \mathcal{B}_{(0,1)} = \text{Pic } \mathcal{B}_{(1,2)} = \mathbb{Z}H + \mathbb{Z}E$ and $\text{Pic } \mathcal{B}_{(0,1,2)} = \mathbb{Z}H + \mathbb{Z}E_1 + \mathbb{Z}E_2$ (recall all three spaces are smooth by Theorem 2.4).

Theorem 5.1. *Let $X \subset \mathcal{B}_{(0)} = \mathbb{P}^{s_0}$ be a smooth, irreducible variety of dimension r that satisfies (K_2^1) . Assume that X is embedded by a complete linear system $|L|$ and that the following conditions are satisfied:*

1. *L is $(5+r)$ -very ample*
2. *If $r \geq 2$, then for every point $p \in X$, $H^1(X, L \otimes \mathcal{I}_p^3) = 0$*
3. *$\text{Sec}^2 X = \text{Sec}_3^1(\text{Sec}^1 X)$ as schemes*
4. *The projection of X into \mathbb{P}^m , $m = s_0 - 1 - r$, from any embedded tangent space is such that the image is projectively normal and satisfies (K_2)*

Then the morphism $\widetilde{\varphi}_2^+ : \mathcal{B}_{(1,2)} \rightarrow \mathcal{B}_{(2)}$ induced by $\mathcal{O}_{\mathcal{B}_{(1,2)}}(3H - 2E)$ is an embedding off the transform of $\text{Sec}^2 X$, and the restriction of $\widetilde{\varphi}_2^+$ to the transform of $\text{Sec}^2 X$ has fibers isomorphic to \mathbb{P}^2 .

As the proof of Theorem 5.1 is somewhat involved, we break it into several pieces. We begin with a Lemma and a crucial observation, followed by the proof of the Theorem. The observation invokes a technical lemma whose proof is postponed until the end.

Remark on the hypotheses 5.2 Note that if X is a smooth curve embedded by a line bundle of degree at least $2g + 5$, then conditions 1 – 4 are automatically satisfied. Conjecture 3.10 would imply condition (K_2^1) holds also. Furthermore, if $r = 2$ and $H^1(X, L) = 0$ then condition 1 implies condition 2.

If $r \geq 2$, then the image of the projection from the space tangent to X at p is $\text{Bl}_p(X) \subset \mathbb{P}^m$. Furthermore, by the discussion after Remark 3.9 any such projection of X will be generated as a scheme by quadrics when $\text{Sec} X$ is defined by cubics, hence condition 4 is not unreasonable. \square

Lemma 5.3. *With hypotheses as in Theorem 5.1, the image of the projection of X into \mathbb{P}^m , $m = s_0 - 1 - r$, is $\text{Bl}_p(X)$, hence is smooth. Furthermore, it contains no lines and it contains no plane quadrics except for the exceptional divisor, which is the quadratic Veronese embedding of \mathbb{P}^{r-1} .*

Proof. If $r = 1$ the statement is clear. Otherwise, let $X' \subset \mathbb{P}^m$ denote the closure of the image of projection from the embedded tangent space to X at p . As mentioned above, $X' \cong \text{Bl}_p(X)$, hence is smooth. Let $E_p \subset X'$ denote the exceptional divisor. The existence of a line or plane quadric not contained in E_p is immediately seen to be impossible by the $(5 + r)$ -very ampleness hypothesis.

As $\mathbb{P}^m = \mathbb{P}\Gamma(X', L \otimes \mathcal{O}(-2E_p))$ and as $E_p \cong \mathbb{P}^{r-1}$, we have $L \otimes \mathcal{O}(-2E_p)|_{E_p} \cong \mathcal{O}_{\mathbb{P}^{r-1}}(2)$. Condition 2 implies this restriction is surjective on global sections. \square

Observation 5.4 Let $\mathcal{B}_{(0,1,2)} \rightarrow \mathcal{B}_{(0)}$ be the projection and let F_p be the fiber over $p \in X$; hence F_p is the blow up of \mathbb{P}^m along a copy of $\text{Bl}_p(X)$. We again denote this variety by $X' \subset \mathbb{P}^m$, and the embedding of X' into \mathbb{P}^m satisfies (K_2) by hypothesis. The restriction of $\mathcal{O}_{\mathcal{B}_{(0,1,2)}}(3H - 2E_1 - E_2)$ to F_p can thus be identified with $\mathcal{O}_{\text{Bl}_{X'}(\mathbb{P}^m)}(2H' - E')$, and, noting Lemma 5.3, it seems that Theorem 2.2 could be applied. Unfortunately, it is not clear that this restriction should be surjective on global sections. However, by Lemma 5.7 below, the image of the morphism on F_p induced by the restriction of global sections is isomorphic to the image of the morphism given by the complete linear system $|\mathcal{O}_{\text{Bl}_{X'}(\mathbb{P}^m)}(2H' - E')|$. Hence by the fourth hypothesis and Lemma 5.3, the only collapsing that occurs in F_p under the morphism $\mathcal{B}_{(0,1,2)} \rightarrow \mathcal{B}_{(2)}$ is that of secant lines to $X' \subset \mathbb{P}^m$.

Now, for some $p \in X$, suppose that a secant line S in F_p is collapsed to a point by the projection $\mathcal{B}_{(0,1,2)} \rightarrow \mathcal{B}_{(2)}$. Then S is the proper transform of a secant line to $X' \subset \mathbb{P}^m$, but every such secant line is the intersection of F_p with a 3-secant \mathbb{P}^2 through $p \in X$. For example, if $S \subset F_p$ is the secant line through $q, r \in X'$, $q, r \notin E_p$, then S is the intersection of F_p with the proper transform of the plane spanned by p, q, r . It should be noted that the two dimensional fiber associated to the collapsing of a plane spanned by a quadric in the exceptional divisor (Lemma 5.3) will take the place of a 3-secant \mathbb{P}^2 spanned by a non-curvilinear scheme contained in the tangent space at p .

Therefore, all the collapsing in the exceptional locus over a point $p \in X$ is associated to the collapsing of 3-secant 2-planes. \square

Proof(of Theorem 5.1). Let $a \in \mathcal{B}_{(2)}$ be a point in the image of $\widetilde{\varphi}_2^+$. The fiber over a is mapped isomorphically into $\mathcal{B}_{(1)}$ by the projection $\mathcal{B}_{(1,2)} \rightarrow \mathcal{B}_{(1)}$. We are therefore able to study $(\widetilde{\varphi}_2^+)^{-1}(a)$ by looking at the fiber of the projection $\mathcal{B}_{(0,1,2)} \rightarrow \mathcal{B}_{(2)}$, and projecting to $\mathcal{B}_{(0,1)}$ and to $\mathcal{B}_{(1)}$.

By applying Lemma 4.3 to the map $\mathcal{B}_{(0,2)} \rightarrow \mathcal{B}_{(2)}$, the projection to $\mathcal{B}_{(0,1)}$ is contained as a scheme in the *total* transform of one of the following (note the more refined division of possibilities):

1. a point in $\mathbb{P}^{s_0} \setminus \text{Sec}^2 X$
2. a 3-secant 2-plane to X not contained in $\text{Sec}X$
3. a linear subspace of $\text{Sec}X$ not tangent to X
4. a linear subspace of $\text{Sec}X$ tangent to X

In the first case, there is nothing to show as the total transform of a point in $\mathbb{P}^{s_0} \setminus \text{Sec}^2 X$ is simply a reduced point and the map $\widetilde{\varphi}_1^+$ to $\mathcal{B}_{(1)}$ is an embedding in a neighborhood of this point.

If the projection is a 3-secant 2-plane, then by Observation 5 the projection to $\mathcal{B}_{(0,1)}$ is a 3-secant 2-plane blown up at the three points of intersection, and so the image in $\mathcal{B}_{(1)}$ is a \mathbb{P}^2 that has undergone a Cremona transformation.

In the third case, Observation 5 shows that either the projection to $\mathcal{B}_{(0,1)}$ is the *proper* transform of a secant line to X , or that the projection to $\mathcal{B}_{(0)}$ is a linear subspace of $\text{Sec}X$ that is not a secant line. In the first case, every such space is collapsed to a point by $\widetilde{\varphi}_1^+$. The second implies $\widetilde{\varphi}_2^+$ has a fiber of dimension d that is contained in $\mathbb{P}(\mathcal{F}) \subset \mathcal{B}_{(1,2)}$. Because $E_2 \rightarrow \mathbb{P}(\mathcal{F})$ is a \mathbb{P}^1 -bundle, this implies the projection of the fiber to $\mathcal{B}_{(0)}$ is contained in a linear subspace M of $\text{Sec}X$ of dimension $d + 1$. Furthermore, the proper transform of M is collapsed to a d dimensional subspace of $\mathcal{B}_{(1)}$, in particular the general point of M lies on a secant line *contained in* M by Theorem 2.3. Therefore $Y = M \cap X$ has $\text{Sec}Y = M$, hence $\text{Sec}^2 Y = M$ but this is impossible by Lemma 4.2 and the restriction that M not be tangent to X .

In the final case, the proper transform in $\mathcal{B}_{(0,1)}$ of a linear space $M \cong \mathbb{P}^k$ tangent to X at a point p is $\text{Bl}_p(\mathbb{P}^k)$. Denote the exceptional \mathbb{P}^{k-1} by Q ; Lemma 5.3 implies $Q \cong E_p$ is the quadratic Veronese embedding of $\mathbb{P}^{k-1} \subset \mathbb{P}(\Gamma(\text{Bl}_p(X), L(-2E_p)))$. A simple dimension count shows that the restriction to Q of the projective bundle $E_2 \rightarrow \widetilde{\text{Sec}X}$ arising from the blow up of $\mathcal{B}_{(0,1)}$ along $\text{Sec}X$ is precisely the restriction to Q of the projective bundle arising from the *induced* blow up of $\mathbb{P}(\Gamma(\text{Bl}_p(X), L(-2E_p)))$ along $\text{Bl}_p(X)$; denote this variety \mathbb{P}_Q . Furthermore, the transform of $\text{Bl}_p(\mathbb{P}^k)$ in $\mathcal{B}_{(0,1,2)}$ is a \mathbb{P}^1 -bundle over $\mathbb{P}_Q \subset \mathcal{B}_{(1,2)}$. Now by Lemma 5.7, every fiber of $\widetilde{\varphi}_2^+$ contained in $\mathbb{P}_Q \subset \mathcal{B}_{(1,2)}$ is either a point or is isomorphic to a \mathbb{P}^2 spanned by a plane quadric in Q . \square

Remark 5.5. For curves, parts 3 and 4 of the proof can also be concluded by showing that any line contained in $\text{Sec}X$ must be a secant or tangent line (this is immediate from the 6-very ample hypothesis). \square

To complete the proof, we need Lemma 5.7 which itself requires a general result:

Lemma 5.6. *Let $\pi : X \rightarrow Y$ be a flat morphism of smooth projective varieties. Let $F = \pi^{-1}(p)$ be a smooth fiber and let L be a locally free sheaf on X . If $R^i\pi_*L = 0$ and $H^i(F, L \otimes \mathcal{O}_F) = 0$ for all $i > 0$, then $R^i\pi_*(\mathcal{J}_F \otimes L) = 0$ for all $i > 0$.*

Proof. The hypotheses easily give the vanishing $R^i\pi_*(\mathcal{J}_F \otimes L) = 0$ for all $i > 1$. For $i = 1$, take the exact sequence on Y

$$0 \rightarrow \pi_*(\mathcal{J}_F \otimes L) \rightarrow \pi_*L \rightarrow \pi_*(\mathcal{O}_F \otimes L) \rightarrow R^1\pi_*(\mathcal{J}_F \otimes L) \rightarrow 0$$

Because $\pi_*(\mathcal{O}_F \otimes L)$ is supported at the point p , it suffices to check that $H^1(F, \mathcal{J}_F \otimes \mathcal{O}_F \otimes L) = H^1(F, N_{F/X}^* \otimes L) = 0$. π flat implies $N_{F/X}^* \cong \pi^*(N_{p/Y}^*)$, hence $N_{F/X}^*$ is trivial. Now, $H^1(F, L \otimes \mathcal{O}_F) = 0$ implies $H^1(F, N_{F/X}^* \otimes L) = 0$. \square

Lemma 5.7. *Under the hypotheses of Theorem 5.1, The image of F_p under the projection $\mathcal{B}_{(0,1,2)} \rightarrow \mathcal{B}_{(2)}$ is isomorphic to the image of F_p under the morphism induced by the complete linear system associated to $\mathcal{O}_{F_p}(2H' - E')$.*

Proof. **Step 1.** If $a, b \in F_p$ are mapped to the same point under the projection to $\mathcal{B}_{(2)}$, then a and b map to the same point under the projection to $\mathcal{B}_{(0,2)}$. This is clear from the construction of the maps in question as the projections $\mathbb{P}^{s_0} \times \mathbb{P}^{s_1} \times \mathbb{P}^{s_2} \rightarrow \mathbb{P}^{s_2}$ and $\mathbb{P}^{s_0} \times \mathbb{P}^{s_1} \times \mathbb{P}^{s_2} \rightarrow \mathbb{P}^{s_0} \times \mathbb{P}^{s_2}$ respectively.

Step 2. Re-embed $\mathcal{B}_{(0,2)} \hookrightarrow \mathbb{P}^N \times \mathbb{P}^{s_2}$ via the map associated to $\mathcal{O}_{\mathbb{P}^{s_0}}(k) \boxtimes \mathcal{O}_{\mathbb{P}^{s_2}}(1)$. This gives a map $\mathcal{B}_{(0,1,2)} \rightarrow \mathbb{P}^N \times \mathbb{P}^{s_2}$ induced by a subspace of $H^0(\mathcal{B}_{(0,1,2)}, \mathcal{O}_{\mathbb{P}^{s_0}}(k) \boxtimes \mathcal{O}_{\mathbb{P}^{s_2}}(1) \otimes \mathcal{O}_{\mathcal{B}_{(0,1,2)}})$ where $\mathcal{O}_{\mathbb{P}^{s_0}}(k) \boxtimes \mathcal{O}_{\mathbb{P}^{s_2}}(1) \otimes \mathcal{O}_{\mathcal{B}_{(0,1,2)}} \cong \mathcal{O}_{\mathcal{B}_{(0,1,2)}}((k+3)H - 2E_1 - E_2)$. As $\mathcal{B}_{(0,2)} \hookrightarrow \mathbb{P}^N \times \mathbb{P}^{s_2}$ is an embedding, the induced maps on F_p have isomorphic images for all $k \geq 1$. We have, therefore, only to show $H^0(\mathbb{P}^{s_0} \times \mathbb{P}^{s_1} \times \mathbb{P}^{s_2}, \mathcal{O}_{\mathbb{P}^{s_0}}(k) \boxtimes \mathcal{O}_{\mathbb{P}^{s_2}}(1))$ surjects onto $H^0(F_p, \mathcal{O}_{F_p}(2H' - E'))$ for some k .

Step 3. *The map*

$$\begin{aligned} & H^0(\mathbb{P}^{s_0} \times \mathbb{P}^{s_1} \times \mathbb{P}^{s_2}, \mathcal{O}_{\mathbb{P}^{s_0}}(k) \boxtimes \mathcal{O}_{\mathbb{P}^{s_2}}(1)) \\ & \rightarrow H^0(\mathcal{B}_{(0,1,2)}, \mathcal{O}_{\mathcal{B}_{(0,1,2)}}((k+3)H - 2E_1 - E_2)) \end{aligned}$$

is surjective for all $k \gg 0$. This follows directly from the fact that $SecX$ is scheme theoretically defined by cubics and the construction of \mathbb{P}^{s_2} as $\mathbb{P}(\Gamma(\mathcal{B}_{(0,1,2)}, \mathcal{O}_{\mathcal{B}_{(0,1,2)}}(3H - 2E_1 - E_2)))$.

Step 4. *The map*

$$\begin{aligned} H^0(\mathcal{B}_{(0,1,2)}, \mathcal{O}_{\mathcal{B}_{(0,1,2)}}((k+3)H - 2E_1 - E_2)) \\ \rightarrow H^0(E_1, \mathcal{O}_{E_1}((k+3)H - 2E_1 - E_2)) \end{aligned}$$

is surjective for all $k \gg 0$.

We show $H^1(\mathcal{B}_{(0,1,2)}, \mathcal{O}_{\mathcal{B}_{(0,1,2)}}((k+3)H - 3E_1 - E_2)) = 0$. Let $\rho : \mathcal{B}_{(0,1,2)} \rightarrow \mathcal{B}_{(0)}$ be the projection. By the projective normality assumption of Theorem 5.1, $R^i \rho_* \mathcal{O}_{E_1}((k+3)H - \ell E_1 - E_2) = 0$ for all $i, \ell > 0$ since $E_1 \rightarrow X$ is flat. Ampleness of $\mathcal{O}_{\mathbb{P}^{s_0}}(H)$ implies $H^1(E_1, \mathcal{O}_{E_1}(mH - \ell E_1 - E_2)) = 0$ for all $m \geq m_0$, where m_0 may depend on ℓ . From the exact sequence

$$\begin{aligned} 0 \rightarrow \mathcal{O}_{\mathcal{B}_{(0,1,2)}}(mH - (\ell+1)E_1 - E_2) \\ \rightarrow \mathcal{O}_{\mathcal{B}_{(0,1,2)}}(mH - \ell E_1 - E_2) \rightarrow \mathcal{O}_{E_1}(mH - \ell E_1 - E_2) \end{aligned}$$

a finite induction shows that if $H^1(\mathcal{B}_{(0,1,2)}, \mathcal{O}_{\mathcal{B}_{(0,1,2)}}(mH - (\ell+1)E_1 - E_2)) = 0$ for $m \gg 0$, some $\ell > 1$ then $H^1(\mathcal{B}_{(0,1,2)}, \mathcal{O}_{\mathcal{B}_{(0,1,2)}}((k+3)H - 3E_1 - E_2)) = 0$ for all $k \gg 0$.

As $K_{\mathcal{B}_{(0,1,2)}} = \mathcal{O}_{\mathcal{B}_{(0,1,2)}}((-s_0-1)H + (s_0-r-1)E_1 + (s_0-2r-2)E_2)$, we have

$$\begin{aligned} \mathcal{O}_{\mathcal{B}_{(0,1,2)}}(mH - (\ell+1)E_1 - E_2 - K) \\ = \mathcal{O}_{\mathcal{B}_{(0,1,2)}}((m+s_0+1)H - (\ell+s_0-r)E_1 - (s_0-2r-1)E_2) \end{aligned}$$

As soon as $\ell \geq s_0-3r-2$, the right side is ρ -nef and, because ρ is birational, the restriction of the right side to the general fiber of ρ is big. Hence by [Ko, 2.17.3], $R^i \rho_* \mathcal{O}_{\mathcal{B}_{(0,1,2)}}(mH - (\ell+1)E_1 - E_2) = 0$ for $i \geq 1$. Again by the ampleness of $\mathcal{O}_{\mathbb{P}^{s_0}}(H)$, we have $H^1(\mathcal{B}_{(0,1,2)}, \mathcal{O}_{\mathcal{B}_{(0,1,2)}}(mH - (\ell+1)E_1 - E_2)) = 0$ for $m \gg 0$, ℓ as above.

Step 5. *The map $H^0(E_1, \mathcal{O}_{E_1}((k+3)H - 2E_1 - E_2)) \rightarrow H^0(F_p, \mathcal{O}_{F_p} \times (2H' - E'))$ is surjective for all $k \gg 0$.* This is immediate by Lemma 5.6 and the projective normality assumption of Theorem 5.1. \square

As in Theorem 2.3, we show that the restriction of $\widetilde{\varphi_2}^+$ to the transform of $\widehat{\text{Sec}^2 X}$ is a projective bundle over $\mathcal{H}^3 X$. By a slight abuse of notation, write $\widehat{\text{Sec}^2 X} \subset \mathcal{B}_{(1,2)}$ for the image of the proper transform of $\text{Sec}^2 X$. Note the following:

Lemma 5.8. *Let $S_Z = (\widetilde{\varphi_2}^+)^{-1}(Z) \cong \mathbb{P}^2$ be a fiber over a point $Z \in \mathcal{H}^3 X$. Then $\mathcal{O}_{S_Z}(H) = \mathcal{O}_{\mathbb{P}^2}(2)$ and $\mathcal{O}_{S_Z}(E) = \mathcal{O}_{\mathbb{P}^2}(3)$.*

Proof. This is immediate from the restrictions $\mathcal{O}_{S_Z}(2H - E) = \mathcal{O}_{\mathbb{P}^2}(1)$ and $\mathcal{O}_{S_Z}(3H - 2E) = \mathcal{O}_{\mathbb{P}^2}$ \square

Lemma 5.9. *There exists a morphism $\widetilde{\text{Sec}^2 X} \rightarrow \mathbb{G}(2, s_0)$ whose image is $\mathcal{H}^3 X$.*

Proof. A point $p \in \widetilde{\text{Sec}^2 X}$ determines a unique 2-plane S_Z in $\widetilde{\text{Sec}^2 X}$ by Theorem 5.1. For every such p , the homomorphism $H^0(\mathcal{B}_{(1,2)}, \mathcal{O}_{\mathcal{B}_{(1,2)}}(H)) \rightarrow H^0(\mathcal{B}_{(1,2)}, \mathcal{O}_{S_Z}(H))$ has rank 3, hence gives a point in $\mathbb{G}(2, s_0)$. The image of the associated morphism clearly coincides with the natural embedding of $\mathcal{H}^3 X$ into $\mathbb{G}(2, s_0)$ described in [CG]. \square

As in [V2, 3.5], there is a morphism $\mathcal{H}^3 X \rightarrow \mathcal{B}_{(2)}$ so that the composition factors $\widetilde{\varphi_2}^+ : \widetilde{\text{Sec}^2 X} \rightarrow \mathcal{B}_{(2)}$. This is constructed by associating to every $Z \in \mathcal{H}^3 X$ the rank 1 homomorphism:

$$H^0(\mathcal{B}_{(1,2)}, \mathcal{O}_{\mathcal{B}_{(1,2)}}(3H - 2E)) \rightarrow H^0(\mathcal{B}_{(1,2)}, \mathcal{O}_{S_Z}(3H - 2E))$$

where S_Z is the \mathbb{P}^2 in $\mathcal{B}_{(1,2)}$ associated to Z .

Exactly as in Theorem 2.3, this allows the identification of $\widetilde{\text{Sec}^2 X}$ with a \mathbb{P}^2 -bundle over $\mathcal{H}^3 X$. Specifically, $\mathcal{E}_2 = (\widetilde{\varphi_2}^+)_*(\mathcal{O}_{\widetilde{\text{Sec}^2 X}}(2H - E))$ is a rank 3 vector bundle on $\mathcal{H}^3 X$ and:

Proposition 5.10. *With notation as above, $\widetilde{\varphi_2}^+ : \widetilde{\text{Sec}^2 X} \rightarrow \mathcal{H}^3 X$ is the \mathbb{P}^2 -bundle $\mathbb{P}_{\mathcal{H}^3 X}(\mathcal{E}_2) \rightarrow \mathcal{H}^3 X$.* \square

We wish to show further that blowing up $\text{Sec}^2 X$ along X and then along $\text{Sec} X$ resolves the singularities of $\text{Sec}^2 X$. By Theorem 2.4, $h_1 : \mathcal{B}_{(0,1,2)} \rightarrow \mathcal{B}_{(1,2)}$ is the blow up of $\mathcal{B}_{(1,2)}$ along $\mathbb{P}(\mathcal{F})$, hence it suffices to show $\mathbb{P}(\mathcal{F}) \cap \mathbb{P}(\mathcal{E}_2)$ is a smooth subvariety of $\mathbb{P}(\mathcal{E}_2)$.

Proposition 5.11. *$\mathcal{D} = \mathbb{P}(\mathcal{F}) \cap \mathbb{P}(\mathcal{E}_2)$ is the nested Hilbert scheme $Z_{2,3}(X) \subset \mathcal{H}^2 X \times \mathcal{H}^3 X$, hence is smooth. Therefore $\text{Bl}_{\widetilde{\text{Sec} X}}(\text{Bl}_X(\text{Sec}^2 X)) \subset \mathcal{B}_{(0,1,2)}$ is smooth and $\text{Sec}^2 X \subset \mathbb{P}^{s_0}$ is normal.*

Proof. Let $\mathcal{U}_i \subset X \times \mathcal{H}^i X$ denote the universal subscheme. We have morphisms $\widetilde{\varphi_2}^+ : \mathcal{D} \rightarrow \mathcal{H}^3 X$ and $\widetilde{\varphi_1}^- : \mathcal{D} \rightarrow \mathcal{H}^2 X$, and it is routine to check that $(id_X \times \widetilde{\varphi_1}^-)^{-1}(\mathcal{U}_2) \subset (id_X \times \widetilde{\varphi_2}^+)^{-1}(\mathcal{U}_3)$. Hence (Cf. [L, §1.2]) $\widetilde{\varphi_1}^- \times \widetilde{\varphi_2}^+$ maps \mathcal{D} to the nested Hilbert scheme $Z_{2,3}(X) \subset \mathcal{H}^2 X \times \mathcal{H}^3 X$, where closed points of $Z_{2,3}(X)$ correspond to pairs of subschemes (α, β) with $\alpha \subset \beta$. Furthermore, via the description of the structure of the map $\widetilde{\varphi_2}^+$, it is clear that the morphism of $\mathcal{H}^3 X$ -schemes $\mathcal{D} \rightarrow Z_{2,3}(X)$ is finite and birational. It is shown in [C, 0.2.1] that $Z_{2,3}(X)$ is smooth, hence this is an isomorphism. \square

Let $\mathcal{B}_{(1,2,3)}$ be the blow up of $\mathcal{B}_{(1,2)}$ along $\mathbb{P}(\mathcal{E}_2)$; note $\mathcal{B}_{(1,2,3)}$ is smooth. To construct $\mathcal{B}_{(2,3)}$, we first construct the exceptional locus as a projective bundle over $\mathcal{H}^3 X$. Write $\text{Pic } \mathcal{B}_{(1,2,3)} = \mathbb{Z}H + \mathbb{Z}E_1 + \mathbb{Z}E_3$.

Lemma 5.12. Let $p_3 : E_3 \rightarrow \mathcal{H}^3 X$ be the composition $E_3 \rightarrow \mathbb{P}(\mathcal{E}_2) \rightarrow \mathcal{H}^3 X$. Then $\mathcal{F}_2 = \widetilde{(p_3)_*} \mathcal{O}_{E_3}(4H - 3E_1 - E_3)$ is locally free of rank $s_0 - 3r - 2 = \text{codim } (\text{Sec}^2 X, \mathcal{B}_{(1,2)})$.

Proof. Each fiber F_x of p_3 is isomorphic to $\mathbb{P}^2 \times \mathbb{P}^t$, $t + 1 = \text{codim } (\text{Sec}^2 X, \mathcal{B}_{(1,2)})$. Furthermore $H^0(F_x, \mathcal{O}_{F_x}(4H - 3E_1 - E_3)) = H^0(\mathbb{P}^t, \mathcal{O}_{\mathbb{P}^t}(1))$ follows easily from Lemma 5.8. \square

There is a map $E_3 \rightarrow \mathbb{P}(\mathcal{F}_2)$ given by the surjection

$$p_3^* \mathcal{F}_2 \rightarrow \mathcal{O}_{E_3}(4H - 3E_1 - E_3) \rightarrow 0$$

hence a diagram of exceptional loci:

$$\begin{array}{ccc} & E_3 & \\ p_3 & \swarrow \quad \searrow & \\ \mathbb{P}(\mathcal{E}_2) & & \mathbb{P}(\mathcal{F}_2) \\ & \searrow \quad \swarrow & \\ & \mathcal{H}^3 X & \end{array}$$

It is important to note that

$$\mathbb{P}(\mathcal{F}_2) \cong \mathbb{P}(p_{3*} \mathcal{O}_{E_3}(4H - 3E_1 - E_3 + t(3H - 2E_1)))$$

for all $t \geq 0$ as the direct image on the right will differ from \mathcal{F}_2 by a line bundle. Hence for all $t \geq 0$ the same morphism $E_3 \rightarrow \mathbb{P}(\mathcal{F}_2)$ is induced by the surjection

$$\begin{aligned} p_3^* p_{3*} \mathcal{O}_{E_3}(4H - 3E_1 - E_3 + t(3H - 2E_1)) \\ \rightarrow \mathcal{O}_{E_3}(4H - 3E_1 - E_3 + t(3H - 2E_1)) \end{aligned}$$

One can now repeat almost verbatim [V2, 4.7-4.10] to construct the second flip; i.e. the space $\mathcal{B}_{(2,3)}$. Recall the following:

Proposition 5.13. [V2, 4.5] Let \mathcal{L} be an invertible sheaf on a complete variety X , and let \mathcal{B} be any locally free sheaf. Assume that the map $\lambda : X \rightarrow Y$ induced by $|\mathcal{L}|$ is a birational morphism and that λ is an isomorphism in a neighborhood of $p \in X$. Then for all n sufficiently large, the map

$$H^0(X, \mathcal{B} \otimes \mathcal{L}^n) \rightarrow H^0(X, \mathcal{B} \otimes \mathcal{L}^n \otimes \mathcal{O}_p)$$

is surjective. \square

Taking $\mathcal{B} = \mathcal{O}_{\mathcal{B}_{(1,2,3)}}(4H - 3E_1 - E_3)$ and $\mathcal{L} = \mathcal{O}_{\mathcal{B}_{(1,2,3)}}(3H - 2E_1)$, the map induced by the linear system associated to

$$\begin{aligned} & \mathcal{O}_{\mathcal{B}_{(1,2,3)}}((4H - 3E_1 - E_3) + (k-2)(3H - 2E_1)) \\ &= \mathcal{O}_{\mathcal{B}_{(1,2,3)}}((3k-2)H - (2k-1)E_1 - E_3) \end{aligned}$$

is base point free off E_3 for $k \gg 3$. To show this gives a morphism, one shows the restriction of above linear system to the divisor E_3 induces a surjection on global sections, hence restricts to the map $E_3 \rightarrow \mathbb{P}(\mathcal{F}_2)$ above. For this, define $\mathcal{L}_\rho = \mathcal{O}((3\rho-2)H - (2\rho-1)E_1 - E_3)$ and write

$$\mathcal{O}_{\mathcal{B}_{(1,2,3)}}((3k-2)H - (2k-1)E_1 - 2E_3) \otimes K_{\mathcal{B}_{(1,2,3)}}^{-1} = \mathcal{L}_\alpha^{s_0-3r-1} \otimes \mathcal{A}$$

where $\alpha = \frac{2k+s_0-r-1}{2s_0-6r-2}$ and $\mathcal{A} = \mathcal{O}_{\mathcal{B}_{(1,2,3)}}\left(\left(\frac{3s_0-9r-4}{2}\right)H - (s_0-3r-2)E_1\right)$. By the above discussion, $\mathcal{L}_\alpha^{s_0-3r-1}$ is nef for $k \gg 0$ and it is routine to verify that \mathcal{A} is a big and nef \mathbb{Q} -divisor; hence $H^1(\mathcal{B}_{(1,2,3)}, \mathcal{O}((3k-2)H - (2k-1)E_1 - 2E_3)) = 0$.

The variety $\mathcal{B}_{(2,3)}$ is defined to be the image of this morphism. This gives:

Proposition 5.14. *With hypotheses as in Theorem 5.1 and for k sufficiently large, the morphism $h_2 : \mathcal{B}_{(1,2,3)} \rightarrow \mathcal{B}_{(2,3)}$ induced by the linear system $|\mathcal{L}_k|$ is an embedding off of E_3 and the restriction of h_2 to E_3 is the morphism $E_3 \rightarrow \mathbb{P}(\mathcal{F}_2)$ described above. \square*

Remark 5.15. The best (smallest) possible value for k is $k = 3$. This will be the case if $\text{Sec}^3 X \subset \mathbb{P}^{s_0}$ is scheme theoretically cut out by quartics. \square

Lemma 5.16. $\mathcal{B}_{(2,3)}$ is smooth.

Proof. Because $\mathcal{B}_{(2,3)}$ is the image of a smooth variety with reduced, connected fibers it is normal (Cf. [V1, 3.2.5]). Let $Z \cong \mathbb{P}^2$ be a fiber of h_2 over a point $p \in \mathbb{P}(\mathcal{F}_2)$. $Z \times \{p\}$ is a fiber of a $\mathbb{P}^2 \times \mathbb{P}^t$ bundle over $\mathcal{H}^3 X$, hence the normal bundle sequence becomes:

$$0 \rightarrow \bigoplus_1^{s_0-3} \mathcal{O}_{\mathbb{P}^2} \rightarrow N_{Z/\mathcal{B}_{(1,2,3)}} \rightarrow \mathcal{O}_{\mathbb{P}^2}(-1) \rightarrow 0$$

This sequence splits, and allowing the elementary calculations $H^1(Z, S^r N_{Z/\mathcal{B}_{(1,2,3)}}) = 0$ and $H^0(Z, S^r N_{Z/\mathcal{B}_{(1,2,3)}}) = S^r H^0(Z, N_{Z/\mathcal{B}_{(1,2,3)}})$ for all $r \geq 1$, $\mathcal{B}_{(2,3)}$ is smooth by a natural extension of the smoothness portion of Castelnuovo's contractibility criterion for surfaces given in [AW, 2.4]. \square

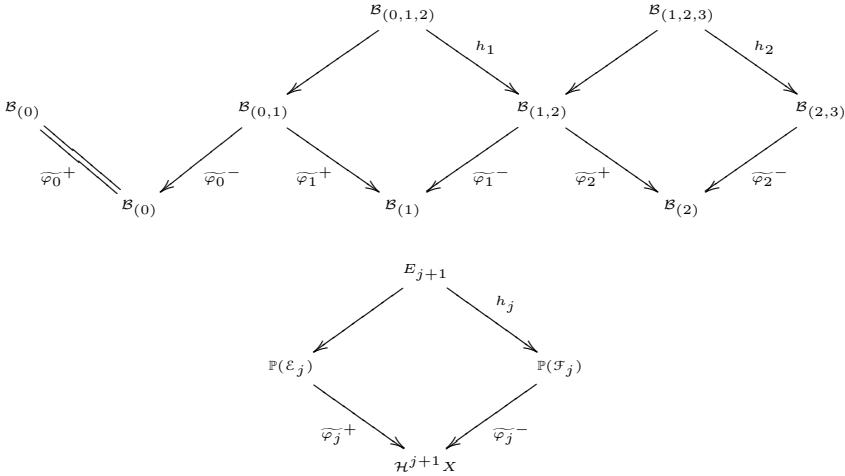
Letting $\mathbb{P}(\mathcal{F}_0) = \mathbb{P}_X(N_{X/\mathbb{P}^{s_0}}^*) = E_1$ and $\mathbb{P}(\mathcal{E}_0) = X$, the analogue of Theorem 2.4 is:

Theorem 5.17. Let $X \subset \mathcal{B}_{(0)} = \mathbb{P}^{s_0}$ be a smooth, irreducible variety of dimension r that satisfies (K_2^1) , with $s_0 \geq 3r + 4$. Assume that X is embedded by a complete linear system $|L|$ and that the following conditions are satisfied:

1. L is $(5+r)$ -very ample and $\text{Sec}^2 X = \text{Sec}_3^1(\text{Sec}^1 X)$ as schemes
2. The projection of X into \mathbb{P}^m , $m = s_0 - 1 - r$, from any embedded tangent space is such that the image is projectively normal and satisfies (K_2)
3. If $r \geq 2$, then for every point $p \in X$, $H^1(X, L \otimes \mathcal{I}_p^3) = 0$

Then there is a pair of flips as pictured below with:

1. $\mathcal{B}_{(i,i+1)}$ and $\mathcal{B}_{(i,i+1,i+2)}$ smooth
2. $\mathcal{B}_{(i,i+1)} \setminus \mathbb{P}(\mathcal{E}_{i+1}) \cong \mathcal{B}_{(i+1,i+2)} \setminus \mathbb{P}(\mathcal{F}_{i+1})$; as $s_0 \geq 3r + 4$, $\text{Pic } \mathcal{B}_{(0,1)} \cong \text{Pic } \mathcal{B}_{(i+1,i+2)}$
3. $\mathbb{P}\mathcal{E}_i = \mathbb{P}\widetilde{\varphi_{i*}}^+ \mathcal{O}_{\widetilde{\text{Sec}^i X}}(iH - (i-1)E)$ and $\mathbb{P}\mathcal{F}_i = \mathbb{P}\widetilde{\varphi_{i*}}^- \mathcal{O}_{\widetilde{\text{Sec}^i X}}((i+2)H - (i+1)E)$
4. h_i is the blow up of $\mathcal{B}_{(i,i+1)}$ along $\mathbb{P}(\mathcal{F}_i)$
5. $\mathcal{B}_{(i,i+1,i+2)} \rightarrow \mathcal{B}_{(i,i+1)}$ is the blow up along $\mathbb{P}(\mathcal{E}_{i+1})$
6. $\widetilde{\varphi}_i^-$, induced by $\mathcal{O}_{\mathcal{B}_{(i,i+1)}}((i+1)H - iE)$, is an embedding off of $\mathbb{P}(\mathcal{F}_i)$, and the restriction of $\widetilde{\varphi}_i^-$ is the projection $\mathbb{P}(\mathcal{F}_i) \rightarrow \mathcal{H}^{i+1} X$
7. $\widetilde{\varphi}_i^+$, induced by $\mathcal{O}_{\mathcal{B}_{(i-1,i)}}((i+1)H - iE)$, is an embedding off of $\mathbb{P}(\mathcal{E}_i)$, and the restriction of $\widetilde{\varphi}_i^+$ is the projection $\mathbb{P}(\mathcal{E}_i) \rightarrow \mathcal{H}^{i+1} X$
8. $\mathbb{P}(\mathcal{F}_i) \cap \mathbb{P}(E_{i+1}) \subset \mathcal{B}_{(i,i+1)}$ is isomorphic to the nested Hilbert scheme $Z_{i+1,i+2} \subset \mathcal{H}^i X \times \mathcal{H}^{i+1} X$, hence is smooth.



□

References

- [AW] M Andreatta, J A Wiśniewski: A View on Contractions of Higher-Dimensional Varieties. Algebraic Geometry - Santa Cruz 1995, Proc. Sympos. Pure Math. **62**, Part 1, Amer. Math. Soc., Providence 1997, 153–183
- [B1] A Bertram: Moduli of rank-2 vector bundles, Theta divisors, and the geometry of curves in projective space. *J. Diff. Geom.* **35** (1992), 429–469
- [B2] A Bertram: Complete extensions and their map to moduli space. In: Complex Projective Geometry, pp.81–91. London Math. Soc. Lecture Note Series 179, Cambridge University Press, Cambridge 1992
- [B3] A Bertram: Stable pairs and log flips. In: Algebraic geometry - Santa Cruz 1995, pp. 185–201. Proc. Sympos. Pure Math., 62, Part 1, Amer. Math. Soc., Providence, RI 1997
- [B4] A Bertram: An application of general Kodaira vanishing to embedded projective varieties. (preprint, alg-geom/9707001)
- [CG] F Catanese, L Göttsche: d -very-ample line bundles and embeddings of Hilbert schemes of 0-cycles. *Manuscripta Math.* **68** (1990), 337–341
- [C] J Cheah: Cellular decompositions for nested Hilbert schemes of points. *Pac. Journ. Math.* **183** (1998), 39–90
- [EKS] D Eisenbud, J Koh, M Stillman: Determinantal equations. *Amer. Journ. Math.* **110** (1988), 513–539
- [Go] Lothar Göttsche: Hilbert schemes of zero-dimensional subschemes of smooth varieties. Springer Verlag Lecture Notes in Mathematics 1572, 1994
- [G] M Green: Koszul cohomology and the geometry of projective varieties. *J. Diff. Geom.* **19** (1984), 125–171
- [H] Joe Harris: Algebraic geometry, a first course. Springer, New York 1992
- [JPW] T Józefiak, P Pragacz, J Weyman: Resolutions of determinantal varieties and tensor complexes associated with symmetric and antisymmetric matrices. In: Tableaux de Young et Foncteurs de Schur en Algèbre et Géométrie, pp. 109–189. Astérisque 87–88, Société Mathématique de France, 1981
- [Ka] V Kanev: Chordal varieties of Veronese varieties and Catalecticant matrices. *Algebraic Geometry*, 9. *J. Math. Sci. (New York)* **94** (1999), 1114–1125
- [Ko] J Kollar: Singularities of pairs. In: Algebraic geometry – Santa Cruz 1995, pp. 221–287. Proc. Sympos. Pure Math. 62, Part 1, Amer. Math. Soc., Providence, RI 1997
- [L] M Lehn: Chern classes of tautological sheaves on Hilbert schemes of points on surfaces. *Invent. Math.* **136** (1999), 157–207
- [R] MS Ravi: Determinantal equations for secant varieties of curves. *Comm. in Algebra* **22** (1994), 3103–3106
- [T1] M Thaddeus: Stable pairs, linear systems, and the Verlinde formula. *Invent. Math.* **117** (1994), 317–353
- [T2] MThaddeus: Toric quotients and flips. In: K Fukaya, M Furuta, T Kohno (eds) Topology, geometry, and field theory, pp. 193–213. World Scientific 1994
- [T3] M Thaddeus: Complete collineations revisited. *Math. Ann.* **315** (1999), 469–495
- [V1] P Vermeire: University of North Carolina PhD Thesis, 1998
- [V2] P Vermeire: Some results on secant varieties leading to a geometric flip construction. *Compositio Math.* **125** (2001), 263–282.
- [V3] P Vermeire: On the regularity of powers of ideal sheaves. (To appear in *Compositio Math.*, available at math.AG/9911171)