1. *(12 points)*

For each of the following statements, either prove it is true, or provide a counterexample to show that it is false.

(a) If a sequence (s_n) is unbounded, then (s_n) cannot have a convergent subsequence.
FALSE. Let

$$s_n = \begin{cases} n & \text{if } n \text{ is odd,} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

Then for given $M > 0$ find an odd integer $2n - 1 > M$. We then have $s_{2n-1} = 2n - 1 > M$, so that the sequence (s_n) is unbounded. On the other hand $(s_{2n}) = (1)$ a constant sequence, which converges to 0.

(b) Let $f : [a, b] \to \mathbb{R}$, and let $c \in (a, b)$. If the limit of $f(x)$ as x goes to c exists, then f is continuous at c.
FALSE. Defin $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 2x & \text{if } x \neq \frac{1}{2}, \\ 2 & \text{if } x = \frac{1}{2}. \end{cases}$$

We note $\lim_{x \to 1/2} f(x) = 2 \cdot (1/2) = 1$, but $f(\frac{1}{2}) = 2$. Therefore f is not continuous at $x = \frac{1}{2}$.

(c) If I is an interval, and $f : I \to \mathbb{R}$ is uniformly continuous on I, then f is continuous on I.
TRUE: Let $c \in I$ be fixed, and let $\epsilon > 0$ be fixed. Since f is uniformly continuous on I, there exist $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $x, y \in I$ and $|x - y| < \delta$. Take $y = c$, then $|f(x) - f(c)| < \epsilon$ whenever $x,\ c \in I$, and $|x - c| < \delta$. Therefore f is continuous at c. Since $c \in I$ was arbitrary, f is continuous on c.

2. *(13 points)*

(a) Suppose that $f : [a, b] \to \mathbb{R}$, and that f is continuous at $c \in (a, b)$, with $f(c) > 0$. Prove that there is a $\delta > 0$ such that $f(x) > 0$ for all $x \in (c - \delta, c + \delta)$.
Take $\epsilon = f(c) > 0$ in the definition of continuity. Then since c is an interior point of I there exists $\delta > 0$ such that whenever $|x - c| < \delta$ (i.e. whenever $x \in (c - \delta, c + \delta)$),

$$|f(x) - f(c)| < \epsilon = f(c).$$

Therefore, whenever $x \in (c - \delta, c + \delta)$,

$$-f(d) < f(x) - f(c) < f(c).$$

Adding $f(c)$ to all sides of the inequality, we see that whenever $|x - c| < \delta$,

$$0 < f(x) < f(c) + f(c) = 2f(c).$$

Thus $f(x) > 0$ whenever $|x - c| < \delta$, as desired.

(b) Let

$$f(x) = \begin{cases} 3x^2 + 1 & \text{if } x \geq 1, \\ 6x & \text{if } x < 1. \end{cases}$$

Determine whether or not f is differentiable at $x = 1$, and if it is differentiable, compute the derivative $f'(1)$.
We note that

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1-} 6x = 6 \neq f(1) = 4.$$

Therefore, f is not continuous at $x = 1$. It follows from Theorem 6.1.3 that f is not differentiable at $x = 1$, either.

3. *(12 points)*

Define $s_1 = 1$ and for $n \geq 1$, let $s_{n+1} = \sqrt{2 + s_n}$.

(a) Prove that $s_n < 2$, $\forall n \in \mathbb{N}$.
We use mathematical induction for this result.
The base case is $n = 1$ and $s_1 = 1 < 2$. Hence the statement holds for the base case $n = 1$. We now assume the statement holds for $n = k$ so that $s_k < 2$. Then adding 2 to both sides, $2 + s_k < 2 + 2 = 4$. We note all s_k are non negative so $2 + x_k \geq 0$. Taking square roots gives $\sqrt{2 + s_k} < \sqrt{4} = 2$. Therefore $s_{k+1} = \sqrt{2 + s_k} < 2$. The statement is thus true for $n = k + 1$, establishing the induction step. It follows that $s_n < 2$ for all $n \in \mathbb{N}$.

(b) It is also possible to show that (s_n) is an increasing sequence. **Assuming this fact**, deduce that (s_n) is a convergent sequence and compute its limit. Be sure to justify your reasoning.
We know that (s_n) is bounded, and we are allowed to assume (s_n) is monotone increasing as a sequence. By the Monotone Convergence Theorem for sequences, we obtain that (s_n) is a convergent sequence. Let $\lim_{n\to\infty} s_n = L$. Since (s_{n+1}) is a subsequence of (s_n), we know $\lim_{n\to\infty} s_{n+1} = L$, too. Therefore,

$$L = \lim_{n\to\infty} s_{n+1} = \lim_{n\to\infty} \sqrt{2 + s_n}$$

$$= \sqrt{2 + \lim_{n\to\infty} s_n} = \sqrt{2 + L}.$$

It follows that the limit L satisfies the equation

$$L = \sqrt{2 + L}.$$

Squaring both sides we get $L^2 = 2 + L$ or

$$:^2 - L - 2 = 0,$$

i.e.

$$(L - 2)(L + 1) = 0; \; L = 2 \text{ or } L = -1.$$

But (s_n) is an increasing sequence of positive numbers so that $L > 0$. It follows that $L = 2$ so that

$$\lim_{n\to\infty} s_n = 2.$$

4. *(13 points)*

(a) Prove that the equation $x + \frac{1}{2} = \cos x$ has a solution in $[0, \frac{\pi}{2}]$.
Let functions $g : [0, \frac{\pi}{2}] \to \mathbb{R}$ and $h : [0, \frac{\pi}{2}] \to \mathbb{R}$ be defined by

$$g(x) = x + \frac{1}{2}, \text{ and } h(x) = \cos x.$$

Note that g and h are continuous on $[0, \frac{\pi}{2}]$, so that $f : [0, \frac{\pi}{2}] \to \mathbb{R}$ defined by $f(x) = g(x) - h(x)$ is also continuous on $[0, \frac{\pi}{2}]$. We note that $f(0) = +\frac{1}{2} - \cos 0 = \frac{1}{2} - 1 = -\frac{1}{2} < 0$, and $f(\frac{\pi}{2}) = \frac{\pi}{2} + \frac{1}{2} - \cos\frac{\pi}{2} = \frac{\pi+1}{2} - 0 = \frac{\pi+1}{2} > 0$. By the Intermediate Value Theorem, there exists $x_0 \in (0, \frac{\pi}{2})$ such that $f(x_0) = 0$, i.e. there exists $x_0 \in (0, \frac{\pi}{2})$ such that

$$x_0 + \frac{1}{2} - \cos x_0 = 0.$$

But this means here exists $x_0 \in (0, \frac{\pi}{2})$ such that

$$x_0 + \frac{1}{2} = \cos x_0,$$

so that the given equation has a solution within the desired interval.

(b) Prove that f and g are uniformly continuous on $D \subset \mathbb{R}$, then the sum function $f + g$ is uniformly continuous on $D \subset \mathbb{R}$.
Fix $\epsilon > 0$. Since f is uniformly continuous on D, there exists $\delta_1 >$ such that whenever $x, y \in D$ and $|x - y| < \delta_1$,

$$|f(x) - f(y)| < \frac{\epsilon}{2}, \quad (1).$$

Similarly, since g is uniformly continuous on D, there exists $\delta_2 >$ such that whenever $x, y \in D$ and $|x - y| < \delta_2$,

$$|g(x) - g(y)| < \frac{\epsilon}{2}, \quad (2).$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then $0 < \delta \leq \delta_1$ and $0 < \delta \leq \delta_2$. It follows that if $x, y \in D$ and $|x - y| < \delta$,

$$|(f+g)(x) - (f+g)(y)| = |f(x) + g(x) - (f(y) + g(y))| = |f(x) + g(x) - f(y) - g(y)|$$

$$= \ |f(x) - f(y) + g(x) - g(y)| \ \leq \ |f(x) - f(y)| + |g(x) - g(y)|$$

(by the Triangle inequality)

$$< \ \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Therefore if $x, y \in D$ and $|x - y| < \delta$,

$$|(f+g)(x) - (f+g)(y)| < \epsilon,$$

so that $f + g$ is uniformly continuous on D.