Lie Group Actions and Differentiability Beyond Manifolds

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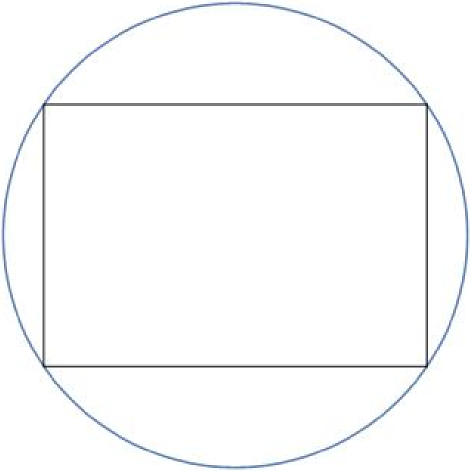
Motivation

Question: What is the area of the largest rectangle inscribed in the unit circle?



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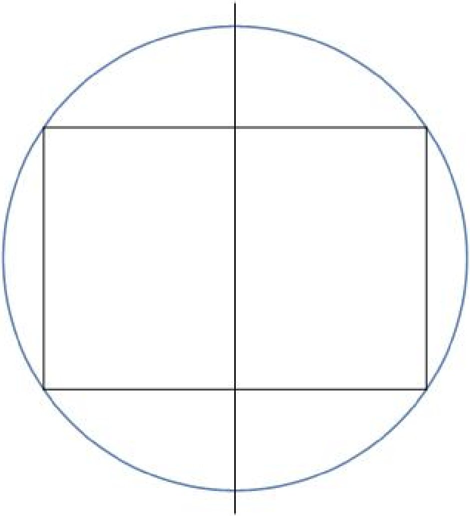

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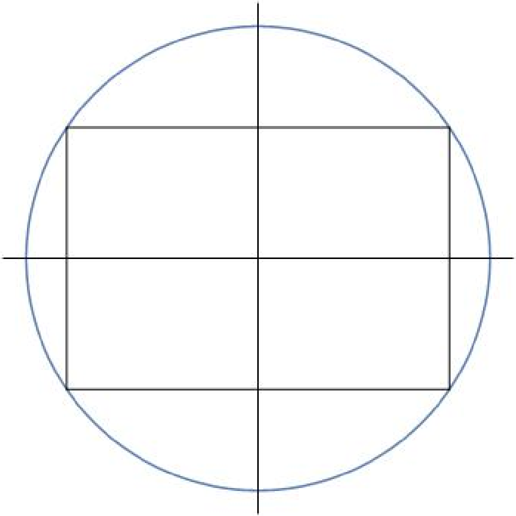

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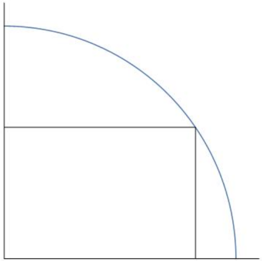


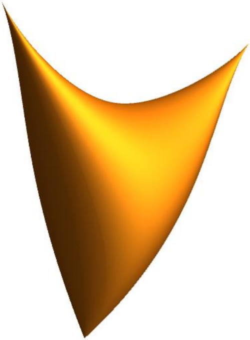
Motivation

Question: What are the equations of motions of a spherical pendulum on a rigid wire (no gravity)?

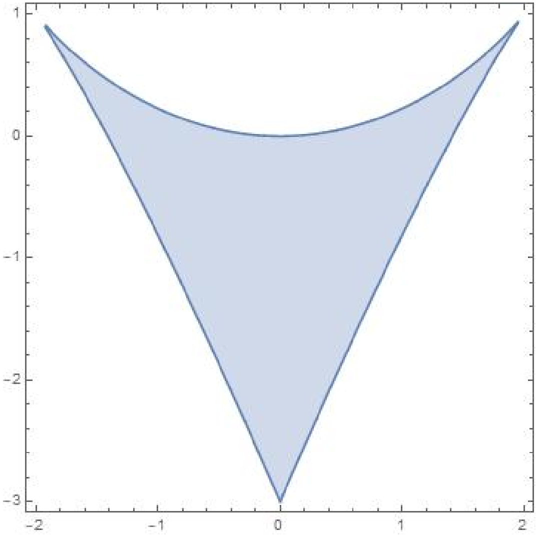
The configuration (position) space is $\mathbb{S}^2$, and the phase (position and momentum) space is $T^*\mathbb{S}^2$, a rank-2 vector bundle over $\mathbb{S}^2$. To simplify calculations, we may want to reduce the symmetry, and so we could “divide out” by the obvious $\mathbb{S}^1$ symmetry, or even the $\mathsf{SO}(3)$ symmetry.

We could also fix some invariants of the system, such as total energy, which would give a subset of $T^*\mathbb{S}^2$ as the new phase space, and we could then further divide out by some $\mathbb{S}^1$ or $\mathsf{SO}(3)$ symmetry. This process is called **symplectic reduction**.

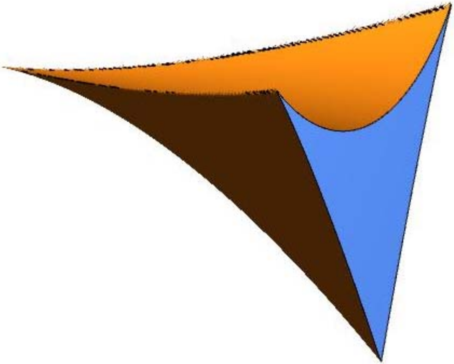
$\mathbb{S}^2/A_4$



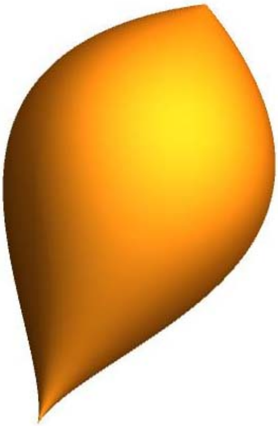
$\mathbb{S}^2$ Modulo Tetrahedral Group



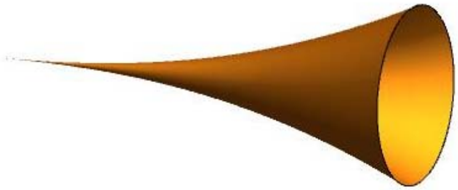
$\mathbb{R}^3$ Modulo Tetrahedral Group



$\mathbb{S}^3/\mathbb{S}^1$ with Weights -2 and 3



$\mathbb{C}^2 /\!\!/_0 \mathbb{S}^1$ with Weights -2 and 3



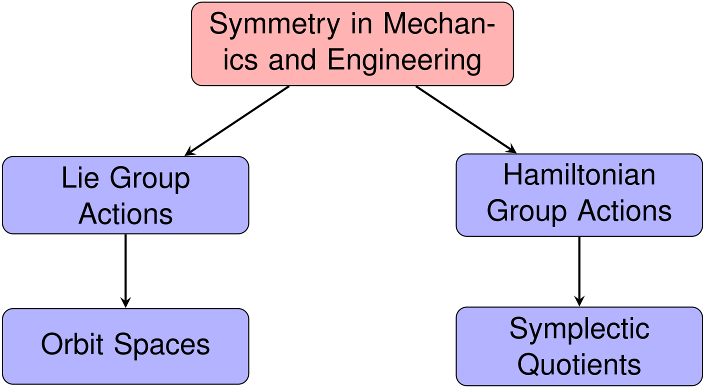
The Study of Orbit Spaces and Symplectic Quotients

The study of orbit spaces and symplectic quotients have obvious applications in Hamiltonian dynamics, symplectic geometry, Poisson geometry, as well as mathematical physics and engineering.

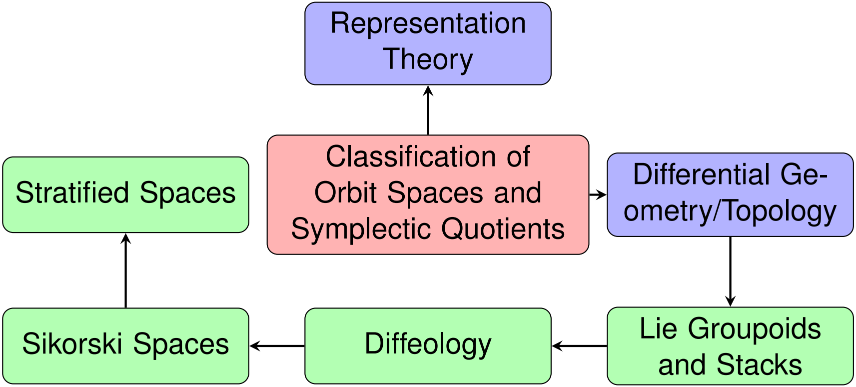
In particular, analysis on these spaces allow you to decrease the number of degrees of freedom in which to do computations, which theoretically should make computations easier.

However, studying the geometry of orbit spaces and symplectic quotients, *e.g.* their classification and invariants, is interesting in its own right.

The Geometry of Symmetry



Connections to Other Mathematical Areas



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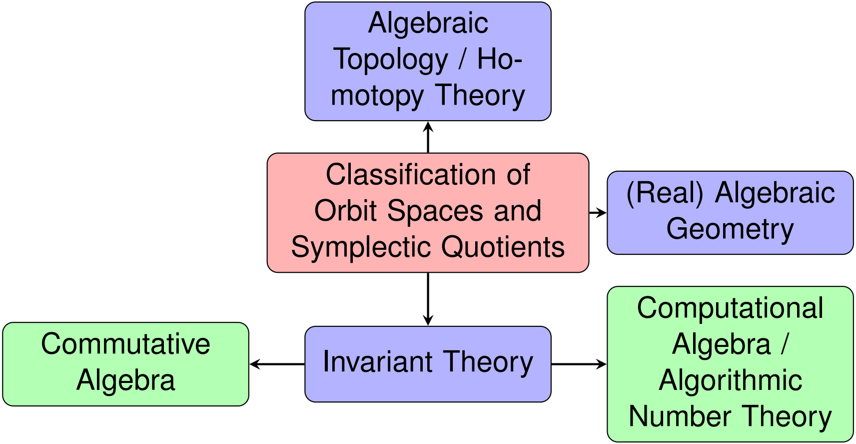

This Talk:

Question 1: Given an orbit space, can we tell which group action it came from?

Question 2: When are symplectic quotients the same as orbit spaces of Lie group actions?