HOLONOMY

JORDAN ALAN WATTS

1. Introduction

This paper is comprised of notes for a talk I gave for the Calabi Conjecture Reading Group. The purpose of this seminar series was to familiarise ourselves with Calabi-Yau manifolds and the Calabi Conjecture. The notion of holonomy was one of many steps to this end. In particular, we wished to understand the following definition of a Calabi-Yau manifold.

Definition 1.1. A smooth manifold M of dimension $2m$ is *Calabi-Yau* if it admits a Kähler metric with holonomy group contained in $\mathrm{SU}(m)$.

2. Some Riemannian Geometry

The main reference for this section is Lee's book on riemannian geometry, [2], and Besse's book on Einstein manifolds, [1].

Fix a smooth manifold M.

Definition 2.1. A *(linear) connection* on a vector bundle $\pi : E \to M$ is a linear map $\nabla : \Gamma(E) \to \Omega^1(M) \otimes \Gamma(E)$ satisfying Leibnitz' rule

$$\nabla(f\sigma) = df \otimes \sigma + f\nabla\sigma$$

for any $f \in C^\infty(U)$, $\sigma \in \Gamma(E|_U)$ and $U \subseteq M$ open. Note that $\nabla(\sigma)$ acts on a vector field X on M to give a section of E. Denote this section $\nabla_X\sigma$. Thus we have a linear map $\nabla_X : \Gamma(E) \to \Gamma(E)$ that obeys Leibnitz' rule.

Now fix a riemannian metric g on M.

Definition 2.2. The *Levi-Civita connection* on (M, g) is a connection ∇ such that

(1) ∇ is *g-compatible*; that is for any vector fields X, Y on M,

$$d(g(X, Y)) = g(\nabla X, Y) + g(X, \nabla Y),$$

(2) ∇ is *torsion-free*; that is for any vector fields X, Y on M,

$$\nabla_X Y - \nabla_Y X - [X, Y] = 0.$$

Let I be an interval in $\mathbb{R}$; that is, a path-connected subset of $\mathbb{R}$. We do not require I to be open or closed.

Definition 2.3. Let $\gamma : I \to M$ be a piecewise smooth curve. A *vector field X along γ* is a smooth map $X : I \to M : t \mapsto X|_t \in T_{\gamma(t)}M$. In other words, it is a smooth section of γ^*TM.

A linear connection ∇ on M determines a unique connection $D_t = \gamma^*\nabla$ on γ^*TM such that $D_tX = \nabla_{\dot\gamma(t)}\tilde{X}$ if an extension $\tilde{X} \in \mathrm{Vect}(M)$ of γ^*X exists.

Fix a linear connection ∇ on M and a piecewise smooth curve γ on M. Let D_t be the induced connection on γ^*TM.

Definition 2.4. A vector field X along γ is *parallel along* γ if $D_tX = 0$.

Theorem 2.5. *Let $t_0 \in I$ and $v \in T_{\gamma(t_0)}M$. Then there exists a unique parallel vector field X along γ such that $X|_{\gamma(t_0)} = v$.*

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Sketch of proof. If $\gamma(I)$ lives in a chart, then using coordinates, solving for X reduces to a linear system of ODE's. The ODE theorem provides us with the existence and uniqueness of X. If $\gamma(I)$ needs more than one chart to cover it, then let β be the supremum of all $b > t_0$ such that there exists a unique parallel transport of v on $[t_0, b]$. Then, there exists a unique parallel transport of v on $[t_0, \beta)$. If $\beta \notin I$, then we are done. But if $\beta \in I$, then apply the above argument in a small chart centred at $\gamma(\beta)$, and uniqueness implies that β was in fact not a supremum, giving us a contradiction. Thus, such a $\beta \notin I$. □

Fix a connected riemannian manifold (M, g) equipped with its Levi-Civita connection ∇. Fix $x \in M$, and let $\gamma : [0, 1] \to M$ be a piecewise smooth path starting at x and ending at $y = \gamma(1)$. Then, parallel transport along γ induces a linear map $P_\gamma : T_xM \to T_yM$ defined by extending $v \in T_xM$ to the (unique) vector field X along γ such that $X|_{\gamma(0)} = v$, and setting $P_\gamma(v) := X|_y$. Note that if γ is a loop (*i.e.* $x = y$), then P_γ is a linear endomorphism of T_xM.

Remark 2.6. Fix a piecewise smooth loop $\gamma : [0, 1] \to M$ starting and ending at x. Let X and Y be the unique vector fields along γ induced by v and w in T_xM, respectively. Then,

$$\begin{aligned} & d(g(X,Y)) = g(D_tX, Y) = g(X, D_tY) = 0 + 0 = 0 \\ \Rightarrow\ & g(P_\gamma(v), P_\gamma(w)) = g(v, w) \\ \Rightarrow\ & P_\gamma \in \mathrm{O}(n) \end{aligned}$$

where $n = \dim M$. Also, it is not hard to show that

- $P_{\gamma^{-1}} = P_\gamma^{-1}$ where $\gamma^{-1}(t) := \gamma(1 - t)$
- $P_{\gamma_1 * \gamma_2} = P_{\gamma_1} \circ P_{\gamma_2}$ for any two loops γ_1 and γ_2.

Definition 2.7. The subgroup of $\mathrm{O}(n)$ of all linear maps P_γ defined over all piecewise smooth loops γ based at x is called the *holonomy group*, denoted $\mathrm{Hol}(x)$. If we restrict to loops homotopic to a point, we obtain the *local holonomy group*, denoted $\mathrm{Hol}^0(x)$.

Choose $y \neq x \in M$, and let $\rho : [0, 1] \to M$ be a path from x to y (recalling that M is connected). Parallel transport along ρ induces a group homomorphism $\mathrm{Hol}(x) \to \mathrm{Hol}(y)$ sending P_γ to $P_{\rho * \gamma * \rho^{-1}}$. This map turns out to be an isomorphism of groups. Thus, up to isomorphism, the holonomy group is independent of the point x chosen, depending only on g. The same result occurs for Hol^0 as well. Thus, we shall henceforth denote these groups $\mathrm{Hol}(g)$ and $\mathrm{Hol}^0(g)$.

Example 2.8. Consider $\mathbb{S}^n \cong \mathrm{SO}(n+1)/\mathrm{SO}(n)$. Let γ be a loop starting and ending at $x \in \mathbb{S}^n$, and let $v \in T_x\mathbb{S}^n$. Viewing $\mathbb{S}^n$ as an orbit of the action $\mathrm{SO}(n+1)$ on $\mathbb{R}^{n+1}$, γ corresponds to a path in $\mathrm{SO}(n+1)$ from the identity e back to itself. $P_\gamma(v)$ corresponds to the linear action of $\mathrm{Stab}(x) \cong \mathrm{SO}(n)$ on T_xM. Since one can find a loop γ that corresponds to any element of $\mathrm{SO}(n)$ in this fashion, the result is that $\mathrm{Hol}(\mathbb{S}^n, g_{std}) \cong \mathrm{SO}(n)$.

Definition 2.9. Let $\alpha \in \bigwedge^k T_x^*M$, and let ρ be a piecewise smooth curve starting at x and ending at y. We can parallel transport α along ρ in the following way:

$$P_\rho^*\alpha(u_1, ..., u_k) = \alpha(P_\rho^{-1}(u_1), ..., P_\rho^{-1}(u_k))$$

for all $u_1, ..., u_k \in T_yM$.

With a little thought, we now know how to parallel transport any tensor (at a point) along a piecewise smooth curve.

Definition 2.10. Now, let A be a tensor field on M. We say that A is *parallel* if for every $x, y \in M$ and every path ρ starting at x and ending at y,

$$P_\rho(A|_x) = A_y.$$

Theorem 2.11 (Fundamental Principle of Holonomy)**.** *Let (M, g) be a connected riemannian manifold. Then the following are equivalent.*

1. *There exists a tensor field A of type (r, s) which is parallel.*
2. *There exists on (M, g) a tensor field A of type (r, s) such that $\nabla A = 0$.*
3. *There exists $x \in M$ and a tensor α at x such that $\mathrm{Hol}(x)$ fixes α under the action of parallel transport.*

Sketch of proof. (1) $\Rightarrow$ (3): This is easy; just use the definition of parallel.
(3) $\Rightarrow$ (1): Define the tensor field A by $A|_y = P_\rho(\alpha)$ for some path ρ from x to y. This is well-defined due to the invariance of α under $\mathrm{Hol}(x)$.
(1) $\Leftrightarrow$ (2): This is a calculation, and will be omitted. □

Remark 2.12. The above principle has a local version.

3. Examples

Example 3.1. Let (M,g) be a connected riemannian manifold of dimension n. Fix $x \in M$ and let $\alpha \in \bigwedge^n(T_x^*M)$ be nonzero. Extend α via parallel transport to an n-form $A \in \Omega^n(M)$. This is well-defined if and only if α is invariant under the holonomy action, by the fundamental principle. But any such linear action must be based on a subgroup of $\mathrm{SO}(n)$ (since we must have $\det B = 1$ for any element B of this subgroup). Thus, (M,g) is **orientable** if and only if $\mathrm{Hol}(g) \leq SO(n)$.

Example 3.2. Let (M,g) be a connected riemannian manifold of dimension $2m$. Let $x \in M$ and let J_x be a complex structure on T_xM such that $g(J_xv, J_xw) = g(v,w)$ for all $v, w \in T_xM$. To extend J_x to all of M as a parallel complex structure J, we require $\mathrm{Hol}(g) \subseteq \mathrm{GL}(m,\mathbb{C}) \cap \mathrm{O}(2m) = U(m)$. This is equivalent to the Hermitian manifold (M,g,J) admitting a (compatible) Kähler structure. Briefly, the torsion-free property of ∇ can be used to show that $\omega(\cdot,\cdot) := g(J\cdot,\cdot)$ is closed and $\nabla J = 0$ if and only if $N_J = 0$, where N_J is the Nijenhuis tensor. See [3]. Thus, (M,g,J) admits a **Kähler** structure if and only if $\mathrm{Hol}(g) \leq U(m)$.

Example 3.3. Let (M,g,J,ω) be a connected Kähler manifold of dimension $2m$. Let $\alpha \in \bigwedge_{\mathbb{C}}^m T^*_{(m,0),x}M$ be a nonzero holomorphic m-covector at some $x \in M$. α extends to a parallel $A \in \Omega^{(n,0)}(M)$ (*i.e.* a nonvanishing section of the canonical line bundle) if and only if for every $P_\gamma \in \mathrm{Hol}(g)$, we have $P_\gamma \in \mathrm{U}(m)$ such that $\det P_\gamma = 1$. Otherwise, P_γ may change the type of A along paths: $d\bar{z_1} \wedge \ldots \wedge d\bar{z_m}$ on $\mathbb{C}^m$ is invariant under $B \in U(m)$ if and only if $\det B = 1$. Thus, (M,g,J,ω) is **Calabi-Yau** if and only if $\mathrm{Hol}(g) \subseteq SU(m)$.

Intuitively, a Calabi-Yau manifold is an "orientable" manifold in the Kähler category.

References

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2. J. M. Lee, *Riemannian Manifolds: an Introduction to Curvature*, Springer-Verlag, New York, 1997.
3. D. McDuff and D. Salamon, *Introduction to Symplectic Topology*, 2 ed., Oxford Science Publications, 2005.