UNIVERSITY OF COLORADO AT BOULDER
MATH 2300 880 - MIDTERM 2

October 19, 2015
Instructor: Jordan Watts

Name: Solutions

Read all of the following information before starting the exam:

- **Show all work**, clearly and in order, if you want to get full credit. The graders reserve the right to take off points if they cannot see how you arrived at your answer (even if your final answer is correct). Points will also be removed for answers that are not clear.
- **Calculators, notes, phones, and other aids including all electronic devices, are not permitted for this test.**
- This test has 6 problems, 9 pages, and is worth 60 points. It is your responsibility to make sure that you have all of the pages!
- Good luck!

Problem	Grade
1. (/10)	
2. (/15)	
3. (/5)	
4. (/10)	
5. (/10)	
6. (/10)	
Total: (/60)	

Problem 1 (10 points). Determine whether the following series converge or diverge. **Show your work, and justify your claims.**

/5 (a) $\sum_{n=0}^{\infty} \frac{n}{5^n}$

$n < 2^n$ for all n.

$\Rightarrow \frac{n}{5^n} < \frac{2^n}{5^n}$ for all n

$\sum_{n=0}^{\infty} \frac{2^n}{5^n}$ is a geometric series, and $\left|\frac{2}{5}\right| < 1$. Thus, this series converges.

By the basic comparison test, $\sum_{n=0}^{\infty} \frac{n}{5^n}$ converges.

/5 (b) $\sum_{n=1}^{\infty} \frac{\ln n}{n^2+1}$

$\ln n < \sqrt{n}$

$\Rightarrow \frac{\ln n}{n^2+1} < \frac{\sqrt{n}}{n^2+1} < \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$

Since $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges (p-series), by the basic comparison test, $\sum_{n=1}^{\infty} \frac{\ln n}{n^2+1}$ converges.

Problem 2 (15 points). Determine whether the following sequences and series converge or diverge. If they converge, what do they converge to? **Show your work, and justify your claims.**

/5 (a) $\left\{\left(1+\frac{4}{n}\right)^{3n}\right\}$

$$\lim_{n\to\infty} \ln\left[\left(1+\frac{4}{n}\right)^{3n}\right] = \lim_{n\to\infty} \frac{\ln\left(1+\frac{4}{n}\right)}{1/3n} \underset{\text{l'Hôpital}}{=} \lim_{n\to\infty} \frac{\frac{4}{1+\frac{4}{n}}\left(\frac{-1}{\cancel{n^2}}\right)}{\frac{-1}{3}\frac{1}{\cancel{n^2}}} \to 12$$

$$\Rightarrow \lim_{n\to\infty}\left(1+\frac{4}{n}\right)^{3n} = e^{12}$$

converges

/5 (b) $\sum_{n=4}^{\infty} \frac{1}{\sqrt{n+3}}$

Applying the limit comparison test to the series, comparing with $\frac{1}{\sqrt{n}}$:

$$\lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+3}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+3/\sqrt{n}}} = 1 > 0.$$ And so, since $\frac{1}{\sqrt{n}}$ diverges so does $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+3}}$.

diverges

/5 (c) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^3 + 3n^2 + 2n}$

$$\frac{2}{n^3+3n^2+2n} = \frac{2}{n(n+1)(n+2)} = \frac{A}{n} + \frac{B}{n+1} + \frac{C}{n+2}$$

$$\Rightarrow 2 = A(n+1)(n+2) + Bn(n+2) + Cn(n+1)$$

$$= An^2 + 3An + 2A + Bn^2 + 2Bn + Cn^2 + Cn$$

$$\Rightarrow \quad 0 = A+B+C \quad \Rightarrow \quad 1 + B - 2B - 3 = 0 \Rightarrow B = -2$$

$$0 = 3A + 2B + C \quad \Rightarrow \quad C = -2B - 3 \quad \Rightarrow C = 1$$

$$2 = 2A \quad \Rightarrow \quad A = 1$$

So, $\displaystyle\frac{2}{n^3+3n^2+2n} = \frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}$

$$\left(1 - \frac{2}{2} + \cancel{\frac{1}{3}}\right) + \left(\frac{1}{2} - \cancel{\frac{2}{3}} + \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{2}{4}} + \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{2}{5}} + \frac{1}{6}\right) \ldots$$

This is a telescoping sum: all terms cancel except for $1 - \frac{2}{2} + \frac{1}{2} = \frac{1}{2}$.

So, the series is equal to $\frac{1}{2}$.

Problem 3 (5 points). Find the arc-length of the graph of $y = x^{3/2}$ for $x \in [0, 4]$.

$$s = \int_0^4 \sqrt{1 + \left(\frac{d}{dx}\left(x^{3/2}\right)\right)^2}\, dx = \int_0^4 \sqrt{1 + \left(\frac{3}{2}x^{1/2}\right)^2}\, dx$$

$$= \int_0^4 \sqrt{1 + \frac{9}{4}x}\, dx$$

$u = 1 + \frac{9}{4}x$

$du = \frac{9}{4}dx$

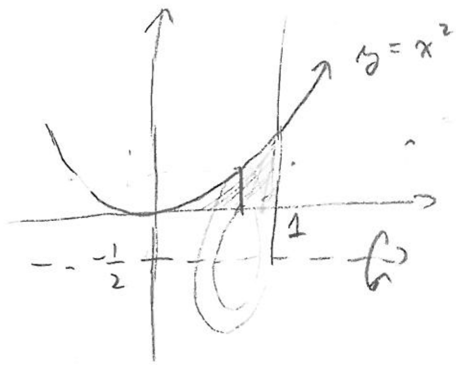
$x = 0 \Rightarrow u = 1$

$x = 4 \Rightarrow u = 10$

$$= \int_1^{10} \sqrt{u}\ \frac{4}{9}du$$

$$= \frac{4}{9} \cdot \frac{2}{3} u^{3/2} \Big|_1^{10} = \frac{8}{27}\left(10^{3/2} - 1\right)$$

Problem 4 (10 points). Find the volume of the solid of revolution given by rotating the region bounded by $y = x^2$, $x = 1$, and the x-axis about the line $y = -\frac{1}{2}$.



Intersect. x bounded between $x=0$ and $x=1$.

Area $A(x) = \pi R^2 - \pi r^2$

$= \pi\left(\frac{1}{2} + x^2\right)^2 - \pi\left(\frac{1}{2}\right)^2$

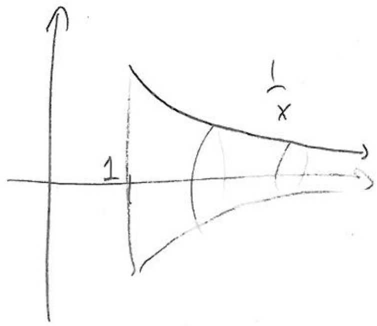
Volume $= \pi\int_0^1 \left(\frac{1}{4} + x^2 + x^4 - \frac{1}{4}\right) dx$

$= \pi\int_0^1 (x^4 + x^2)\,dx$

$= \pi\left(\frac{x^5}{5} + \frac{x^3}{3}\right)\Big|_0^1 = \pi\left(\frac{1}{5} + \frac{1}{3}\right)$

$= \frac{8\pi}{15}$

Problem 5 (10 points). "Gabriel's Horn" is the solid of revolution given by rotating the region bounded above by $y = \frac{1}{x}$, below by the x-axis, and on the left by $x = 1$ (there is no right bound, the region goes off to infinity), about the x-axis. Find its volume. (What makes this shape interesting is that while its volume is finite, its surface area is infinite.)



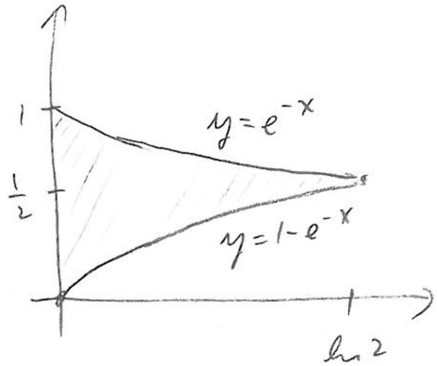
$$A(x) = \pi \frac{1}{x^2}$$

$$\text{Volume} = \int_1^\infty \frac{\pi\, dx}{x^2} = \lim_{t \to \infty} \int_1^t \frac{\pi\, dx}{x^2}$$

$$= \lim_{t \to \infty} -\pi x^{-2} \Big|_1^t = \lim_{t \to \infty} \left((-\pi) t^{-2} + \pi \right)$$

$$= \pi .$$

Problem 6 (10 points). Find the centroid (or centre of mass, assuming a uniform density) of the region bounded above by $y = e^{-x}$, below by $y = 1 - e^{-x}$, and on the left by the y-axis.



Intersect: $e^{-x} = 1 - e^{-x}$

$\Rightarrow e^{-x} = \frac{1}{2}$

$\Rightarrow x = \ln 2$

$$M_{x=0} = \int_0^{\ln 2} x\left(e^{-x} - (1-e^{-x})\right) dx$$

$$= \int_0^{\ln 2} (2xe^{-x} - x)\,dx = 2\int_0^{\ln 2} xe^{-x}\,dx - \frac{1}{2}x^2\Big|_0^{\ln 2}$$

$u = x \quad dv = e^{-x}dx$

$du = dx \quad v = -e^{-x}$

$$= 2\left(-xe^{-x}\Big|_0^{\ln 2} + \int_0^{\ln 2} e^{-x}\,dx\right) - \frac{1}{2}(\ln 2)^2$$

$$= 2\left(-\frac{1}{2}\ln 2 - e^{-x}\Big|_0^{\ln 2}\right) - \frac{1}{2}(\ln 2)^2$$

$$= \ln\frac{1}{2} - 1 + 2 - \frac{1}{2}(\ln 2)^2 = 1 - \ln 2 - \frac{1}{2}(\ln 2)^2$$

$$\bar{x} = \frac{M_{x=0}}{\int_0^{\ln 2}(e^{-x} - 1 + e^{-x})\,dx} = \frac{1 - \ln 2 - \frac{1}{2}(\ln 2)^2}{(-2e^{-x} - x)\Big|_0^{\ln 2}}$$

$$= \frac{1 - \ln 2 - \frac{1}{2}(\ln 2)^2}{-1 - \ln 2 + 2} = \frac{1 - \ln 2 - \frac{1}{2}(\ln 2)^2}{1 - \ln 2}$$

$\bar{y} = \frac{1}{2}$ by symmetry. So, the centre of mass

$$= \left(\frac{1 - \ln 2 - \frac{1}{2}(\ln 2)^2}{1 - \ln 2}, \frac{1}{2}\right)$$

Scrap Page
(Please do not remove this page from the test packet.)

$$\begin{aligned}
\sin^2(x) + \cos^2(x) &= 1 \\
\sin(a \pm b) &= \sin(a)\cos(b) \pm \sin(b)\cos(a) \\
\cos(a \pm b) &= \cos(a)\cos(b) \mp \sin(a)\sin(b) \\
\sin(2x) &= 2\sin(x)\cos(x) \\
\cos(2x) &= \cos^2(x) - \sin^2(x) \\
&= 1 - 2\sin^2(x) \\
&= 2\cos^2(x) - 1
\end{aligned}$$