Homework #05

Math 6230 - Section 001

Due: Wednesday, April 5, 2017.

Instructions. Prove the following statements. **All of your assignments must be typed up using LaTeX.** Either way, your solutions must be legible, and the grader must be able to follow the logic. (It may be helpful to write out a rough draft of a proof first, and then make a good copy.) Utter nonsense will receive negative points, and so if you do not know how to prove a problem, do not just make things up and pass it in. Finally, while you are encouraged to work together, each person must pass in their own work. If you copy a solution off of the internet, this is pretty easy to figure out, is considered cheating, and will be treated as such.

1. **Read:** - Chapter 9 (especially pages 205–217 and 227–231) and Appendix D (especially pages 663–671), and the Lie groupoid notes found on the course website.
2. **(Global Derivations Versus Vector Fields)** Fix a manifold M, possibly with boundary. A **global derivation** on M is a linear map $X\colon C^\infty(M) \to C^\infty(M)$ that satisfies Leibniz' Rule. A **vector field** on M is a global derivation that admits a local flow at every point. Consider the manifold-with-boundary $[0,\infty)$. Find a global derivation on $[0,\infty)$ that is *not* a vector field.
3. **(The Exponential Map of a Lie Group)** Let G be a Lie group. Define the **exponential map** exp as the map sending $\xi \in \mathfrak{g} \cong T_{1_G}G$ to the (maximal) integral curve of the left-invariant vector field associated to ξ, through 1_G, denoted $t \mapsto \exp(t\xi)$.
   (a) Let $G = \mathbb{S}^1$. Find $\mathfrak{g}$ (remember to say what the Lie bracket is), and describe the corresponding exponential map.
   (b) Is $\exp(t(\xi+\zeta)) = \exp(t\xi)\exp(t\zeta)$ for any Lie group G? If so, prove it. If not, give a counterexample (justify your counterexample).
4. **(Lie Groupoids)** Let $\mathcal{G}$ be a Lie groupoid.
   (a) Show that the domain of the multiplication map m is a smooth manifold. (**Hint:** transversality!)
   (b) Show that the unit map $u\colon \mathcal{G}_0 \to \mathcal{G}_1$ is an embedding.
   (c) Let $\mathcal{H}$ be another Lie groupoid that is Morita equivalent to $\mathcal{G}$, and let $\pi_{\mathcal{G}}$ and $\pi_{\mathcal{H}}$ be the quotient maps to the orbit spaces of $\mathcal{G}$ and $\mathcal{H}$, respectively. Show that there exists a homeomorphism $\Psi\colon \mathcal{G}_0/\mathcal{G}_1 \to \mathcal{H}_0/\mathcal{H}_1$, and for any $x \in \mathcal{G}_0/\mathcal{G}_1$, $y \in \pi_{\mathcal{G}}^{-1}(x)$, and $z \in \pi_{\mathcal{H}}^{-1}(\Psi(x))$, the stabiliser of $\mathcal{G}$ at y is isomorphic as a group to the stabiliser of $\mathcal{H}$ at z.
5. **(Alternating and Symmetric Tensors)** Let V be a vector space.
   (a) Consider the subspace $V \odot V \subseteq V \otimes V$ of all symmetric 2-tensors. Let $\sim_S$ be the equivalence relation on $V \otimes V$ generated linearly by $v \otimes w \sim_S w \otimes v$. Prove that there is a natural linear isomorphism between $(V \otimes V)/\sim_S$ and $V \odot V$.

(b) Consider the subspace $V \wedge V \subseteq V \otimes V$ of all alternating 2-tensors. Let $\sim_A$ be the equivalence relation on $V \otimes V$ generated linearly by $v \otimes w \sim_A -w \otimes v$. Prove that there is a natural linear isomorphism between $(V \otimes V)/\sim_A$ and $V \wedge V$.