Homework #03

Math 6230 - Section 001

Due: Wednesday, March 1, 2017.

Instructions. Prove the following statements. **All of your assignments must be typed up using LaTeX.** Either way, your solutions must be legible, and the grader must be able to follow the logic. (It may be helpful to write out a rough draft of a proof first, and then make a good copy.) Utter nonsense will receive negative points, and so if you do not know how to prove a problem, do not just make things up and pass it in. Finally, while you are encouraged to work together, each person must pass in their own work. If you copy a solution off of the internet, this is pretty easy to figure out, is considered cheating, and will be treated as such.

1. **Read:** Chapter 4, and Chapter 5 (especially pages 98–120) of the text.
2. **(Immersions and Submersions)**
   (a) Show that there is no immersion $\mathbb{S}^1 \to \mathbb{R}$.
   (b) Consider the function $f\colon \mathbb{R}^3 \to \mathbb{R}^2 : (x, y, z) \mapsto (x^3 z, xy + z)$. At which point(s) is f a submersion, if any? Determine the regular values of f.
3. **(Local Diffeomorphisms)** Show that the following maps are local diffeomorphisms.
   (a) $\mathbb{R} \to \mathbb{S}^1\colon\ : t \mapsto e^{it} = (\cos(t), \sin(t))$,
   (b) $\mathbb{S}^n \to \mathbb{RP}^n$, which factors as the composition given by the inclusion $\mathbb{S}^n \to \mathbb{R}^{n+1}$ and the quotient map $(\mathbb{R}^{n+1} \smallsetminus \{0\}) \to \mathbb{RP}^n$.
4. **(The Kronecker Flow on $\mathbb{T}^2$)** Recall that the 2-torus $\mathbb{T}^2$ is defined as $\mathbb{S}^1 \times \mathbb{S}^1$. Another definition is given by $\mathbb{R}^2/\mathbb{Z}^2$; *i.e.* we put an equivalence relation on $\mathbb{R}^2$ and consider the quotient: $(x_1, y_1) \sim (x_2, y_2)$ if there exist integers m and n such that $x_2 - x_1 = m$ and $y_2 - y_1 = n$.
   (a) Show that the two definitions of $\mathbb{T}^2$ yield diffeomorphic smooth manifolds.
   (b) Fix $\alpha \in \mathbb{R} \smallsetminus \mathbb{Q}$. Let $\gamma\colon \mathbb{R} \to \mathbb{R}^2$ be given by $t \mapsto (t, \alpha t)$. Show that this curve descends to a *dense* immersed submanifold of $\mathbb{T}^2$.
5. **(Open Submanifolds)** Prove that a smooth map between manifolds $U \to M$ is an open submanifold if and only if it is an embedded submanifold of codimension 0. [Here we are abusing terminology, identifying the image of a map with the map itself.]
6. #5-3 from the text.
7. #5-17 from the text.