MATH 3001 2016/12/05

WEEK 15

Review

CARDINALITY

#1. Let S be the set of odd integers (both positive and negative). Show that S is countable.

SOL'N:

• METHOD 1: We know $\mathbb{Z}$ is countable, and $S \subseteq \mathbb{Z}$. So, S is countable.

• METHOD 2: Define a function $f: \mathbb{N} \to S$ by:

$$f(n) = \begin{cases} -\cancel{2}n-1 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd.} \end{cases}$$

We need to show that f is a bijection.

Starting with injectivity:

Let $n, m \in \mathbb{N}$ s.t. $f(n) = f(m)$.

If n is odd, then $f(n) > 0$, and so $f(m) > 0$. Since $-\cancel{2}p-1 < 0$ for all $p \in \mathbb{N}$, we have that m is odd. So, $n = f(n) = f(m) = m$.

If n is even, then $f(n) = -n-1 < 0$. Since $f(m) = f(n)$, $f(m) < 0$, and so m is even, and $-n-1 = -m-1 \Rightarrow n = m$.

For surjectivity, fix an odd integer p. If $p > 0$, ~~so~~ then $f(p) = p$. If $p < 0$, then $f(p+1) = -(p+1)-1 = p$. (Note that since p is odd, $p-1 \in \mathbb{N}$.)

$\square$

②

COMPLETENESS AXIOM

*) 2. Let $k \geq 0$, and let $S \subseteq \mathbb{R}$ (bounded below). Show that $\inf(kS) = k \inf S$.

SOL'N:

- $kS := \{x \in \mathbb{R} \mid \frac{x}{k} \in S\}$. (assume $k > 0$).
- $\inf kS$ is the greatest lower bound of kS.
- Since S is bounded below, $\inf S$ exists.
- $\forall x \in S$, we have $\inf S \leq x$.
- $\forall x \in S$, we have $k \inf S \leq kx$.
- $\forall x \in kS$, we have $k \inf S \leq x$.
- Thus kS is bounded below by $k \inf S$, and so $\inf kS$ exists, with $k \inf S \leq \inf kS$.
- If $k \inf S < \inf kS$, then $\exists y$ s.t. $k \inf S < y \leq \inf kS$.
- Then, $\inf S < \frac{y}{k}$.
- There exists $x \in S$ s.t. $\inf S \leq x < \frac{y}{k}$, and so $\frac{y}{k}$ is not a lower bound for S.
- So, y is not a lower bound of kS.
- This contradicts $y \leq \inf kS$.
- If $k = 0$, then $kS = \{0\}$, and $\inf(kS) = k \inf S$ is immediate. $\square$

TOPOLOGY OF $\mathbb{R}$

#3. (a) Let $S \subseteq \mathbb{R}$. Is it true that $\overline{\text{int}(\bar{S})} = \bar{S}$?
If so, prove it; otherwise, give a counterexample.

SOL'N:

This is false. Consider $S = \{\frac{1}{n} \mid n \in \mathbb{N}\}$.
Then $\bar{S} = S \cup \{0\}$. But $\text{int}(\bar{S}) = \emptyset$, and so
$\overline{\text{int}(\bar{S})} = \emptyset \neq S \cup \{0\} = \bar{S}$.

(b) Is it true that $\text{int}(\overline{\text{int}(\bar{S})}) = \text{int}(\bar{S})$?

SOL'N:

This is true. Let K be a closed set (in place of $\bar{S}$). ~~Let $x \in \text{int}(\bar{S})$. Then~~
If $\text{int} K = \emptyset$, then $\text{int}(\cancel{\text{int}\,\emptyset}\ \overline{\emptyset}) = \emptyset$, and we are done.

If $\emptyset \neq \text{int} K$, then fix $x \in \text{int} K$. Then, $\exists$ a nbhd U of x s.t. $U \subseteq \text{int} K \subseteq \overline{\text{int} K}$.
So, $\exists$ a nbhd U of x contained in $\overline{\text{int} K}$, so $x \in \text{int}(\overline{\text{int} K})$.

On the other hand, if $x \in \text{int}(\overline{\text{int} K})$, then $\exists$ nbhd U of x in $\overline{\text{int} K}$. So, every point of U is in $\text{int} K$ or in $\text{bdry}(\text{int} K)$.

~~If $x \in \text{int} K$, we are done. If $x \in \text{bdry}(\text{int} K)$, then $\forall$ nbhd V of x, $V \cap \text{int} K \neq \emptyset$ and $(V \cap (\mathbb{R} \setminus \text{int} K) \neq \emptyset$.~~

~~Let $V = U$~~ Then, every point of U is an accumulation pt of $\text{int} K$ (since $\text{int} K$ has no isolated pt's).
In particular, $U \subseteq K$. Hence, $x \in \text{int} K$. $\square$

SEQUENCES

#4. Consider the sequence $\left(\frac{x^n}{n!}\right)$ where $x \in \mathbb{R}$ is fixed, $n > 0$.

(a) Is the sequence monotone for any x? Which ones?

(b) Does the sequence converge? If so, to what?

HINT: initially assume that $x \in \mathbb{N}$.

SOL'N:

(a). If $x < 0$, then the sequence alternates, and so is not monotone. If $x = 0$, it is constant, and so monotone. If $x = 1$, it is the sequence $\left(\frac{1}{n!}\right)$, which is monotone.

For $x \in (0,1)$, $\quad \left(\frac{x^n}{n!}\right) / \left(\frac{x^{n+1}}{n+1!}\right) = \frac{n+1}{x}$

Since $n+1 > 1$, we have $\frac{n+1}{x} > 1$ for all such x.

Hence, $\frac{x^n}{n!} > \frac{x^{n+1}}{(n+1)!}$ for all such x.

In fact, this proof works $\forall x > 0$ s.t. $\frac{n+1}{x} \geq 1$.

That is $2 \geq x$. Hence, the sequence is monotone for all $x \in [0, 2]$. If $x > 2$, then

$\frac{x}{1!} / \frac{x^2}{2!} = \frac{2}{x} < 1$. However, by A.P., $\exists n \in \mathbb{N}$ s.t.

$n > x$, and so $\frac{n+1}{x} > 1$, and so the sequence is not monotone.

(b). If $x \in \mathbb{N}$, then $\frac{x^n}{n!} = \frac{x^x}{x!}\left(\frac{x}{(x+1)!} \cdots \frac{x}{(n-1)}\right)\frac{x}{n}$ for any $n > x$

$\leq \frac{x^{x+1}}{x!}\frac{1}{n} \to 0$ as $n \to \infty$.

If $x \in \mathbb{R}$, then $\exists k \in \mathbb{N}$ s.t. $k > |x|$, by A.P.

Then, $\left|\frac{|x|^n}{n!} - 0\right| \leq \frac{k^n}{n!} \to 0$, and so $\frac{|x|^n}{n!} \to 0$.

But $\frac{|x|^n}{n!} = \left|\frac{x^n}{n!} - 0\right|$, and so $\frac{x^n}{n!} \to 0$. $\square$

SEQUENCES CONT.

#5. A sequence (s_n) is *contractive* if $\exists k \in (0,1)$ s.t. $\forall n \in \mathbb{N}$, $|s_{n+2} - s_{n+1}| \leq k|s_{n+1} - s_n|$. Show that every such sequence converges.

SOL'N:

It is enough to show that it is Cauchy.

Fix $\varepsilon > 0$.

Note: $|s_{n+2} - s_{n+1}| \leq k|s_{n+1} - s_n| \leq k^2|s_n - s_{n-1}|$, etc.

Hence, $\forall m, n \in \mathbb{N}$, (wolog, $n \geq m$),

$$|s_n - s_m| \leq |s_n - s_{n-1}| + |s_{n-1} - s_{n-2}| + \dots + |s_{m+1} - s_m|$$

by $\triangle$-ineq.

$$\leq k^{n+1-m}|s_{m+1} - s_m| + k^{n-m}|s_{m+1} - s_m| + \dots + k^0|s_{m+1} - s_m|$$

$$= \frac{1 - k^{n+2-m}}{1-k}|s_{m+1} - s_m|$$

$$\leq \frac{1}{1-k}|s_{m+1} - s_m|.$$

~~Since $k \in (0,1)$~~ Need $N \in \mathbb{N}$ s.t. $k^N < \frac{\varepsilon(1-k)}{|s_1 - s_0|} =: A$.

That is, $\left(\frac{1}{k}\right)^N > \frac{1}{A}$. That is, $N > \frac{\ln \frac{1}{A}}{\ln \frac{1}{k}} = \frac{\ln A}{\ln k}$.

This exists by the A.P. Fix such an N.

Then, $|s_n - s_m| \leq \frac{1}{1-k}|s_{m+1} - s_m| \leq \frac{1}{1-k}k^N|s_{m+1-N} - s_{m-N}|$

$\cancel{< \varepsilon, \ \forall n, m}$

$$\leq \frac{1}{1-k}k^N k^{m-N}|s_1 - s_0|$$

$$< k^{m-N}\varepsilon < \varepsilon.$$

actually, this argument can be made much shorter

So, (s_n) is Cauchy. $\square$

CONTINUITY

#6. Show that $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ -x & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$ is continuous at $x = 0$, but discontinuous elsewhere.

SOL'N:

Fix $\varepsilon > 0$. We want $\delta > 0$ s.t. if $|x - 0| < \delta$, then $|f(x) - 0| < \varepsilon$. Note that $|f(x) - 0| = |x|$ for all x. Thus, set $\delta = \varepsilon$. Then if $|x| < \delta$, we have $|f(x) - 0| < \varepsilon$.

Fix $x_0 \neq 0$. Assume $x_0 \in \mathbb{Q}$. Set $\varepsilon = 2|x_0|$. For any $\delta > 0$, $\exists$ irrationals in $(x_0 - \delta, x_0 + \delta)$. Fix such an irrational number y (for a fixed δ) such that $|y + x_0| > 2|x_0|$. Then, $|f(y) - f(x_0)| = |-y - x_0| = |y + x_0| > 2|x_0| = \varepsilon$.

A similar proof works for $x_0 \in \mathbb{R} \setminus \mathbb{Q}$. □

#7. Let $D \subseteq \mathbb{R}$ be compact, and $f: D \to \mathbb{R}$ a cont. injection. Show that $f^{-1}: f(D) \to \mathbb{R}$ is continuous.

HINT: use the fact that if D is closed, then a function $f: D \to \mathbb{R}$ is cont. iff $f^{-1}(K)$ is closed for all closed sets K.

SOL'N: Since D is cpct and f is cont, $f(D)$ is cpct. Thus $f(D)$ is closed. Fix a closed set $K \subset \mathbb{R}$. Then $(f^{-1})^{-1}(K) = f(K \cap D)$. We claim that $K \cap D$ is cpct, from which we conclude $f(K \cap D)$ is cpct, hence closed. Now, $K \cap D$ is closed, and contained in D. Since D is cpct, it is bounded. Thus $K \cap D$ is bd'd. By Heine-Borel, $K \cap D$ is cpct, which proves the claim. □

DIFFERENTIATION & INTEGRATION

#8. Let $F(x) = \int_0^x f$, where $f : [0,b] \to \mathbb{R}$ is cont. Show that $|F(x)| \le b \max\{f(x) \mid x \in [0,b]\}$ for all $x \in [0,b]$, using ⓐ the mean-value theorem, and ⓑ using ~~the~~ ~~order~~ of integration techniques.

SOL'N:

ⓐ. By the fundamental th'm, $F : [0,b] \to \mathbb{R}$ is diff, with $F'(x) = f(x)$.

By the mean-value th'm, $\exists\, c \in [0,b]$ s.t.

$$F(x) - F(0) = f(c)(x-0).$$

Since $F(0) = 0$, we have

$$|F(x)| = |f(c)||x|.$$

Since $|f(c)| \le \max\{f(x) \mid x \in [0,b]\}$ and $|x| \le b$, we have $|F(x)| \le \max\{\text{—''—}\}\, b$. $\square$

ⓑ.

$$\begin{aligned} |F(x)| = \left|\int_0^x f\right| &\le \int_0^x |f| \le \int_0^x |f| + \int_x^b |f| \\ &\le \int_0^b |f| \le \int_0^b \max\{|f(x)| \mid x \in [0,b]\} \\ &= b \max\{|f(x)| \mid x \in [0,b]\}. \end{aligned}$$ $\square$

#9. ⓐ Let $f : [0,1] \to \mathbb{R}$ be cont. Prove (carefully) that $\int_0^1 f(\sqrt{1-x^2})\,dx = \int_0^{\pi/2} f(\cos\theta)\cos\theta\,d\theta$.

SOL'N:

Note that $\sin\theta$ is diff on $[0, \frac{\pi}{2}]$, and its derivative $\cos\theta$ is integrable on $[0, \frac{\pi}{2}]$. Then the change of variable formula guarantees that $\int_0^1 f(\sqrt{1-x^2})\,dx = \int_0^{\pi/2} f(\cos\theta)\cos\theta\,d\theta$

b. Prove (carefully) that if $f: \mathbb{R} \to \mathbb{R}$ is cont, $g: (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ cont.

$$\int_a^b f(\tan\theta)\sec^2\theta\, g(\theta)\,d\theta = \int_c^d f(x)\, g(\arctan x)\,dx$$

where $\tan a = c$ and $\tan b = d$, and $[a,b] \subseteq (-\frac{\pi}{2}, \frac{\pi}{2})$.

sol'n: We want to apply the change-of-variable formula. To do this we need to set $x = \tan\theta$, and show that $\tan\theta$ is diff with derivative $\sec^2\theta$, which is integrable on $[a,b]$.

To get $\cos\theta$ to be diff, we need to show $\lim\limits_{h\to 0} \frac{\cos(h+\theta) - \cos\theta}{h}$ exists, and is equal to $-\sin\theta$.

$$= \lim_{h\to 0} \frac{\cos h\cos\theta - \sin h\sin\theta - \cos\theta}{h}$$

$$= \lim_{h\to 0} \cos\theta \overbrace{\left(\frac{\cos h - 1}{h}\right)}^{0} - \overbrace{\left(\lim_{h\to 0}\frac{\sin h}{h}\right)}^{1}\sin\theta = -\sin\theta.$$

Hence, $\tan\theta$ is diff on $(-\frac{\pi}{2}, \frac{\pi}{2})$, as is $\sec^2\theta$ on $[a,b]$. Thus, $\sec^2\theta$ is integrable on $[a,b]$. We now can apply the change-of-variable formula.

Alternatively, set $\theta = \arctan x$. Since $\tan x$ is diff and injective on $(-\frac{\pi}{2}, \frac{\pi}{2})$, and $\frac{d}{dx}\tan x = \sec^2 x$, which is non-zero on $(-\frac{\pi}{2}, \frac{\pi}{2})$, we have by the inverse function theorem that $\arctan x$ is differentiable. ~~with integrable~~ Its derivative is $\frac{1}{1+x^2}$, which is cont., hence integrable. Thus, we may use the change-of-variables formula in the other direction.

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#10. For which $\alpha \in \mathbb{R}$ does the following integral converge?

$$\int_0^1 x^\alpha \, dx$$

SOL'N

For $\alpha \geq 0$, we have $\int_0^1 x^\alpha dx = \frac{1}{\alpha+1} x^{\alpha+1} \Big|_0^1 = 1 \cdot \frac{1}{\alpha+1} = \frac{1}{\alpha+1}$.

For $\alpha \in (-1,0)$, we have $\int_0^1 x^\alpha dx = \lim_{c \to 0^+} \int_c^1 x^\alpha dx = \lim_{c\to 0^+} \frac{1}{\alpha+1} x^{\alpha+1} \Big|_c^1$

$$= \frac{1}{\alpha+1} \lim_{c \to 0^+} 1 - c^{\alpha+1} = \frac{1}{\alpha+1}$$

For $\alpha = -1$, we have $\int_0^1 x^\alpha dx = \lim_{c \to 0^+} \int_c^1 x^\alpha dx = \lim_{c \to 0^+} \ln x \Big|_c^1$

$$= +\infty.$$

For $\alpha < -1$, we have $\int_0^1 x^\alpha dx = \lim_{c \to 0^+} \int_c^1 x^\alpha dx = \lim_{c \to 0^+} \frac{x^{\alpha+1}}{\alpha+1} \Big|_c^1$

$$= +\infty.$$

Thus, $\int_0^1 x^\alpha dx$ converges for $\alpha \in (-1, \infty)$, and diverges to ∞ otherwise.

#11. Find $\lim_{x \to \infty} x^{1/x}$.

SOL'N:

Assume $\lim_{x \to \infty} x^{1/x}$ exists. Then, $\lim_{x \to \infty} x^{1/x} = e^{\lim_{x \to \infty} \ln x^{1/x}}$

since e^x is cont. But $\lim_{x \to \infty} \ln x^{1/x} = \lim_{x \to \infty} \frac{1}{x} \ln x$.

By L'Hôpital, this is equal to $\lim_{x \to \infty} \frac{1/x}{x} = \lim_{x \to \infty} \frac{1}{x^2} = 0$.

Hence, obtain $\lim_{x \to \infty} x^{1/x} = e^0 = 1$.