Homework #04

Math 6230 - Section 001

Due: Wednesday, March 15, 2017.

Instructions. Prove the following statements. **All of your assignments must be typed up using LaTeX.** Either way, your solutions must be legible, and the grader must be able to follow the logic. (It may be helpful to write out a rough draft of a proof first, and then make a good copy.) Utter nonsense will receive negative points, and so if you do not know how to prove a problem, do not just make things up and pass it in. Finally, while you are encouraged to work together, each person must pass in their own work. If you copy a solution off of the internet, this is pretty easy to figure out, is considered cheating, and will be treated as such.

1. **Read:** Chapter 6 (especially pages 125–136, the tubular neighbourhood theorem part 137–141, and 143–147), Chapter 7 (especially pages 150–154, 156–168), and pages 544–547 (the quotient manifold theorem) of the text. In particular, be familiar with the statements of the propositions and theorems, as well as the terminology, of these two chapters. Finally, read Chapter 8.
2. **(Endomorphisms of Compact Manifolds)** Let M be a compact smooth manifold, and let $F\colon M \to M$ be smooth. Prove that there exists $x \in M$ such that $F^{-1}(x)$ is finite.
3. Problem #6-13 from the text.
4. **(Conjugation for Lie Groups)** Prove that the conjugation map on $\mathrm{GL}(n;\mathbb{R})$ is smooth and compute its pushforward map. Recall that conjugation is defined as the map $\mathrm{GL}(n;\mathbb{R}) \times \mathrm{GL}(n;\mathbb{R}) \to \mathrm{GL}(n;\mathbb{R})$ sending (g,h) to ghg^{-1}.
5. **(Orbit-Stabiliser Theorem)** Let G be a compact Lie group acting smoothly on a manifold M. Fix $x \in M$, and let H be the stabiliser of the action at x. Prove that G/H is a smooth manifold diffeomorphic to the orbit $G \cdot x$. Conclude that $G \cdot x$ is a properly embedded submanifold of M.
6. **(Parallelisable Manifolds)** Let M be a parallelisable manifold of dimension m. Prove that TM is diffeomorphic to $M \times \mathbb{R}^m$.
7. **(F-Related Vector Fields)** Let $F\colon \mathbb{R} \to \mathbb{R}^2$ be the map $t \mapsto (\cos t, \sin t)$. Prove that $\frac{d}{dt} \in \mathrm{vect}(\mathbb{R})$ is F-related to $x\partial_y - y\partial_x \in \mathrm{vect}(\mathbb{R}^2)$. (Exercise 8.18 of the text.)
8. **(Lie Algebra of $\mathrm{U}(n)$)** Prove that $\mathfrak{u}(n)$, the Lie algebra of the unitary group $\mathrm{U}(n)$, is given by
$$\mathfrak{u}(n) := \{A \in \mathrm{Mat}_n(\mathbb{C}) \mid A + A^* = 0\}.$$