Homework #6 (Continued)

Math 2300 - Section 880

Due: Thursday, Oct 1

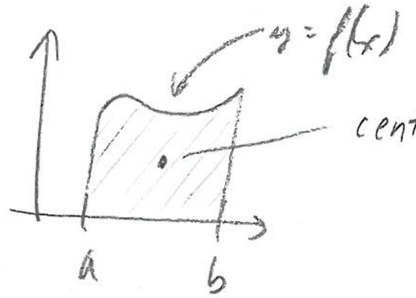
Instructions. Be sure to show your work and explain your reasoning for full credit. Be aware that this homework assignment also has problems from the textbook (as indicated on the course website).

NAME Solutions

In this part of Homework 6, you will develop Pappus' Theorem using similar techniques used to develop the formulas for volumes of solids of revolution, arc-length, work, pressure, center of mass, etc. Pappus' Theorem states the following.

> **Pappus' Theorem:** Let a planar region R be on the right side of the y-axis. Rotate R about the y-axis to form a solid of revolution. Then the volume V of the solid of revolution is
> $$V = 2\pi\overline{x}A,$$
> where $\overline{x}$ is the x-coordinate of the centroid of R, and A is the area of R.

(a) Assume that R is the region under a graph $y = f(x)$ where the domain of f is $[a, b]$ (with $0 \leq a < b$), and $f(x) \geq 0$ for each $x \in [a, b]$. What is the centroid of R in terms of f, a, and b? (Your answer should be in terms of integrals.)



centroid $(\bar{x}, \bar{y})$ where

$$\bar{x} = \frac{\int_a^b x f(x)\,dx}{\int_a^b f(x)\,dx} \qquad \bar{y} = \frac{\int_a^b \frac{1}{2} f(x)^2\,dx}{\int_a^b f(x)\,dx}$$

(b) Divide the interval $[a, b]$ into small subintervals $[x_{i-1}, x_i]$ where

$$a = x_0 < x_1 < \cdots < x_n = b.$$

Consider the rectangle with base $[x_{i-1}, x_i]$ and height $f(x_i)$. What is the volume of the cylindrical shell obtained by revolving this rectangle about the y-axis?

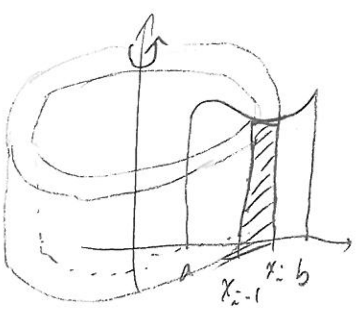
Volume $\approx 2\pi x_i f(x_i)(x_i - x_{i-1})$

OR

$\approx 2\pi x_{i-1} f(x_{i-1})(x_i - x_{i-1})$

OR

$\approx 2\pi x_i^* f(x_i^*)(x_i - x_{i-1})$ where $x_i^* \in [x_{i-1}, x_i]$.



(c) Sum up the volumes you obtained in part (b) for $i = 0$ to $i = n$, and take a limit as $n \to \infty$. What formula do you get for the volume?

$$\text{Volume} = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi x_i^* f(x_i^*)(x_i - x_{i-1}) = \int_a^b 2\pi x f(x)\,dx$$

(d) Compare the answer you obtained in part (c) with the centroid formula in part (a). From this, derive the equation in Pappus' Theorem.

Volume $= 2\pi \int_a^b x f(x)\,dx$, but $\int_a^b x f(x)\,dx = \bar{x}\int_a^b f(x)\,dx$

Also, $\int_a^b f(x)\,dx$ = area of the region $R = A$. So,

Volume $= V = 2\pi \bar{x} A$. This proves Pappus' theorem.