UNIVERSITY OF COLORADO AT BOULDER
MATH 3430 002 - MIDTERM 2

October 16, 2015
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Name: Solutions

Read all of the following information before starting the exam:

- **Show all work**, clearly and in order, if you want to get full credit. The graders reserve the right to take off points if they cannot see how you arrived at your answer (even if your final answer is correct). Points will also be removed for answers that are not clear.
- **Calculators, notes, phones, and other aids including all electronic devices, are not permitted for this test.**
- This test has 4 problems, 7 pages, and is worth 40 points. It is your responsibility to make sure that you have all of the pages!
- Good luck!

Problem	Grade
1. (/5)	
2. (/10)	
3. (/10)	
4. (/15)	
Total: (/40)	

Problem 1 (5 points). You are a crew member on the USS *Enterprise* (NCC-1701-D). A spatial anomaly has appeared in front of the *Enterprise*, rendering all power supplies nearly depleted and all crew unconscious except for you and Captain Picard. There is enough power for you to use the sensors briefly, and to fire phasers for several seconds.

You: Sir, sensors indicate that a phaser burst at the correct time-dependent frequency $y(t)$ would be sufficient to destroy the anomaly. The shape of the anomaly indicates that the frequency must satisfy the differential equation

$$\frac{dy}{dt} = y^2 + t$$

with initial condition $y(0) = 0$. However, the computer ran out of power before it could compute what $y(t)$ is.

Picard: An approximate frequency will have to do, perhaps a degree-~~4~~5 polynomial. We will only have enough power to try this once. Make it so!

Find a ~~4~~5$^{\text{th}}$ degree polynomial that approximates the solution to the initial value problem above.

$y_0 = 0$

$y_1 = 0 + \int_0^t \left(y_0^2 + \tilde{t}\right) d\tilde{t} = \frac{1}{2}t^2$

$y_2 = 0 + \int_0^t \left(\left(\frac{1}{2}\tilde{t}^2\right)^2 + \tilde{t}\right) d\tilde{t} = \frac{1}{20}t^5 + \frac{1}{2}t^2$

Set $y(t) = \frac{1}{20}t^5 + \frac{1}{2}t^2.$

Check: $y(0) = 0.$

$\frac{dy}{dt} = \frac{1}{4}t^4 + t$

$y^2 + t = \underbrace{\frac{1}{400}t^{10} + \frac{1}{20}t^7}_{\text{small.}} + \frac{1}{4}t^4 + t$

Problem 2 (10 points). Given an object with mass equal to $m = 1$ unit, hanging from a spring with spring constant equal to $k = 9$ units, with the system immersed in a fluid with drag (resistance) coefficient $c = 6$, what is the position $y(t)$ of the object if the spring starts at $y(0) = 0$ with velocity $y'(0) = 1$ (with the y-axis pointing downward)? Note that there is **no** applied force.

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 0$$

$$r^2 + 6r + 9 = 0$$

$$\Rightarrow r = \frac{-6 \pm \sqrt{36-36}}{2} = -3$$

$$y(t) = c_1 e^{-3t} + c_2 t e^{-3t}$$

$$y(0) = 0 = c_1$$

$$y'(t) = -3\cancel{c_1 e^{-3t}} + c_2 e^{-3t} + 3c_2 t e^{-3t}$$

$$y'(0) = c_2 = 1$$

$$\text{So, } y(t) = t e^{-3t}$$

Problem 3 (10 points).

/5 (a) Find the general solution to the following homogeneous differential equation.

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 4y = 0.$$

$$r^2 + 3r + 4 = 0 \rightarrow r = \frac{-3 \pm \sqrt{9-16}}{2} = -\frac{3}{2} \pm i\frac{\sqrt{7}}{2}$$

$$\text{So, } y(t) = e^{-3t/2}\left(c_1 \cos\left(\tfrac{\sqrt{7}}{2}t\right) + c_2 \sin\left(\tfrac{\sqrt{7}}{2}t\right)\right)$$

/5 (b) Find the general solution to the following non-homogeneous differential equation.

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 4y = e^{-3t/2}.$$

$$W(t) = \det\begin{bmatrix} e^{-3t/2}\cos\left(\frac{\sqrt{7}}{2}t\right) & e^{-3t/2}\sin\left(\frac{\sqrt{7}}{2}t\right) \\ -\frac{3}{2}e^{-3t/2}\cos\left(\frac{\sqrt{7}}{2}t\right) - \frac{\sqrt{7}}{2}e^{-3t/2}\sin\left(\frac{\sqrt{7}}{2}t\right) & -\frac{3}{2}e^{-3t/2}\sin\left(\frac{\sqrt{7}}{2}t\right) + \frac{\sqrt{7}}{2}e^{-3t/2}\cos\left(\frac{\sqrt{7}}{2}t\right) \end{bmatrix}$$

$$= e^{-3t}\left(-\tfrac{3}{2}\sin\left(\tfrac{\sqrt{7}}{2}t\right)\cos\left(\tfrac{\sqrt{7}}{2}t\right) + \tfrac{\sqrt{7}}{2}\cos^2\left(\tfrac{\sqrt{7}}{2}t\right)\right)$$
$$- e^{-3t}\left(-\tfrac{3}{2}\sin\left(\tfrac{\sqrt{7}}{2}t\right)\cos\left(\tfrac{\sqrt{7}}{2}t\right) - \tfrac{\sqrt{7}}{2}\sin^2\left(\tfrac{\sqrt{7}}{2}t\right)\right)$$
$$= \frac{\sqrt{7}}{2}e^{-3t}$$

$$u_1(t) = -\int \frac{\cancel{e^{-3t/2}\,e^{-3t/2}}\sin\left(\frac{\sqrt{7}}{2}t\right)dt}{\frac{\sqrt{7}}{2}\cancel{e^{-3t}}} = \frac{2}{\sqrt{7}}\frac{2}{\sqrt{7}}\cos\left(\frac{\sqrt{7}}{2}t\right)$$

$$u_2(t) = \int \frac{\cancel{e^{-3t/2}\,e^{-3t/2}}\cos\left(\frac{\sqrt{7}}{2}t\right)dt}{\frac{\sqrt{7}}{2}\cancel{e^{-3t}}} = \frac{2}{\sqrt{7}}\frac{2}{\sqrt{7}}\sin\left(\frac{\sqrt{7}}{2}t\right)$$

$$y(t) = e^{-3t/2}\left(c_1\cos\left(\tfrac{\sqrt{7}}{2}t\right) + c_2\sin\left(\tfrac{\sqrt{7}}{2}t\right) + \tfrac{4}{7}\right)$$

4

Problem 4 (15 points).

/5 (a) Given a homogeneous second-order linear ordinary differential equation

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y,$$

with p and q continuous on their domains, if $y_1(t)$ is a specific solution for the differential equation, what is a formula for a second, linearly independent solution $y_2(t)$?

$$y_2(t) = y_1(t)v(t) \quad \text{where} \quad v(t) = \int \frac{1}{y_1^2(t)} e^{-\int p(\tilde{t})\,d\tilde{t}}\,dt$$

/5 (b) From your solution to part (a), prove that $\{y_1, y_2\}$ is a fundamental set of solutions for the differential equation; that is, prove that this is a linearly independent set.

$$W[y_1, y_2](t) = \det \begin{bmatrix} y_1(t) & y_1(t)v(t) \\ y_1'(t) & y_1'(t)v(t) + y_1(t)v'(t) \end{bmatrix}$$

$$= \cancel{y_1(t)y_1'(t)v(t)} + y_1^2(t)v'(t) - \cancel{y_1(t)y_1'(t)v(t)}$$

$v'(t) = \frac{1}{y_1^2(t)} e^{-\int p(t)\,dt}$, and so, $W[y_1, y_2](t) = e^{-\int p(t)\,dt} > 0$ for all t.

Since the Wronskian is nonzero for all t, it follows that $\{y_1, y_2\}$ is a fundamental set of solutions.

/5 (c) Use your solution for part (a) to solve the following differential equation, where $y_1(t) = t$.

$$(1-t^2)\frac{d^2y}{dt^2} + 2t\frac{dy}{dt} - 2y = 0.$$

$$v(t) = \int \frac{1}{t^2} e^{-\int \frac{+2t}{1-t^2}\,dt}\,dt \qquad u = 1-t^2, \quad du = -2t\,dt$$

$$= \int \frac{1}{t^2} e^{+\int \frac{du}{u}}\,dt$$

$$= \int \frac{1}{t^2}(1-t^2)\,dt$$

$$= \int (t^{-2} - 1)\,dt$$

$$= -t^{-1} - t$$

$$y_2(t) = -t\left(\frac{-1}{t} - t\right) = 1 + t^2$$

$y(t) = c_1 t + c_2(1+t^2)$ is the general sol'n.

<u>Check</u>: $y_2'' = 2, \quad y_2' = 2t$

$$(1-t^2)\cdot 2 + 2t(2t) - 2(1+t^2)$$

$$= \cancel{2} - \cancel{2t^2} + \cancel{4t^2} - 2 - 2t^2 = 0 \checkmark$$

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$$\sin^2(x) + \cos^2(x) = 1$$

$$\begin{aligned}
\sin(a \pm b) &= \sin(a)\cos(b) \pm \sin(b)\cos(a) \\
\cos(a \pm b) &= \cos(a)\cos(b) \mp \sin(a)\sin(b) \\
\sin(2x) &= 2\sin(x)\cos(x) \\
\cos(2x) &= \cos^2(x) - \sin^2(x) \\
&= 1 - 2\sin^2(x) \\
&= 2\cos^2(x) - 1
\end{aligned}$$

$$\int u\,dv = uv - \int v\,du$$