MATH 3430
FINAL REVIEW

Final Exam Review

2015/12/09

#1 (a). $t y' + y = \cos t$, $y(1) = 2$ $(t>0)$

SOL'N:

$$y' + \frac{1}{t} y = \frac{\cos t}{t}$$ FOLODE, non-homog.

$$\mu(t) = e^{\int dt/t} = t$$

$$y(t) = t^{-1}\left(\int t \frac{\cos t}{t}\, dt + C\right)$$

$$= t^{-1}(\sin t + C)$$

$$= t^{-1}\sin t + C t^{-1}$$

$$y(1) = 2 = \sin 1 + C \Rightarrow C = 2 - \sin 1$$

$$y(t) = t^{-1}\sin t + (2-\sin 1)t^{-1}$$

DOMAIN: $t \in (0, \infty)$

$$\lim_{t\to 0^-} y(t) = \lim_{t\to 0^-}(2-\sin 1 + \sin t)t^{-1} = \infty$$

$$\lim_{t\to \infty} y(t) = \lim_{t\to\infty}(2-\sin 1 + \sin t)t^{-1} = 0$$

(2)

(b) $y' = (y^2+1)t^2, \quad y(0)=1$

SOL'N:

Separable! $\quad \int \frac{dy}{y^2+1} = \int t^2\,dt$

$\Rightarrow \quad \arctan y = \frac{1}{3}t^3 + C$

$\Rightarrow \quad y = \tan\left(\frac{1}{3}t^3 + C\right)$

$y(0) = 1 = \tan C \Rightarrow C = \frac{\pi}{4}$, so

$y(t) = \tan\left(\frac{1}{3}t^3 + \frac{\pi}{4}\right)$

DOMAIN: From $\tan\theta$, we know $\frac{1}{3}t^3 + \frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (since we need the connected interval in the domain of $\tan\theta$ containing $\frac{1}{3}(0)^3 + \frac{\pi}{4} = \frac{\pi}{4}$.).

So, $\frac{1}{3}t^3 \in \left(-\frac{3\pi}{4}, \frac{\pi}{4}\right)$

$\Rightarrow t^3 \in \left(-\frac{9\pi}{4}, \frac{3\pi}{4}\right)$

$\Rightarrow t \in \left(-\sqrt[3]{\frac{9\pi}{4}}, \sqrt[3]{\frac{3\pi}{4}}\right)$.

$\lim_{t \to -\sqrt[3]{9\pi/4}} y(t) = -\infty, \quad \lim_{t \to +\sqrt[3]{3\pi/4}} = \infty$

①. $2xy + (1+x^2+3y^2)\frac{dy}{dx} = 0, \quad y(0) = 1.$

SOL'N:

$$\frac{\partial}{\partial y}(2xy) = 2x \qquad \frac{\partial}{\partial x}(1+x^2+3y^2) = 2x$$

So, this is an exact ODE.

$$\phi(x) = \int 2xy\,dx = x^2 y + h(y)$$

$$\frac{\partial \phi}{\partial y} = x^2 + h'(y) = 1 + x^2 + 3y^2$$

$$\Rightarrow h'(y) = 1 + 3y^2 \Rightarrow h(y) = y + y^3 + C$$

So, $\phi(x) = x^2 y + y + y^3 = \text{const}$ is the general solution.

#2. $y' = t^2 + y^2, \quad y(0) = 1.$

SOL'N:

The formula for Picard iterates is:

$$y_n = y_0 + \int_{t_0}^{t} f(y_{n-1}(\tilde{t}), \tilde{t})\,d\tilde{t}$$

$$y_1(t) = 1 + \int_0^t (\tilde{t}^2 + 1)\,d\tilde{t}$$

$$= 1 + t + \frac{1}{3}t^3$$

$$y_2(t) = 1 + \int_0^t \left(\tilde{t}^2 + (1 + \tilde{t} + \tfrac{1}{3}\tilde{t}^3)^2\right) d\tilde{t}$$

$$= 1 + \int_0^t \left(1 + 2\tilde{t}^2 + \tfrac{1}{9}\tilde{t}^6 + 2\tilde{t} + \tfrac{2}{3}\tilde{t}^3 + \tfrac{2}{3}\tilde{t}^4\right) d\tilde{t}$$

$$= 1 + t + t^2 + \frac{2}{3}t^3 + \frac{1}{6}t^4 + \frac{2}{15}t^5 + \frac{1}{63}t^7$$

#3. $y' = t + y^2$, $y(0) = 0$, $t \in [0, (\frac{1}{2})^{2/3}]$

SOL'N:

$a = (\frac{1}{2})^{2/3}$

$b = ?$

$$M = \max_{\substack{t \in [0,a] \\ y \in [-b,b]}} |t + y^2| = a + b^2 = (\tfrac{1}{2})^{2/3} + b^2$$

$$\alpha = \min\{a, \tfrac{b}{M}\} = \min\left\{(\tfrac{1}{2})^{2/3}, \frac{b}{(\frac{1}{2})^{2/3} + b^2}\right\}$$

Try $b = (\frac{1}{2})^{1/3}$: $\frac{b}{M} = \frac{(1/2)^{1/3}}{(\frac{1}{2})^{2/3} + (\frac{1}{2})^{2/3}} = \frac{1}{2}(\frac{1}{2})^{-1/3} = (\frac{1}{2})^{2/3}$

This forces $\alpha = (\frac{1}{2})^{2/3}$. So, letting $b = (\frac{1}{2})^{2/3}$, by the existence and uniqueness theorem for FOODEs, the differential equation has a (unique) solution.

#4, ⓐ. $y'' - 4y' + 4y = te^{2t}$, $y(0) = y'(0) = 1$.

$y_H(t) = ?$ $\quad r^2 - 4r + 4 = 0 \Rightarrow (r-2)^2 = 0 \Rightarrow r = 2$
(repeated root)

So, $y_H(t) = c_1 e^{2t} + c_2 t e^{2t}$

$\psi(t) = u_1 e^{2t} + u_2 t e^{2t}$, $W(t) = \det \begin{bmatrix} e^{2t} & te^{2t} \\ 2e^{2t} & e^{2t} + 2te^{2t} \end{bmatrix}$

$= e^{4t} + 2te^{4t} - 2te^{4t} = e^{4t}$

$u_1 = -\int \frac{te^{2t} \cdot te^{2t}}{e^{4t}} dt$

$= -\int t^2 dt = -\frac{1}{3}t^3$

$u_2 = \int \frac{e^{2t} \cdot te^{2t}}{e^{4t}} dt = \frac{1}{2}t^2$

So, $\psi(t) = -\frac{1}{3}t^3 e^{2t} + \frac{1}{2}t^3 e^{2t} = \frac{1}{6}t^3 e^{2t}$.

The general solution is $y(t) = c_1 e^{2t} + c_2 te^{2t} + \frac{1}{6}t^3 e^{2t}$

$y(0) = 1 = c_1$, $\quad y'(t) = 2e^{2t} + c_2 e^{2t} + 2c_2 te^{2t} + \frac{1}{2}t^2 e^{2t}$
$+ \frac{1}{3}t^3 e^{2t}$

$y'(0) = 1 = 2 + c_2 \Rightarrow c_2 = -1$

So, $y(t) = e^{2t} - te^{2t} + \frac{1}{6}t^3 e^{2t}$

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(b) $y'' - 4y' + 4y = te^{2t}$, $y(0) = y'(0) = 1$

SOL'N: $y_H(t) = c_1 e^{2t} + c_2 t e^{2t}$ (from part (a)).

Let $y_2(t)$ be the solution to the (homog) IVP with $y(0) = 0$ and $y'(0) = 1$. Then, a particular solution is given by

$$\psi(t) = (te^{2t}) * y_2(t).$$

So, $y_H(0) = c_1 = 0$ and $y_H'(0) = c_2 = 1$, and so $y_2(t) = te^{2t}$. Thus,

$$\begin{aligned}
\psi(t) &= (te^{2t}) * (te^{2t}) \\
&= \int_0^t (t-u)e^{2t}e^{-2u}\, u e^{2u}\, du \\
&= \int_0^t tu - u^2 e^{2t}\, du = e^{2t}\left(t\frac{u^2}{2} - \frac{u^3}{3}\right)\Bigg|_0^t \\
&= \frac{1}{6} t^3 e^{2t}
\end{aligned}$$

The general solution is thus $y(t) = c_1 e^{2t} + c_2 t e^{2t} + \frac{1}{6} t^3 e^{2t}$.

(The specific solution follows from the work in part (a):)

$$y(t) = e^{2t} - te^{2t} + \frac{1}{6} t^3 e^{2t}$$

#5. $t^2 y'' + t y' - y = 0$

Sol'n: Euler's differential equation

Set $y(t) = t^r$. Then,

$$0 = (r(r-1) + r - 1)t^r = (r^2 - 1)t^r$$

$$\Rightarrow r = \pm 1$$

So, $y(t) = c_1 t + c_2 t^{-1}$ is the general solution.

#6. (a). $\mathcal{L}\{t^3\} = \frac{6}{s^4}$

(b) $\mathcal{L}\{t \sin t\} = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{+2s}{(s^2+1)^2}$

(c) $\mathcal{L}\{t H_2(t)\}$

$= \mathcal{L}\{H_2(t)((t-2)+2)\}$

$= e^{-2s}\left(\frac{1}{s^2} + \frac{2}{s}\right)$

(d). $\mathcal{L}\{\delta(t-3)\} = e^{-3s}$

(e) $\mathcal{L}^{-1}\left\{\frac{-1}{(s-3)^2}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s-3}\right)\right\}$

$= -t e^{3t}$

(f) $\mathcal{L}^{-1}\left\{\frac{2}{(s-3)^2+4}\right\}$

$= e^{3t} \sin(2t)$

(g) $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}$

$= \cos(t) * \sin(t)$

$= \int_0^t \cos(t-u)\sin(u)\,du$

$= \int_0^t (\cos t \cos u + \sin t \sin u) \sin u \,du$

$= \int_0^t \left(\frac{\cos t \sin(2u)}{2} + \frac{\sin t\,(1-\cos(2u))}{2}\right) du$

$= \frac{1}{2}\cos(t)\left(-\frac{1}{2}\cos(2u)\Big|_0^t\right) + \frac{1}{2}\sin(t)\left(u - \frac{1}{2}\sin(2u)\right)\Big|_0^t$

$= -\frac{1}{4}\cos t\,(\cos 2t - 1) + \frac{1}{2}\sin t\left(t - \frac{1}{2}\sin(2t)\right) = \frac{1}{2} t \sin t$

(h) $\mathcal{L}^{-1}\left\{\frac{e^{-t}}{s-5}\right\}$

$= H_1(t)\left(e^{5(t-1)}\right)$

#7. (a). $\dot{X} = \begin{bmatrix} -7 & 2 \\ 4 & -5 \end{bmatrix} X$

SOL'N:

Eigenvalues:

$$0 = \det \begin{bmatrix} -7-\lambda & 2 \\ 4 & -5-\lambda \end{bmatrix} = \lambda^2 + 12\lambda + 35 - 8$$
$$= (\lambda+9)(\lambda+3)$$
$$\Rightarrow \lambda_1 = -9, \ \lambda_2 = -3$$

Eigenvectors: $(\lambda_1 = -9)$

$$\begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 = -x_2$, x_2 arbitrary (set $=1$)

$$X_1(t) = e^{-9t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$(\lambda_2 = -3)$

$$\begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -2x_1 + x_2 = 0$, x_2 arbitrary (set $=1$)

$$X_2(t) = e^{-3t} \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

General solution: $X(t) = c_1 e^{-9t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$

#7 (b). $\dot{X} = \begin{bmatrix} -1 & -2 \\ 2 & 0 \end{bmatrix} X$

SOL'N:

Eigenvalues: $0 = \det \begin{bmatrix} -1-\lambda & -2 \\ 2 & -\lambda \end{bmatrix} = \lambda + \lambda^2 + 4 \Rightarrow \lambda = \dfrac{-1 \pm \sqrt{1-16}}{2}$

$= -\frac{1}{2} \pm \frac{i}{2}\sqrt{15}$

Eigenvectors: $\left(\lambda = -\frac{1}{2} - \frac{i}{2}\sqrt{15}\right)$

$$\begin{bmatrix} -1+\frac{1}{2}+\frac{i}{2}\sqrt{15} & -2 \\ 2 & \frac{1}{2}+\frac{i}{2}\sqrt{15} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1+i\sqrt{15} & -4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-x_1 - x_1(i\sqrt{15}) - 4x_2 = 0$, x_2 arbitrary (set $=1$).

$$X(t) = e^{-t/2}\left(\cos\left(\tfrac{-t}{2}\sqrt{15}\right) + i\sin\left(\tfrac{-t}{2}\sqrt{15}\right)\right)\left(\begin{bmatrix} -1/4 \\ 1 \end{bmatrix} + i\begin{bmatrix} 1/4\sqrt{15} \\ 0 \end{bmatrix}\right)$$

$$\boxed{\frac{4}{-1-i\sqrt{15}} = \frac{4(-1+i\sqrt{15})}{1+15} = -\frac{1}{4} + \frac{i}{4}\sqrt{15}}$$

$$= e^{-t/2}\left(\begin{bmatrix} -\frac{1}{4}\cos\left(\frac{t}{2}\sqrt{15}\right) + \frac{1}{4}\sin\left(\frac{t}{2}\sqrt{15}\right)\sqrt{15} \\ \cos\left(\frac{t}{2}\sqrt{15}\right) \end{bmatrix} + i\begin{bmatrix} \frac{1}{4}\cos\left(\frac{t}{2}\sqrt{15}\right)\sqrt{15} + \frac{1}{4}\sin\left(\frac{t}{2}\sqrt{15}\right) \\ \sin\left(\frac{-t}{2}\sqrt{15}\right) \end{bmatrix}\right)$$

So, the general sol'n is:

$$X(t) = e^{-t/2}\left(c_1\begin{bmatrix} -1/4\cos\left(\frac{t}{2}\sqrt{15}\right) + \frac{1}{4}\sin\left(\frac{t}{2}\sqrt{15}\right)\sqrt{15} \\ \cos\left(\frac{t}{2}\sqrt{15}\right) \end{bmatrix} + c_2\begin{bmatrix} \frac{1}{4}\cos\left(\frac{t}{2}\sqrt{15}\right)\sqrt{15} + \frac{1}{4}\sin\left(\frac{t}{2}\sqrt{15}\right) \\ -\sin\left(\frac{t}{2}\sqrt{15}\right) \end{bmatrix}\right)$$

(10)

#7ⓒ $\dot{X} = \begin{bmatrix} -4 & 2 \\ -2 & 0 \end{bmatrix} X$

SOL'N:

Eigenvalues: $D = \det \begin{bmatrix} -4-\lambda & 2 \\ -2 & -\lambda \end{bmatrix} = \lambda^2 + 4\lambda + 4 = (\lambda+2)^2$

$\Rightarrow \lambda = -2$ (mult. 2)

Eigenvectors: $(\lambda = -2)$

$$\begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -x_1 + x_2 = 0$, x_2 arbitrary (set $= 1$)

$$X_1(t) = e^{-2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Generalised Eigenvectors (Rank 2):

$$\begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_1, x_2$ arbitrary. Choose $x_1 = 1$, $x_2 = 0$.

$$V = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \quad X(t) = e^{-2t}\left(I + t\begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix}\right)\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= e^{-2t}\begin{bmatrix} 1-2t \\ -2t \end{bmatrix}$$

General solution:

$$X(t) = c_1 e^{-2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-2t}\begin{bmatrix} 1-2t \\ -2t \end{bmatrix}$$

#8. $$\dot{X} = \underbrace{\begin{bmatrix} -7 & 2 \\ 4 & -5 \end{bmatrix}}_{A} X + \begin{bmatrix} te^{t} \\ t \end{bmatrix}, \quad X(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

SOL'N:

From #7ⓐ, we have a homog. sol'n:

$$X_H(t) = c_1 e^{-9t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{-3t}\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}.$$

So, a fundamental matrix sol'n is given by

$$\mathcal{X}(t) = \begin{bmatrix} -e^{-9t} & \frac{1}{2}e^{-3t} \\ e^{-9t} & e^{-3t} \end{bmatrix}, \quad \mathcal{X}(0) = \begin{bmatrix} -1 & \frac{1}{2} \\ 1 & 1 \end{bmatrix}$$

$$\mathcal{X}(0)^{-1} = \frac{-2}{3}\begin{bmatrix} 1 & -1/2 \\ -1 & -1 \end{bmatrix}$$

$$e^{At} = \mathcal{X}(t)\,\mathcal{X}(0)^{-1} = -\frac{2}{3}\begin{bmatrix} -e^{-9t} & \frac{1}{2}e^{-3t} \\ e^{-9t} & e^{-3t} \end{bmatrix}\begin{bmatrix} 1 & -1/2 \\ -1 & -1 \end{bmatrix}$$

$$= -\frac{2}{3}\begin{bmatrix} -e^{-9t} - \frac{1}{2}e^{-3t} & \frac{1}{2}e^{-9t} - \frac{1}{2}e^{-3t} \\ e^{-9t} - e^{-3t} & -\frac{1}{2}e^{-9t} - e^{-3t} \end{bmatrix}$$

$$X(t) = e^{At}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{At}\int_0^t e^{-As}\begin{bmatrix} se^{s} \\ s \end{bmatrix} ds$$

$$= -\frac{2}{3}\begin{bmatrix} -\frac{1}{2}e^{-9t} - e^{-3t} \\ \frac{1}{2}e^{-9t} - 2e^{-3t} \end{bmatrix} - \frac{2}{3}e^{At}\int_0^t \begin{bmatrix} -e^{9s} - \frac{1}{2}e^{3s} & \frac{1}{2}e^{9s} - \frac{1}{2}e^{3s} \\ e^{9s} - e^{3s} & -\frac{1}{2}e^{9s} - e^{3s} \end{bmatrix}\begin{bmatrix} se^{s} \\ s \end{bmatrix} ds$$

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#9 (a) $\dot{X} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} X$

Sol'n:

$$0 = \det \begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & 4-\lambda & 5 \\ 0 & 0 & 6-\lambda \end{bmatrix} = (1-\lambda)(4-\lambda)(6-\lambda)$$

$$\Rightarrow \lambda_1 = 1,\ \lambda_2 = 4,\ \lambda_3 = 6$$

So, all solutions are unstable since the eigenvalues are positive.

(b) $\dot{X} = \begin{bmatrix} 20x_1 - 2x_1^2 - 4x_1x_2 \\ 40x_2 - 4x_2^2 - 80x_1x_2 \end{bmatrix}$

Sol'n:

Equilibrium sol's: $\begin{bmatrix} 20x_1 - 2x_1^2 - 4x_1x_2 \\ 40x_2 - 4x_2^2 - 80x_1x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{cases} x_1(10 - x_1 - 4x_2) = 0 \\ x_2(10 - x_2 - 20x_1) = 0 \end{cases}$$

$\Rightarrow (x_1 = 0 \text{ and } (x_2 = 0 \text{ or } x_2 = 10))$ or

$(x_1 = 10 - 4x_2 \text{ and } (x_2 = 0 \text{ or } -190 + 79x_2 = 0))$

$$\Rightarrow X(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 10 \end{bmatrix}, \begin{bmatrix} 10 \\ 0 \end{bmatrix}, \begin{bmatrix} 10 - \frac{4 \cdot 190}{79} \\ \frac{190}{79} \end{bmatrix}$$

- $X(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Linearise: $Z = X$.

$$\begin{bmatrix} 20x_1 - 2x_1^2 - 4x_1x_2 \\ 40x_2 - 4x_2^2 - 80x_1x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 20 & 0 \\ 0 & 40 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -2x_1^2 - 4x_1x_2 \\ -4x_2^2 - 80x_1x_2 \end{bmatrix}$$

A has positive eigenvalues $\Rightarrow X(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is unstable.

- $X(t) = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$

Linearise:

$$z = X - \begin{bmatrix} 0 \\ 10 \end{bmatrix}, \quad \dot{z} = \begin{bmatrix} 20z_1 - 2z_1^2 - 4z_1(z_2+10) \\ 40(z_2+10) - 4(z_2+10)^2 - 80z_1(z_2+10) \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} -20 & 0 \\ -80 & -40 \end{bmatrix}}_{A} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} -2z_1^2 - 4z_1z_2 \\ -4z_2^2 - 80z_1z_2 \end{bmatrix}$$

A has two negative eigenvalues $\Rightarrow$ asymptotically stable.

- $X(t) = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$

Linearise:

$$z = X - \begin{bmatrix} 10 \\ 0 \end{bmatrix}, \quad \dot{z} = \begin{bmatrix} 20(z_1+10) - 2(z_1+10)^2 - 4(z_1+10)z_2 \\ 40z_2 - 4z_2^2 - 80(z_1+10)z_2 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} -20 & -40 \\ 0 & 40 \end{bmatrix}}_{A} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} -2z_1^2 - 4z_1z_2 \\ -4z_2^2 - 80z_1z_2 \end{bmatrix}$$

A has one positive eigenvalue $\Rightarrow X(t) = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$ is unstable

The last equilibrium sol'n is left for you!