UNIVERSITY OF COLORADO AT BOULDER
MATH 2300 880 - MIDTERM 1

September 21, 2015
Instructor: Jordan Watts

Name: Solutions

Read all of the following information before starting the exam:

- **Show all work**, clearly and in order, if you want to get full credit. The graders reserve the right to take off points if they cannot see how you arrived at your answer (even if your final answer is correct). Points will also be removed for answers that are not clear.
- **Calculators, notes, phones, and other aids including all electronic devices, are not permitted for this test.**
- This test has 6 problems, 9 pages, and is worth 50 points. It is your responsibility to make sure that you have all of the pages!
- Good luck!

Problem	**Grade**
1. (/5)	
2. (/5)	
3. (/10)	
4. (/10)	
5. (/5)	
6. (/15)	
Total: (/50)	

Problem 1 (5 points). Solve the following integral.

$$\int_1^{\ln 2} \frac{e^x\,dx}{e^x - 1}$$

$u = e^x - 1$

$du = e^x dx$

$x = 1 \Rightarrow u = e-1$

$x = \ln 2 \Rightarrow u = 1$

$= \int_{e-1}^{1} \frac{du}{u} = \ln|u| \Big|_{e-1}^{1} = -\ln|e-1|$

$= \ln\left(\frac{1}{e-1}\right).$

Problem 2 (5 points). Solve the following integral.

$$\int x^2 \ln x \, dx$$

$u = \ln x$ $\qquad dv = x^2 \, dx$

$du = \frac{1}{x} \, dx$ $\qquad v = \frac{x^3}{3}$

$$= \frac{x^3}{3} \ln x - \int \frac{x^2}{3} \, dx$$

$$= \frac{x^3}{3} \ln x - \frac{1}{9} x^3 + C$$

Problem 3 (10 points).

/5 (a) Find the partial fraction decomposition of $\frac{1}{x^3+4x}$.

$$\frac{1}{x^3+4x} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$\Rightarrow \quad 1 = A(x^2+4) + Bx^2 + Cx$$

$$\Rightarrow \quad \begin{cases} A+B=0 \quad \Rightarrow \quad B = \frac{-1}{4} \\ C=0 \\ A = \frac{1}{4} \end{cases}$$

So, $\frac{1}{x^3+4x} = \frac{1}{4x} - \frac{x}{4(x^2+4)}$

/5 (b) Use your result in part (a) to solve the integral $\int \frac{dx}{x^3+4x}$.

$$= \int \frac{dx}{4x} + \frac{1}{4}\int \frac{x\,dx}{x^2+4}$$

$u = x^2+4$

$du = 2x\,dx$

$$= \frac{1}{4}\ln|x| + \frac{1}{8}\int \frac{du}{u}$$

$$= \frac{1}{4}\ln|x| + \frac{1}{8}\ln|x^2+4| + C$$

Problem 4 (10 points). Solve the following integral. ~~Be sure to simplify your final answer in terms of elementary functions~~.

$$\int_0^1 x^2\sqrt{1-x^2}\,dx$$

$x = \sin\theta$

$dx = \cos\theta\,d\theta$

$x = 0 \Rightarrow \theta = 0$

$x = 1 \Rightarrow \theta = \pi/2$

$$= \int_0^{\pi/2} \sin^2\theta \cos^2\theta\,d\theta$$

$$= \int_0^{\pi/2} \left(\frac{1}{2}\sin(2\theta)\right)^2 d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2} \left(\frac{1}{2} - \frac{\cos(4\theta)}{2}\right) d\theta$$

$$= \frac{1}{4}\left(\frac{\theta}{2} - \frac{1}{8}\sin(4\theta)\right)\Big|_0^{\pi/2}$$

$$= \frac{1}{4}\left(\frac{\pi}{4}\right) = \frac{\pi}{16}.$$

Problem 5 (5 points). Solve the following integral. (**Hint:** If these were regular trigonometric functions, how would you solve it?)

$$\int_0^1 \sinh^2(x)\cosh^3(x)\,dx$$

$$= \int_0^1 \sinh^2(x)(1+\sinh^2 x)\cosh x\,dx$$

$u = \sinh x$

$du = \cosh x\,dx$

$$= \int_{x=0}^{1} u^2(1+u^2)\,du$$

$$= \left(\frac{1}{3}u^3 + \frac{1}{5}u^5\right)\Big|_{x=0}^{1}$$

$$= \left(\frac{1}{3}\sinh^3 x + \frac{1}{5}\sinh^5 x\right)\Big|^1$$

$$= \left[\frac{1}{3}\left(\frac{e^x - e^{-x}}{2}\right)^3 + \frac{1}{5}\left(\frac{e^x - e^{-x}}{2}\right)^5\right]\Big|_0^1$$

$$= \frac{1}{3}\left(\frac{e - \frac{1}{e}}{2}\right)^3 + \frac{1}{5}\left(\frac{e - \frac{1}{e}}{2}\right)^5$$

Problem 6 (15 points).

/10 (a) Consider the integral $\int_0^1 \frac{dx}{x^p}$, where $p > 0$. Similar to the p-test we developed in class for integrating over the interval $[1, \infty)$, show for which p the integral above converges, and for which p it diverges.

$p \in (0,1)$

$$\int_0^1 \frac{dx}{x^p} = \lim_{t \to 0^+} \int_t^1 \frac{dx}{x^p} = \lim_{t \to 0^+} \frac{1}{1-p} x^{1-p} \Big|_t^1 = \lim_{t \to 0^+} \left(\frac{1}{1-p} - \frac{t^{1-p}}{1-p} \right)$$

$$= \frac{1}{1-p}$$ converges

$p = 1$ $$\int_0^1 \frac{dx}{x} = \lim_{t \to 0^+} \int_t^1 \frac{dx}{x} = \lim_{t \to 0^+} \ln|x| \Big|_t^1 = \lim_{t \to 0^+} \ln t = -\infty$$

diverges

$p > 1$

$$\int_0^1 \frac{dx}{x^p} = \lim_{t \to 0^+} \int_t^1 \frac{dx}{x^p} = \lim_{t \to 0^+} \frac{1}{1-p} x^{1-p} \Big|_t^1 = \lim_{t \to 0^+} \left(\frac{1}{1-p} - \frac{1}{t^{p-1}} \right) = -\infty$$

diverges

/5 (b) Use your result in part (a) to determine whether the following integral converges or diverges.

$$\int_0^1 \frac{dx}{x^{1/3} + x^{2/3}}$$

$$x^{1/3} + x^{2/3} \geq x^{1/3}$$

$$\Rightarrow \frac{1}{x^{1/3} + x^{2/3}} \leq \frac{1}{x^{1/3}}$$

$$\Rightarrow 0 \leq \int_0^1 \frac{dx}{x^{1/3} + x^{2/3}} \leq \int_0^1 \frac{dx}{x^{1/3}}$$

Since the right integral converges by part ⓐ, by the comparison test, $\int_0^1 \frac{dx}{x^{1/3}+x^{2/3}}$ converges.

Scrap Page
(Please do not remove this page from the test packet.)

$$
\begin{aligned}
\sin^2(x) + \cos^2(x) &= 1 \\
\sin(a \pm b) &= \sin(a)\cos(b) \pm \sin(b)\cos(a) \\
\cos(a \pm b) &= \cos(a)\cos(b) \mp \sin(a)\sin(b) \\
\sin(2x) &= 2\sin(x)\cos(x) \\
\cos(2x) &= \cos^2(x) - \sin^2(x) \\
&= 1 - 2\sin^2(x) \\
&= 2\cos^2(x) - 1
\end{aligned}
$$