EXTRA QUESTIONS

Warning: I do not have answers for all of the following questions. In fact, some of them come from my research and I never bothered to think about them.

Exercise 1 (Euclidean Spaces)**.** Using the tools introduced in this course so far, try showing that $\mathbb{R}^m$ is not homeomorphic to $\mathbb{R}^n$ for $m \neq n$.

Exercise 2 (Topology Induced by Curves and Functions)**.** Fix a topological space $(X, \mathcal{T})$. Let $\mathcal{F} := C^0(X, \mathbb{R})$ be the set of all real-valued continuous functions from X to $\mathbb{R}$ (the latter equipped with the Euclidean topology). Let $\mathcal{T}_{\mathcal{F}}$ be the smallest topology on X such that all functions in $\mathcal{F}$ are continuous. Similarly, let $\mathcal{C} := C^0(\mathbb{R}, X)$ be the set of all continuous curves into X; *i.e.* continuous maps from $\mathbb{R}$ (equipped with the Euclidean topology) to X. Let $\mathcal{T}_{\mathcal{C}}$ be the largest topology on X such that all curves in $\mathcal{C}$ are continuous.

(a) Show that $\mathcal{T}_{\mathcal{F}} \subseteq \mathcal{T} \subseteq \mathcal{T}_{\mathcal{C}}$.
(b) Show that if $X = \mathbb{R}^n$ and $\mathcal{T}$ is the Euclidean topology, then the inclusions in part (a) are, in fact, equalities.

Definition: For the purposes of the following exercises, if $(X, \mathcal{T})$ is a topological space in which

$$\mathcal{T} = \mathcal{T}_{\mathcal{C}} = \mathcal{T}_{\mathcal{F}},$$

then we say that $\mathcal{T}$ is *balanced*.

Exercise 3 (Quotients)**.** We continue to use the notation introduced in Exercise 2. Let $\sim$ be an equivalence relation on X. Equip $X/\sim$ with the quotient topology $\widetilde{\mathcal{T}}$, and let $\pi : X \to X/\sim$ be the quotient map. Define

$$\widetilde{\mathcal{C}} := \{\pi \circ c \mid c \in \mathcal{C}\},$$

and

$$\widetilde{\mathcal{F}} := \{f : X/\sim \to \mathbb{R} \mid f \circ \pi \in \mathcal{F}\}.$$

(a) Show that the curves in $\widetilde{\mathcal{C}}$ are continuous with respect to the quotient topology.
(b) Assume that X is balanced. If $\widetilde{\mathcal{T}}_{\widetilde{\mathcal{C}}}$ is the largest topology in which the curves in $\widetilde{\mathcal{C}}$ are continuous, is $\widetilde{\mathcal{T}} = \widetilde{\mathcal{T}}_{\widetilde{\mathcal{C}}}$? If so, give a proof. If not, give a counterexample. Also, if not, under what conditions are they equal?
(c) Show that the functions in $\widetilde{\mathcal{F}}$ are continuous with respect to the quotient topology.
(d) Assume that X is balanced. If $\widetilde{\mathcal{T}}_{\widetilde{\mathcal{F}}}$ is the smallest topology in which the functions in $\widetilde{\mathcal{F}}$ are continuous, is $\widetilde{\mathcal{T}} = \widetilde{\mathcal{T}}_{\widetilde{\mathcal{F}}}$? If so, give a proof. If not, give a counterexample. Also, if not, under what conditions are they equal?

Exercise 4 (Subsets)**.** We repeat Exercise 3 with quotients replaced by subsets. Let $Y \subseteq X$. Equip Y with the subspace topology, $\mathcal{T}^Y$, and let $i : Y \to X$ be the inclusion map. Define

$$\mathcal{C}^Y := \{c : \mathbb{R} \to Y \mid i \circ c \in \mathcal{C}\},$$

and

$$\mathcal{F}^Y := \{f \circ i \mid f \in \mathcal{F}\}.$$

(a) Show that the curves in $\mathcal{C}^Y$ are continuous with respect to the subspace topology.

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(b) Assume that X is balanced. If $\mathcal{T}^Y_{\mathcal{C}^Y}$ is the largest topology in which the curves in $\mathcal{C}^Y$ are continuous, is $\mathcal{T}^Y = \mathcal{T}^Y_{\mathcal{C}^Y}$? If so, give a proof. If not, give a counterexample. Also, if not, under what conditions are they equal?

(c) Show that the functions in $\mathcal{F}^Y$ are continuous with respect to the subspace topology.

(d) Assume that X is balanced. If $\mathcal{T}^Y_{\mathcal{F}^Y}$ is the smallest topology in which the functions in $\mathcal{F}^Y$ are continuous, is $\mathcal{T}^Y = \mathcal{T}^Y_{\mathcal{F}^Y}$? If so, give a proof. If not, give a counterexample. Also, if not, under what conditions are they equal?