K-Theory and Spectra

The Grothendieck functor

old problem: •) $\mathbb{N}$ is abelian monoid, but does not have negatives

Grothendieck in 50's faced similar problems when studying intersection theory of algebraic varieties:

•) the isomorphism classes of locally free / coherent sheaves on alg. variety X form an abelian monoid (you can add and have zero element) but there are no negatives i.e. additive inverses.

Other abelian monoids with lack of inverses (related to Grothendieck's iso classes of loc. free sheaves)

•) isomorphism classes of vector bundles over compact topological space (Atiyah - Hirzebruch)

•) isomorphism classes of ~~C^*-algebras (unital)~~ finitely generated projective modules over unital C^*-algebra A

•) iso classes of finitely generated projective modules over unital ring R

Assume to be given abelian monoid $(\Pi, +, 0)$ (works for abelian semigroup as well)

define $K^{Gro}(\Pi)$ as follows:

$$K^{Gro}(\Pi) = \Pi \times \Pi / \sim$$

with $(a_1, a_2) \sim (b_1, b_2)$ iff there exists $z \in \Pi$ s.t.

$$a_1 + b_2 + z = b_1 + a_2 + z$$

relation $\sim$ is equivalence relation:

symmetric ✓ reflexive ✓

transitivity: let $(a_1, a_2) \sim (b_1, b_2)$
$(b_1, b_2) \sim (c_1, c_2)$

hence there are $z, w \in \Pi$ s.t.

$$a_1 + b_2 + z = b_1 + a_2 + z \quad \& \quad b_1 + c_2 + w = c_1 + b_2 + w$$

Then

$$a_1 + c_2 + (b_2 + z + w) = b_1 + a_2 + c_2 + (z + w) =$$

$$= c_1 + b_2 + a_2 + (z + w) = c_1 + a_2 + (b_2 + z + w).$$

Denote by $[a_1, a_2]$ equivalence class of (a_1, a_2).

observations: ∘) $[a_1, a_2] + [b_1, b_2] := [a_1 + b_1, a_2 + b_2]$
is well defined, associative and commutative

∘) $[a, a]$ is zero element in $K^{Gro}(\Pi)$,
hence $[a, a] = [b, b]$ for all $a, b \in \Pi$

∘) $[b, a]$ is negative of $[a, b]$.

Thm & Def i) $K^{Gro}(M)$ is an abelian group called the „Grothendieck group" of abelian monoid M (or abelian semigroup M). The mapping

$$h: M \longrightarrow K^{Gro}(M), \quad m \mapsto [m+c, c]$$

is independent of $c \in M$ and called the Grothendieck map, it is additive.

ii) (functoriality) If $f: M \to N$ is morphism of abelian monoids, then there is group hom $K^{Gro}(f): K^{Gro}(M) \to K^{Gro}(N)$ s.t.

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "h"'] & N \arrow[d, "h"] \\ K^{Gro}(M) \arrow[r] & K^{Gro}(N) \end{tikzcd}$$

commutes

iii) The Grothendieck map $h: M \to K^{Gro}(M)$ is injective iff M has cancellation property.

iv) ~~K^0~~ $h(a) = h(b)$ iff there exists $c \in M$ s.t. $a + c = b + c$

v) K^{Gro} is left adjoint to forgetful functor AbGrp $\to$ AMon

v) $K^{Gro}(M) = \{ h(a) - h(b) \mid a, b \in M \}$

vi) The following universal property holds true: For every abelian group A with monoid hom $f: M \to A$ there is unique group hom $\bar{f}: K^{Gro}(M) \to A$ s.t.

$$\begin{tikzcd} M \arrow[r] \arrow[d] & A \\ K^{Gro}(M) \arrow[ur] & \end{tikzcd}$$

commutes

Examples: → $K^{Gr}(\mathbb{N}, +) = (\mathbb{Z}, +)$

→ $K^{Gr}(\mathbb{Z}, \cdot) = \{0\}$, but

$K^{Gr}(\mathbb{Z}^{\neq}, \cdot) = (\mathbb{Q}^{\times}, \cdot)$

→ Topological K-theory of spaces: X compact top. sp.

$K^0(X) := K^{Gr}(\mathrm{Iso}(\mathrm{Vec}_{\mathbb{C}}(X)))$ (Atiyah–Hirzebruch)

→ algebraic K_0 of unital rings:

$K_0(R) := K^{Gr}(\mathrm{Iso}(R\text{-}\mathrm{Mod}_{f_p}))$

Serre–Swan–Thm: Let X be compact top. space

$\Gamma : \mathrm{Vec}_{\mathbb{C}}(X) \longrightarrow C(X)\text{-}\mathrm{Mod}_{f_p}, \ (E \to X) \mapsto \Gamma(X, E)$

space of conts. section

is equivalence of categories.

Remark: works for X C^∞-mfd's and C^∞-vectorbundles as well

Corollary: $K^0(X) \cong K_0(\mathcal{C}(X))$ for compact top. space

Prop. $K_0(\mathcal{C}(X)) \cong K_0(\mathcal{C}^\infty(X))$ for X cpct manifold

(Unpublished Thm by Grauert / Radz, later proved in more generality by Schweitzer et al.)

$\mathrm{Vect}_{\mathbb{C}}(*) \cong \mathbb{N}$, hence $K^0(*) \cong \mathbb{Z}$

$\mathrm{Vect}_{\mathbb{C}}(\coprod X_a) = \cancel{\bigoplus} \prod \mathrm{Vect}_{\mathbb{C}}(X_a)$, X_a compact, finitely many

$K^0(\coprod X_a) = \prod K^0(X_a)$

Reduced & higher K-theory

For X locally compact top. Hausdorff space, let X^+ be its Alexandroff extension (i.e. $X^+ \setminus X = \{\infty\}$)

Define $K^0(X) = \ker\left(i^* : K(X^+) \to K(\infty)\right)$, where $i : \{\infty\} \hookrightarrow X^+$ is canonical embedding. Note that for X compact, $X^+ = X \sqcup \{\infty\}$, so this definition is compatible when X is compact.

If X_* is (locally) compact top. space with basepoint $*$, define reduced K-theory by

$$\widetilde{K}^0(X_*) = \ker\left(i_*^* : K^0(X_*) \longrightarrow K(\{*\})\right)$$

Observe $K^0(X) = \widetilde{K}^0(X^+)$

Negative top. K-theory:

$$\widetilde{K}^{-n}(X_*) = \widetilde{K}^0(\Sigma^n(X_*))$$

$$K^{-n}(X) = \widetilde{K}^0(\Sigma^n(X^+))$$

$$K^{-n}(X, Y) = \widetilde{K}^0(\Sigma^n(X/Y))$$

$Y \subset X$ (locally) compact subspace.

Properties: K^{-n}, $\widetilde{K}^{-n}$ are functorial (contravariantly), since Σ and K^0 are functorial (contravariant)

$f : X \to Y$ continuous, pull-back of vector bundles:

$$f^* : \mathrm{Vect}_{\mathbb{C}}(Y) \to \mathrm{Vect}_{\mathbb{C}}(X)$$

$$f^* : K^0(Y) \to K^0(X)$$

homotopy invariance:

If $H: X \times I \to Z$ is a homotopy then $H_0^* E \cong H_1^* E$ for each vector bundle $E \to X$.

Pf: application of Tietze's extension theorem

LES holds (see Atiyah)

Excision holds as well (see Atiyah as well)

problem: dimension axiom does NOT hold

true, will be replaced by Bott periodicity !?

Remark: $\to$ $\tilde{K}^0(X_+) \oplus \mathbb{Z} \cong K^0(X_+)$

$\to$ every element of $K^0(X)$ can be written as

$[E] - [\mathbb{C}^n]$ for some $E \to X$ and $n \in \mathbb{N}$.

Representability of top. K-Theory

Atiyah - Jänich:

Let H be be infinite dim. separable Hilbert-space (complete inner product space)
Let $\mathcal{F} \subset \mathcal{L}(H)$ be the space of Fredholm operators i.e. of bounded lin. op. with finite dimensional kernel and cokernel
(ind A = dim ker A - dim coker A, $A \in \mathcal{F}$)

Then the following holds true for each compact top. space X:

$$K^0(X) \cong [X, \mathcal{F}]$$

It is given by the index map:

$$\text{ind}: [X, \mathcal{F}] \to K^0(X)$$

$$[F] \mapsto [\ker F] - [\operatorname{coker} F]$$

For a one-point space $\{p\}$,

$$\text{ind}: [\{p\}, \mathcal{F}] \to K^0(\{p\}) = \mathbb{Z}$$

is the real index of an Fredholm operator.
The index map is homotopy invariant?

Group structure on $[X, \mathcal{F}]$;

given by composition of Fredholm operators

rewrite $\mathcal{F}$:

$$[X, \mathcal{F}] = [X, (\mathcal{L}(H)/\mathcal{K})^{\times}]$$

↑

invertible operators in bounded linear operators modulo compact ones

Atiyah - Singer - Index (see back side)

Lemma (Atiyah) $\quad Vect_n(SX) \cong [X, GL(n, \mathbb{C})]$

Indicates that we have underlying spectrum. Actually we will get that from Brown representation theorem (existence). Need extension to negative:

Bott - periodicity

look at tautological ^line^ bundle on $\mathbb{C}P^1 \cong S^2$

$H \to S^2 \cong \mathbb{C}P^1$,

$$H^{*} = \{(\ell, v) \in \mathbb{C}P^1 \times \mathbb{C}^2 \mid v \in \ell\} \longrightarrow \mathbb{C}P^1$$

For example by using transition functions

(recall $[S^1, GL(n,\mathbb{C})] \cong \mathrm{Iso}(\mathrm{Vect}^n_{\mathbb{C}} S^2)$)

one verifies that

$$H^2 \oplus 1 \cong H \oplus H$$

Bott - periodicity:

$$\tilde{K}^0(X_+) \longrightarrow \tilde{K}^0(\Sigma^2 X_+)$$

$$[E] \longmapsto ([H]-[1]) \boxtimes [E]$$

is an isomorphism for all compact top. spaces X.
$[H]-[1]$ is called the Bott class.

Computation of $\tilde{K}^0(S^2)$:

As a consequence of Bott periodicity:

$$\tilde{K}^0(S^2) \cong \mathbb{Z}.$$

$K^0(S^1) = K(\mathbb{N})$
since
$\mathrm{Vect}^{\mathbb{C}}_n(S^1) = \pi_0(U(n))$
$= \{*\}$
hence
$\mathrm{Vect}^{\mathbb{C}}(S^1) \cong \mathbb{N}$
(dimension)

$\tilde{K}^0(S^2)$ is generated by $[H]-[1]$
(note that $[H]-[1]$ is in kernel $i^*: K^0(S^2) \to K(\mathrm{pt})$)

Since $H^2 \oplus 1 \cong H \oplus H = 2H$, one has $([H]-[1])^2 = 0$
Therefore $\tilde{K}^0(S^2) \cong \mathbb{Z}[t]/t^2$, actually this is the full unreduced K-theory ring $K^\bullet(S^2) \cong \mathbb{Z}[t]/t^2$

By Bott periodicity $K^{-n-2}(X) = K^{-n}(X)$.
Therefore define $K^n(X)$ for $n \geq 0$ by continuing 2-periodicity.

6-term exact sequence

$$\begin{tikzcd}
K^0(X,Y) \arrow[r, "j^*"] & K^0(X) \arrow[r, "i^*"] & K^0(Y) \arrow[d, "\delta"] \\
K^1(Y) \arrow[u, "\delta"] & K^1(X) \arrow[l, "i^*"'] & K^1(X,Y) \arrow[l, "j^*"']
\end{tikzcd}$$

<u>Classifying spaces and K-theory spectrum</u>

$$Vect_n^{\mathbb{C}}(X) = [X, Gr_n(\mathbb{C}^\infty)] = [X, BU_n]$$

$Gr_n(\mathbb{C}^\infty) = BU_n$ (CW-complex)

$BU = \mathrm{colim}\, BU_n$ $\qquad$ $U = \mathrm{colim}\, U_n$ (unitary groups)

(have to write down maps)

Spectrum of reduced K-theory:

gluing construction along equator

$$\tilde{K}^0(X_+) \underset{\text{Bott-periodicity}}{=} \tilde{K}^{-2}(X_+) = \tilde{K}^0(\Sigma^2 X_+) = [\Sigma X_+, U]$$

$$= [X_+, \Omega U]$$

Then:

$$\tilde{K}^{-n}(X_+) = [X_+, \Omega^{n+1} U]$$

is a generalized cohomology theory

Bott-periodicity in homotopy theoretic form:

$G \to \Omega BG$ weak homotopy equivalence likewise $\Omega U \to BU \times \mathbb{Z}$

$\Omega U \simeq BU \times \mathbb{Z}$ and since $G \simeq \Omega BG$ for every topological group G

$\Omega^2 U \simeq U$

Then $\pi_{2n+1}(U) = \mathbb{Z}$

$\pi_{2n}(U) = 0$

K-groups of spheres:

$K^0(S^{2n}) = \mathbb{Z} \oplus \mathbb{Z}$ $\quad K^1(S^{2n}) = 0$

and $K^0(S^{2n+1}) = \mathbb{Z}$ $\quad K^1(S^{2n+1}) = \mathbb{Z}$

complex K-theory spectrum (unreduced)

KU: $BU \times \mathbb{Z}$, U, $BU \times \mathbb{Z}$, U ... (2-periodic)

Language:

pre-spectrum: sequence $\{E_n\}_{n \geq 0}$ of pointed top. spaces together with pointed maps $\sigma_n : \Sigma E_n \longrightarrow E_{n+1}$

(adjoints: $\tilde{\sigma}_n : E_n \longrightarrow \Omega E_{n+1}$ need not be weak homotopy equivalences)

Ω-pre spectrum:

pre-spectrum whose adjoints

$$\tilde{\sigma}_n : E_n \longrightarrow \Omega E_{n+1}$$

are weak homotopy-equivalences

Def: Spectrum is prespectrum s.t.

$\tilde{\sigma}_n : E_n \to \Omega E_{n+1}$ are homeomorphisms.

(If $\tilde{\sigma}_n : E_n \to \Omega E_{n+1}$ is injective define associated spectrum by $(LE)_n = \mathrm{colim}_k\, \Omega^k (E_{n+k})$)

AlgK(R1)

$\underline{K_0}$: $K_0^{alg}(R) = K^{\mathrm{Groth}}(R\text{-mod}_{fp})$

for unital ring R

$\underline{K_1}$: Look at $GL_\infty(R) = \operatorname{colim}_n GL_n(R)$

$K_1^{alg}(R) = GL_\infty(R)_{ab} = GL_\infty(R)/[GL_\infty(R), GL_\infty(R)]$

important observation:

$[GL_\infty(R), GL_\infty(R)]$ coincides by Whitehead lemma with $E(R)$, subgroup of elementary matrices

$E(R) = \operatorname{colim}_{n\to\infty} E_n(R)$

$E_n(R) \subset GL_n(R)$ generated by elementary matrices e^λ_{ij} — 1's on diagonal

λ in (i,j) slot

0 elsewhere

Whitehead Lemma

$$[GL_\infty(R), GL_\infty(R)] = E(R)$$

$$[E(R), E(R)] = E(R)$$

$E(R)$ is perfect group!

Prop: $K_1^{alg}(R) \cong H_1(GL(R), \mathbb{Z})$

Thm: For commutative ring R,

$$K_1^{alg}(R) \cong R^\times \oplus SK_1^{alg}(R),$$

where $SK_1^{alg}(R) = SL(R)/E(R)$

$$SL(R) = \det^{-1}(1) \subset GL(R).$$

Thm: R field or commutative local ring or commutative euclidean domain, then

$$K_1^{alg}(R) \cong R^\times$$

Ex. $K_1^{alg}(\mathbb{Z}) \cong \mathbb{Z}/2 = \{1, -1\}$, $K_1(\mathbb{F}[t]) \cong \mathbb{F}^\times$, $\mathbb{F}$ field

Alg Th 3)

K_2^{alg}: can be defined with so-called Steinberg ~~symbols~~ groups (cf. Milnor)

Prop: $K_2^{alg}(R) = H_2(E(R); \mathbb{Z})$

Question (from late 60's):

-) How to define higher K^{alg}-groups?

-) beginning of long exact sequence was known (Milnor et al.)

-) idea was around to use group theory & topology, classifying space etc.

-) realize K_n^{alg} as homotopy groups

Observation: $K^{alg}(\cdot)$ is <u>not</u> homotopy invariant

important for index theory to define secondary invariants like η-invariant which is <u>not</u> homotopy invariant?

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Quillen's plus construction

Let X be connected CW-cplx, $P \subset \pi_1(X)$ a perfect subgroup.
Then exists CW-cplx X_P^+, obtained by attaching 2-cells & 3-cells to X, s.t. $i : X \hookrightarrow X_P^+$ has following props:

i) induced hom $i_* : \pi_1(X) \longrightarrow \pi_1(X_P^+)$

is quotient map $\pi_1(X) \longrightarrow \pi_1(X)/P$

(perfect subgroup P is killed in $\pi_1(X)$)

ii) i induces iso $i_* : H_*(X; A) \xrightarrow{\cong} H_*(X_P^+; A)$

for any local system A on X_P^+

iii) X_P^+ is universal in following sense

If Y is CW-cplx and $f : X \to Y$ any map s.t. induced hom

$f_* : \pi_1(X) \longrightarrow \pi_1(Y)$ fulfills $f_*(P) = 0$, then

there is unique $f^+ : X_P^+ \longrightarrow Y$ s.t. $f^+ i = f$.

In part. X_P^+ is unique up to homotopy equivalence