MTH 744 - TOPICS IN GEOMETRY (FALL 2022)
LECTURE NOTES ON LIE GROUP ACTIONS

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These are lecture notes, with exercises, starred exercises, and presentations, for a graduate topics course on Lie group actions. The more exercises a student attempts, the greater their knowledge of the subject will become. The starred exercises tend to be a bit more difficult, and make good presentation-style problems. The presentations are designed for more serious preparation than the starred exercises, and would make good short talks, perhaps with slides.

The notes are designed for graduate students without assuming a lot of experience with topology or analysis in multiple dimensions. They cover the basics of Lie groups and their actions, as well as important theorems for proper actions such as Bochner's Linearisation Theorem, the Slice Theorem and Equivariant Tubular Neighbourhood Theorem, as well as the orbit type stratification.

If the opportunity arises, it would be beneficial to add a proof of the Quotient Manifold Theorem (as opposed to leaving this as a presentation), study in more detail the orbit type stratification of the orbit space of a proper action, as well as expand on Schwarz' result on the smooth functions on the orbit space of a proper Lie group action (which is only hinted at using polynomials in the introduction). However, all of these things require more time and background, some of which would be found in a standard course on smooth manifolds.

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Group Actions. We are primarily interested in group actions. We often understand groups via their actions on sets. To review groups and their actions, see, for instance, Dummit & Foote [**DF99**].

Definition 1.1 (Group Action)**.** Let Γ be a group, and let S be a set. A **group action** of Γ on S is a map $\alpha\colon \Gamma \times S \to S$ satisfying

(1) $\alpha(1_\Gamma, x) = x$ for all $x \in S$, and

(2) $\alpha(\gamma_2, \alpha(\gamma_1, x)) = \alpha(\gamma_2\gamma_1, x)$ for all $\gamma_1, \gamma_2 \in \Gamma$ and $x \in S$.

Here, the juxtaposition $\gamma_2\gamma_1$ is just multiplication in Γ. $\diamond$

Notation 1.2 (Group Action Notation)**.** In reference to Definition **1.1**, we will usually denote $\alpha(\gamma, x)$ by $\gamma \cdot x$, unless we need to be explicit. Also, to indicate that a group Γ acts on a set S, we often write $\Gamma \circlearrowright S$. $\diamond$

Remark 1.3 (Left Versus Right Actions)**.** Sometimes it will be convenient to refer to the group action defined in Definition **1.1** as a **left group action**. A **right group action**, denoted $S \circlearrowleft \Gamma$, is a map $\beta\colon S \times \Gamma \to S\colon (x, \gamma) \mapsto x \cdot \gamma$ satisfying

(1) $x \cdot 1_\Gamma = x$ for all $x \in S$, and

(2) $(x \cdot \gamma_1) \cdot \gamma_2) = x \cdot (\gamma_1\gamma_2)$ for all $\gamma \in \Gamma$ and $x \in S$.

Again, juxtaposition of group elements is just group multiplication in Γ However, if we want the multiplication to appear on the right, but we still want a left action (which is most common), then we typically replace γ in the definition of a right action with γ^{-1}. $\lrcorner$

Exercise 1.4. Let Γ be a group acting on a set S on the right. Show that $\alpha(\gamma, x) := x \cdot \gamma^{-1}$ is a left action. (In this way, we can turn any right action into a left action.)

Examples 1.5 (Examples of Group Actions)**.**

(1) A group acts on itself on the left via left multiplication, and on the right via right multiplication.

(2) A cyclic group $\mathbb{Z}/n$ acts on the plane $\mathbb{C}$ by $[k]\cdot z := e^{2\pi i k/n} z$. In fact, we often will think of cyclic groups as nth groups of unity $\mathbb{Z}/n = \{e^{2\pi i k/n} \in \mathbb{C} \mid k = 0, \ldots, n-1\}$. //

Definition 1.6 (Orbits, Stabilisers, and Orbit Sets)**.** Given a group action of Γ on a set S, the **orbit** of x, is the set

$$\Gamma \cdot x := \{y \in S \mid y = \gamma \cdot x \text{ for some } \gamma \in \Gamma\}.$$

The **stabiliser** of the action at x, denoted $\mathrm{Stab}_\Gamma(x)$ or Γ_x, is the set

$$\Gamma_x := \{\gamma \in \Gamma \mid \gamma \cdot x = x\}.$$

Finally, the group action induces an equivalence relation $\sim$ on S:

$$x_1 \sim x_2 \Leftrightarrow \exists \gamma \in \Gamma \text{ s.t. } x_2 = \gamma \cdot x_1.$$

The equivalence classes of $\sim$ are exactly the orbits of the action. The set of equivalence classes $S/\sim$ is called the **orbit set** (or **quotient set**), and is denoted by $\Gamma\backslash S$. ⋄

Exercise 1.7. Find the orbits and stabilisers of each action of Examples 1.5. How would you describe the orbit sets?

Exercise 1.8. Given a group action on a set, show that the stabiliser of a point is a subgroup of the group acting.

Theorem 1.9 (Orbit-Stabiliser Theorem)**.** *Given an action of Γ on a set S, the orbits and stabilisers are related by*

$$\Gamma \cdot x \cong \Gamma_x \backslash \Gamma$$

where Γ_x acts on Γ by left multiplication. Here, by $\cong$, we mean there is a bijection between the two sets.

★**Exercise 1.10.** Prove Theorem 1.9.

Examples 1.11 (Dihedral and Permutation Groups)**.** One example of an action of a finite group on a set is that of a dihedral group D_n acting on the vertices of an n-gon. Recall,

$$\begin{aligned} D_n &:= \langle \sigma, \tau \mid \sigma^n = \tau^2 = 1,\ \sigma\tau = \tau\sigma^{n-1} \rangle \\ &\cong \langle \beta_1, \beta_2 \mid \beta_1^2 = \beta_2^2 = (\beta_1\beta_2)^n = 1 \rangle \end{aligned}$$

and that the order of D_n is $2n$. Another example is a permutation group S_n acting on the set $\{1, \dots, n\}$. //

Exercise 1.12. Prove that the two presentations of D_n above are isomorphic.

For now, we are interested in actions of groups on vector spaces; to simplify things, we will only be concerned with $\mathbb{R}^n$ and $\mathbb{C}^n$.

Definition 1.13 (Linear Group Action)**.** Let $\mathrm{GL}(n;\mathbb{R})$ be the **general linear group** of $\mathbb{R}^n$; that is, the set of all $n \times n$ invertible matrices with entries from $\mathbb{R}$; this is a group under matrix multiplication. Let Γ be a subgroup of $\mathrm{GL}(n;\mathbb{R})$. Then an action of Γ on $\mathbb{R}^n$ is a **linear action** if for any $\gamma \in \Gamma$ and $x \in \mathbb{R}^n$, the product $\gamma \cdot x$ is just multiplication of x on the left by a matrix in Γ. More abstractly, an action of a group Γ on $\mathbb{R}^n$ is linear if for any $g \in \Gamma$, scalar $c \in \mathbb{R}$, and points $x, y \in \mathbb{R}^n$,

(1) $g \cdot (x + y) = g \cdot x + g \cdot y$, and

(2) $g \cdot (cx) = c(g \cdot x)$. ⋄

We can *almost* always represent a linear action on $\mathbb{R}^n$ with matrices.

Exercise 1.14. Let Γ be a group acting linearly on $\mathbb{R}^n$. Show that there is a natural group homomorphism $\varphi\colon \Gamma \to \mathrm{GL}(n;\mathbb{R})$ such that up to elements of the kernel $K = \ker(\varphi)$, the action of Γ is the same as the action of the subgroup $\mathrm{im}(\varphi)$. Here, the kernel can be described as

$$K := \{g \in \Gamma \mid \forall x \in \mathbb{R}^n,\ g \cdot x = x\}.$$

Show that Γ/K is isomorphic to a subgroup of $\mathrm{GL}(n;\mathbb{R})$.

We can replace $\mathbb{R}^n$ in Definition **1.13** with $\mathbb{C}^n$; however, we typically will not care about the complex structure or holomorphicity. Complex numbers will merely be convenient to use. Also, typically, we do not care that elements of the group acting linearly in a linear action are matrices; it is more relevant that they are linear transformations.

Exercise 1.15 (Cyclic Groups Acting on $\mathbb{R}^2$)**.** We have already seen that the cyclic group $\mathbb{Z}/n$ acts on $\mathbb{C}$ via multiplication by the associated roots of unity (see Examples **1.5**). By identifying $\mathbb{C}$ with $\mathbb{R}^2$ (*i.e.* forgetting complex multiplication), how can $\mathbb{Z}/n$ be represented as a subgroup of $\mathrm{GL}(2;\mathbb{R})$?

Example 1.16 (Dihedral Groups Acting on $\mathbb{C}$)**.** Let D_n act on $\mathbb{C}$ by

$$\beta_1 \cdot z := \overline{z},$$
$$\beta_2 \cdot z := e^{2\pi i/n}\overline{z}.$$

Here, β_1 and β_2 are as defined in Examples **1.11**. //

Exercise 1.17. Convince yourself that the definitions of $\beta_1 \cdot z$ and $\beta_2 \cdot z$ above give an action of all of D_n on $\mathbb{C}$.

Invariant Polynomials. To study the geometry of linear group actions, we will make use of polynomials that are invariant under the action.

Definition 1.18 (Invariant Polynomials)**.** Let Γ be a group acting linearly on $\mathbb{R}^n$ (or $\mathbb{C}^n$). A real-valued function $f\colon \mathbb{R}^n \to \mathbb{R}$ is **invariant** under the action of Γ if for any $\gamma \in \Gamma$ and $x \in \mathbb{R}^n$, we have

$$f(\gamma \cdot x) = f(x).$$

We denote the set of invariant polynomials $P(\mathbb{R}^n)^\Gamma$, or $\mathbb{R}[x_1, \cdots, x_n]^\Gamma$ if we want to specify the coordinates. ◇

Definition 1.19 (Homogeneous Polynomials)**.** A polynomial is **homogeneous** if all of its terms have the same degree. For example, $x^2y + 3xy^2 - y^3$ is homogeneous, but $x^2y + y^2$ is not. ◇

Proposition 1.20. *Given a linear action of a group Γ on $\mathbb{R}^n$, the invariant polynomials are "graded" by their degree; that is, $P(\mathbb{R}^n)^\Gamma$ is a direct sum of sets of homogeneous invariant polynomials, where the sum is taken over the degrees:*

$$P(\mathbb{R}^n)^\Gamma = \bigoplus_{i\in\mathbb{N}} \{\text{invariant homogeneous polynomials of degree } i\}.$$

What this means is if p is an invariant degree-k polynomial, equal to the sum $\sum_{i=0}^k q_i$ where each q_i is homogeneous of degree i, then each q_i is invariant.

Idea of Proof: [You are not expected to understand this proof.] The action of Γ on $\mathbb{R}^n$ induces an action on $P(\mathbb{R}^n)$ as follows: for any $p \in P(\mathbb{R}^n)$, $\gamma \in \Gamma$ and $x \in \mathbb{R}^n$,

$$(g \cdot p)(x) = g^* p(x) := p(g \cdot x).$$

In particular, it induces an action on $(\mathbb{R}^n)^*$, the dual vector space to $\mathbb{R}^n$. This, in turn, induces an action of Γ on covariant tensors of $\mathbb{R}^n$ which preserves degree; moreover, if α is a symmetric covariant tensor, so is $\gamma^*\alpha$ for any $\gamma \in \Gamma$. Finally, there is a natural linear map Φ from symmetric covariant tensors of degree k to polynomials of degree k on $\mathbb{R}^n$; also, this Φ is Γ-equivariant; that is, it $\Phi(g^*\alpha) = g^*\Phi(\alpha)$. $\square$

Example 1.21 (Invariant Polynomials for $\mathbb{Z}/n \circlearrowright \mathbb{C} \cong \mathbb{R}^2$)**.** We want to find all invariant polynomials of the $\mathbb{Z}/n$ action on $\mathbb{C}$ from Examples **1.5**. Instead of using the coordinates (x, y) of $\mathbb{R}^2 \cong \mathbb{C}$, we will use z and $\overline{z}$ instead:

$$\begin{aligned} z &= x + iy, \\ \overline{z} &= x - iy. \end{aligned}$$

To find the invariant polynomials, it will be sufficient to find a *minimal set of generators* of the ring $P(\mathbb{C})^{\mathbb{Z}/n}$, which turns out to be finite. Consider the polynomials

$$\begin{aligned} p_1(z, \overline{z}) &:= z\overline{z} = |z|^2, \\ p_2(z, \overline{z}) &:= \Re(z^n), \\ p_3(z, \overline{z}) &:= \Im(z^n). \end{aligned}$$

Here, $\Re$ and $\Im$ are the real and imaginary parts, resp. //

Exercise 1.22. Find the linear transformation that sends (x, y) to $(z, \overline{z})$. Prove that this is an element of $\mathrm{GL}(2; \mathbb{C})$. In other words, this is a perfectly good change of coordinates!

★Exercise 1.23. Show that the polynomials p_1, p_2, and p_3 of Example **1.21** are invariant, and that any invariant polynomial is an algebraic combination of them.

Exercise 1.24. Try to find a minimal generating set for the invariant polynomials for the dihedral action of D_n on $\mathbb{C}$ given in Example **1.16**

Next, we want to find relations between the polynomials making up a minimal generating set for $P(\mathbb{R}^n)^\Gamma$.

Example 1.25 (Relations for Invariant Polynomials)**.** Consider the minimal set of generators for $P(\mathbb{R}^2)^{\mathbb{Z}/n}$ given in Example **1.21**. Prove that we have the following relations for all $z \in \mathbb{C}$:

$$\begin{aligned} p_1(z, \overline{z})^n &= p_2(z, \overline{z})^2 + p_3(z, \overline{z})^2, \\ p_1(z, \overline{z}) &\geq 0. \end{aligned}$$

Form the **Hilbert map** $p\colon \mathbb{C} \to \mathbb{R}^3$ by $p = (p_1, p_2, p_3)$. In Exercise **1.26**, you will show that the orbit space $\mathbb{Z}/2\backslash\mathbb{C}$ is in one-to-one correspondence with the image of p! Since the image of p sits inside of $\mathbb{R}^3$, this gives us a way to study the orbit space *analytically*; that is, we can do analysis on this space! //

Exercise 1.26.

(a) What is the image of p?

(b) Show that each orbit lies in exactly one level set of p.

(c) Show that each orbit *is* exactly one level set of p in the case $n = 2$.

Exercise 1.27. Repeat Example **1.25** for the action of D_n on $\mathbb{C}$ given in Example **1.16**.

Some Point-Set Topology. We begin by going over some topology. A topology is a tool used to arrange points in a set; namely, it uses so-called "open sets" to separate points from each other. A good reference is the appendix of Lee [Lee13], but Munkres [Mun00] is also a standard text on "point-set topology".

Definition 2.1 (Topologies)**.** Let X be a set. A **topology** $\mathcal{T}$ on X is a subset of the power set $\mathcal{P}(X)$ of X satisfying:

(1) $\emptyset, X \in \mathcal{T}$;

(2) if $\{U_\alpha\}$ is any family of elements of $\mathcal{T}$, then $\bigcup_\alpha U_\alpha \in \mathcal{T}$;

(3) if $\{U_1, \ldots, U_k\}$ is any (finite) family of elements of $\mathcal{T}$, then $\bigcap_{i=1}^k U_i \in \mathcal{T}$.

We call $(X, \mathcal{T})$ a **topological space** (although we will often drop $\mathcal{T}$ when it is understood), and call the elements of $\mathcal{T}$ **open (sub)sets of** X. ⋄

Example 2.2 (Topology on $\mathbb{R}$)**.** The **Euclidean topology** on $\mathbb{R}$ is the standard one, whose open sets are disjoint unions of open intervals. Recall that these are defined as follows: $S \subseteq \mathbb{R}$ is **open** if for each $x \in S$, there is some $\varepsilon > 0$ so that $(x - \varepsilon, x + \varepsilon) \subseteq S$. //

Example 2.3 (Euclidean Topology of $\mathbb{R}^n$)**.** We define the Euclidean topology of $\mathbb{R}^n$ similarly to that of $\mathbb{R}$, but replacing "intervals" with "balls" (or "disks" in the case $n = 2$). A set $S \subseteq \mathbb{R}^n$ is **open** (or contained in the **Euclidean topology**) if for each $x \in S$, there is some $\varepsilon > 0$ so that the **ball of radius** ε **centred at** x, denoted $B_\varepsilon(x)$, is contained in S. //

Remark 2.4 (Open Neighbourhoods)**.** Instead of using intervals, disks, and balls above, we can instead use "open neighbourhoods". Given a topological space X, an **open neighbourhood** of a point $x \in X$ is an open set containing x. ⌟

Example 2.5 (Trivial/Indiscrete Topology)**.** Given a set X, the **trivial topology** (or **indiscrete** topology) is the topology $\mathcal{T} := \{\emptyset, X\}$. //

Example 2.6 (Discrete Topology)**.** Given a set X, the **discrete topology** is the power set $\mathcal{P}(X)$; that is, all sets are open. //

Example 2.7 (Subspace Topology)**.** Let $(X, \mathcal{T})$ be a topological space and $Y \subseteq X$. The **subspace topology** of Y, denoted $\mathcal{T}_Y$, is given as follows: $U \subseteq Y$ is in $\mathcal{T}_Y$ if there exists $V \in \mathcal{T}$ so that $U = V \cap Y$. //

Exercise 2.8. Show that the Euclidean topology on $\mathbb{R}^n$ is, in fact, a topology.

Exercise 2.9. Given a topological space $(X, \mathcal{T})$, show that the subspace topology on a subset $Y \subseteq X$ is, in fact, a topology.

Now that we have introduced topological spaces, the natural thing to do next is to "topologise" the functions between them.

Definition 2.10 (Continuous Maps)**.** Given two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, a function $f\colon X \to Y$ is **continuous** if for any $V \in \mathcal{T}_Y$, the preimage satisfies $f^{-1}(V) \in \mathcal{T}_X$. That is, preimages of open sets are open. $\diamond$

Exercise 2.11 (Continuous Maps and the Euclidean Topology)**.**

(1) Let U and V be open subsets of $\mathbb{R}$. Show that $f\colon U \to V$ is continuous in the traditional (ε-δ)-sense if and only if it is continuous as defined in Definition **2.10**.

(2) Let $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ be open. The traditional definition of a continuous map $f\colon U \to V$ is given as follows: f is continuous if for any $x \in U$ and $\varepsilon > 0$, there exists $\delta > 0$ such that if $x_0 \in U$ and $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$. (Here, $|\cdot|$ is the standard norm on $\mathbb{R}^n$ from linear algebra: $|x| := \sqrt{x_1^2 + \cdots + x_m^2}$.) Show that $f\colon U \to V$ is continuous in the traditional sense if and only if it is continuous as defined in Definition **2.10**.

Exercise 2.12.

(1) Given a topological space $(X, \mathcal{T})$ and a subset $Y \subseteq X$, show that the **inclusion map** $i_Y\colon Y \hookrightarrow X$ is continuous.

(2) Let X be given the discrete topology and Y be any topological space. Show that any function $f\colon X \to Y$ is continuous.

(3) Let Y be given the trivial topology and X be any topological space. Show that any function $f\colon X \to Y$ is continuous.

Remember that the idea of an "isomorphism" in group theory is a function between two groups that are essentially the same: their underlying sets are in bijection and the group structures behave exactly the same way. In fact, one can think of a bijection as an isomorphism between sets. With this idea in mind, what is an "isomorphism" of topological spaces?

Definition 2.13 (Homeomorphism)**.** Given topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, a function $f\colon X \to Y$ is a **homeomorphism** if it is continuous, bijective, and its inverse $f^{-1}\colon Y \to X$ is also continuous. $\diamond$

Exercise 2.14.

(1) Are a circle and an ellipse, both subsets of the plane, homeomorphic?

(2) Construct a continuous bijection from the interval $[0, 2\pi)$ to the circle

$$\mathbb{S}^1 := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}.$$

Is it a homeomorphism?

Many topological concepts from your study of $\mathbb{R}$ port to our more abstract setting; namely, the definition of interior point, interior, boundary point, boundary, closed set, accumulation point, isolated point, and closure. Moreover, many of the results you would have learned also port to our more abstract setting, typically with the same proofs. Indeed, the complement of an open set is closed, and *vice versa*; a subset $Y \subseteq (X, \mathcal{T})$ is equal to the union of its interior (denoted Y°) and the boundary points contained within it; the closure of Y (denoted $\overline{Y}$) is the union of Y with its boundary (denoted $\mathrm{bdy}(Y)$), as well as the union of Y with its accumulation points; and a function $f\colon (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is continuous if and only if for any closed set $C \subseteq Y$, the preimage $f^{-1}(C)$ is closed in X.

However, many properties *do not* carry forward; in particular, facts about sequences and compactness often do not. For instance, there are topological spaces in which the limit of a sequence may exist, but it may not be unique. Also, the Heine-Borel theorem (that compact sets are precisely the sets that are closed and bounded) no longer makes sense: what does "bounded" mean? To help alleviate the loss of properties (and intuition), we are going to come up with some conditions that help "tame" the topological spaces we are dealing with.

Definition 2.15 (Hausdorff Topological Space)**.** A topological space X is **Hausdorff** if for any two points $x, y \in X$, there are disjoint open neighbourhoods U and V of x and y, resp. $\diamond$

Exercise 2.16. Prove that $\mathbb{R}^n$ (with the Euclidean topology) is Hausdorff. (Hint: draw a picture first.)

Exercise 2.17.

(1) Consider the set $X = \{x, y\}$ with topology $\mathcal{T} = \{\emptyset, \{x\}, X\}$. Is this Hausdorff?

(2) Suppose X is a Hausdorff topological space and $Y \subseteq X$ has the subspace topology. Is Y Hausdorff?

It turns out that if the limit of a sequence exists in a Hausdorff topological space, then it is unique. (Of course, we never defined what precisely this means; try coming up with the definition of convergence of a sequence in an abstract topological space, and then prove this claim!)

Definition 2.18 (Topological Basis)**.** A **basis** $\mathcal{B}$ for a topology $\mathcal{T}$ on a set X is a collection of open sets $\{U_\alpha\}_{\alpha \in A}$ such that for any $U \in \mathcal{T}$, there is some subcollection $\{U_\beta\}_{\beta \in B \subseteq A}$ such that $U = \bigcup_{\beta \in B} U_\beta$. $\diamond$

Exercise 2.19. An equivalent definition for basis $\mathcal{B}$ of a topology on a set X is a collection $\{U_\alpha\}$ of subsets of X such that $X = \bigcup_\alpha U_\alpha$ and for any two $U, V \in \mathcal{B}$, the intersection $U \cap V$ is the union of elements of $\mathcal{B}$. Prove that this is equivalent to the definition above. This is how you can specify a topology by specifying its basis; we say the topology obtained in this way is the topology **generated** by the basis $\mathcal{B}$; its open sets are exactly unions of basis elements (along with $\emptyset$).

Definition 2.20 (Second-Countability)**.** A topological space X is **second-countable** if it admits a countable basis. $\diamond$

Exercise 2.21. If X is a second-countable topological space, and $Y \subseteq X$ has the subspace topology, show that Y is second-countable.

Example 2.22 ($\mathbb{R}^n$ is Second-Countable)**.** The Euclidean topology on $\mathbb{R}^n$ is second-countable: take for a basis all open balls of positive rational radii centred at points $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ such that $x_i \in \mathbb{Q}$ for each $i = 1, \ldots, n$. //

Exercise 2.23. Prove the claim in the previous example.

Topological Manifolds. We are now ready for the definition of a topological manifold. One might argue that this is the nicest type of topological space you can have beyond $\mathbb{R}^n$.

Definition 2.24 (Topological Manifold)**.** A **topological n-manifold** M is a Hausdorff second-countable topological space such that for every $x \in M$ there exists an open neighbourhood U of x and a homeomorphism $\varphi\colon U \to \widetilde{U} \subseteq \mathbb{R}^n$; this latter condition is often summarised as saying that M is "locally homeomorphic to $\mathbb{R}^n$". Here, we call n the **dimension** of M. The homeomorphisms $\varphi\colon U \to \widetilde{U}$ are called **(continuous) charts**, and the collection of all charts a **(continuous) atlas** of M. Sometimes we write M^n to indicate it is n-dimensional (but try not to do this when it might be confused with the power). ⋄

Example 2.25 ($\mathbb{R}^n$ is an n-Manifold)**.** Cartesian space $\mathbb{R}^n$ is an n-manifold (the identity map serves as a chart, the only one needed). //

Example 2.26 (The Circle $\mathbb{S}^1$)**.** The unit circle

$$\mathbb{S}^1 := \left\{(x,y) \in \mathbb{R}^2 \middle| |(x,y)| = 1\right\}$$

is a one-dimensional manifold. //

To see this, it suffices to check whether it is Hausdorff and second-countable, and then construct an atlas for it. It follows from Exercise 2.17 that $\mathbb{S}^1$ is Hausdorff. It follows from Exercise 2.21 that $\mathbb{S}^1$ is second-countable. We construct an atlas with exactly two charts. Let N be the north pole of $\mathbb{S}^1$ and S the south pole. Let φ_1 have domain $\mathbb{S}^1 \setminus \{S\}$, which sends the point (x,y) on the circle to the unique point on the x-axis where the line through S and (x,y) intersects it. Similarly, let φ_2 have domain $\mathbb{S}^1 \setminus \{N\}$, which sends the point (x,y) on the circle to the unique point on the x-axis where the line through N and (x,y) intersects it.

★**Exercise 2.27.** Write down an explicit formula for φ_1, and show that it is a homeomorphism onto $\mathbb{R}$. Do the same for φ_2.

We call φ_1 and φ_2 **stereographic projections** of $\mathbb{S}^1$. This completes the proof that $\mathbb{S}^1$ is a manifold.

Exercise 2.28. Define the n-sphere to be the set

$$\mathbb{S}^n = \left\{x \in \mathbb{R}^{n+1} \;\middle|\; |x| = 1\right\}.$$

Show that this is an n-manifold.

Example 2.29 (Open Subsets of Manifolds)**.** Any open subset of a topological manifold is itself a topological manifold with the subspace topology. //

Definition 2.30 (Product Topology)**.** Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Define the **product topology** on $X \times Y$ to be the topology generated by the basis $\{U \times V\}_{U \in \mathcal{T}_X,\ V \in \mathcal{T}_Y}$. ◇

Exercise 2.31. Prove that the product topology is, in fact, a topology. (You can do this directly through the definition of a topology, or prove that the basis in the definition is, in fact, a basis.)

Exercise 2.32. Show that the Euclidean topology on $\mathbb{R}^n$ is equal to the product topology.

Definition 2.33 (Product Manifolds)**.** Let M and N be two topological manifolds of dimensions m and n, resp. The **product manifold** $M \times N$ is the topological space whose topology is the product topology. This is a topological manifold: if $\{\varphi_\alpha \colon U_\alpha \to \widetilde{U}_\alpha\}$ is an atlas for M and $\{\psi \colon V_\beta \to \widetilde{V}_\beta\}$ an atlas for N, then $\{\varphi_\alpha \times \psi_\beta \colon U_\alpha \times V_\beta \to \widetilde{U}_\alpha \times \widetilde{V}_\beta\}$ is an atlas for $M \times N$. ◇

★**Exercise 2.34.** Complete the proof that the product of two manifolds is again a manifold by showing the following:

(1) If X and Y are Hausdorff topological spaces, then $X \times Y$ is Hausdorff.

(2) If X and Y are second-countable topological spaces, then $X \times Y$ is second-countable.

Example 2.35 (Tori)**.** The torus (or 2-torus) is the manifold $\mathbb{T}^2 := \mathbb{S}^1 \times \mathbb{S}^1$. More generally, we have the n-torus $\mathbb{T}^n := \prod_{i=1}^n \mathbb{S}^1$. //

Definition 2.36 (Topological Group)**.** A topological group G is a group equipped with a topology (often required to be Hausdorff) with a continuous multiplication map $m \colon G \times G \to G \colon (m_1, m_2) \mapsto m_1 m_2$ and continuous inverse map $\mathrm{inv} \colon G \to G \colon g \mapsto g^{-1}$. ◇

Examples 2.37 (Examples of Topological Groups)**.** Any finite group is a topological group (with the discrete topology). The circle $\mathbb{S}^1$ has a group structure making it into a topological group. More generally, many matrix groups are topological groups. Define $\mathrm{Mat}(n; \mathbb{R})$ to be the set of all $n \times n$ matrices with real entries. This has a natural topology making it homeomorphic to $\mathbb{R}^{n^2}$ (and so it is a topological manifold). Define $\mathrm{GL}(n; \mathbb{R})$ to be all invertible matrices in $\mathrm{Mat}(n; \mathbb{R})$. This is a manifold (why?). It turns out that many of the topological groups we care about are topological manifolds (but not always). //

Exercise 2.38. Answer the question "why" above.

Here we review differentiable maps $\mathbb{R}^m \to \mathbb{R}^n$, and use this notion to define a "smooth structure" on a topological manifold. Lee in [**Lee13**, Appendix C] has a review of this material, and Spivak's *Calculus on Manifolds* [**Spi65**] is an excellent text on precisely this topic. However, the following is based off of [**Wad18**, Chapter 11].

Differentiable Maps $\mathbb{R}^m \to \mathbb{R}^n$**.**

Definition 3.1 (Differentiable Map)**.** Let $U \subseteq \mathbb{R}^m$ be open and fix $x_0 \in U$. A function $f\colon U \to \mathbb{R}^n$ is **differentiable at** x_0 if there exists a linear map $T_{x_0}\colon \mathbb{R}^m \to \mathbb{R}^n$ such that for sufficiently small $h \in \mathbb{R}^m$,

$$f(x_0 + h) = f(x_0) + T_{x_0}(h) + \varphi(h) \tag{1}$$

where φ satisfies

$$\lim_{h\to 0} \frac{|\varphi(h)|}{|h|} = 0.$$

More precisely, for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $h \in B_\delta(0)$, then $\frac{\varphi(h)}{|h|} < \varepsilon$. We say that f is **differentiable** if it is differentiable at each point $x_0 \in U$. $\diamond$

When $m = n = 1$, this should remind you of approximating a function near a fixed x_0 with a linear approximation; in terms of graphs, this is the tangent line to the graph of f at $(x_0, f(x_0))$. In fact, this idea generalises: for $m > 1$, the graph is approximated by a tangent hyperplane, which is determined by the graph of the linear map above.

Example 3.2 (Linear Transformation)**.** Let $A\colon \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation. For any $x_0, h \in \mathbb{R}^m$, we have

$$A(x_0 + h) = A(x_0) + A(h) + \varphi(h)$$

where we take $\varphi = 0$. So taking A as T_{x_0} for each $x_0 \in \mathbb{R}^m$ satisfies the definition of differentiability. //

Example 3.3 ($f(x) = e^x$)**.** Since T_{x_0} may remind the reader of the derivative of a function at x_0, the previous example may be confusing. Indeed, we are essentially saying that the derivative of a linear map at a point is itself; but doesn't only the exponential map (up to a constant factor) satisfy this in the case that $m = n = 1$? Take $f(x) = e^x$. Then (1) becomes for any $x_0, h \in \mathbb{R}$:

$$e^{x_0+h} = e^{x_0} + T_{x_0}(h) + \varphi(h)$$

where $T_{x_0}(h) = e^{x_0}h$; the linear map T_{x_0} is scalar multiplication by e^{x_0}. Using l'Hôpital's Rule, one can show that $\lim\limits_{h\to 0} \frac{|\varphi(h)|}{|h|} = 0$. The point here is that there are two inputs to $T_{x_0}(h)$: the point x_0 which determines the linear map, and the input to the linear map, h. //

Proposition 3.4 (Properties of Differentiable Maps)**.** *Let $U \subseteq \mathbb{R}^m$ be open, $f, g\colon U \to \mathbb{R}^n$ differentiable, and $a \in \mathbb{R}$.*

(1) *f is continuous.*

(2) *If $x_0 \in U$, then T_{x_0} in (1) satisfies*

$$T_{x_0} = \left[\frac{\partial f_i}{\partial x_j}(x_0) \right],$$

which is referred to as the ***Jacobian matrix*** *or* ***total derivative*** *of f at x_0. In particular, all partial derivatives of f exist, and T_{x_0} is unique; we often denote it by $Df|_{x_0}$, or $f'(x_0)$. (In the case $n = 1$, this is technically the transpose of the gradient of f, which is a column vector function.)*

(3) *$f + g$ is differentiable and $D(f + g) = Df + Dg$.*

(4) *af is differentiable and $D(af) = aDf$.*

(5) *$f \cdot g$ is differentiable and $D(f \cdot g)|_{x_0} = g(x_0)^{\mathrm{t}} Df|_{x_0} + f(x_0)^{\mathrm{t}} Dg|_{x_0}$ for all $x_0 \in U$.*

(6) *If $V \subseteq \mathbb{R}^n$ is open with $f(U) \subseteq V$, and $h \colon V \to \mathbb{R}^p$ is differentiable, then $h \circ f$ is differentiable and $D(h \circ f)|_{x_0} = Dh|_{f(x_0)} Df|_{x_0}$.*

Proof. This is standard material that can be found in the cited texts. However, to give an idea of how these proofs go, we prove Item **5**. Set

$$T_{x_0} = g(x_0)^{\mathrm{t}} Df|_{x_0} + f(x_0)^{\mathrm{t}} Dg|_{x_0};$$

this is a $1 \times n$ row vector. Since total derivatives are unique, it suffices to show

$$\lim_{h \to 0} \frac{(f \cdot g)(x_0 + h) - (f \cdot g)(x_0) - T(h)}{|h|} = 0.$$

The numerator, which is equal to $\varphi(h)$, can be broken down as follows:

$$\begin{aligned} \varphi(h) &= (f \cdot g)(x_0 + h) - (f \cdot g)(x_0) - g(x_0)^{\mathrm{t}} Df|_{x_0} - f(x_0)^{\mathrm{t}} Dg|_{x_0} h \\ &= (f(x_0 + h) - f(x_0) - Df|_{x_0} h) \cdot g(x_0 + h) \\ &\quad + Df|_{x_0} h \cdot (g(x_0 + h) - g(x_0)) \\ &\quad + f(x_0) \cdot (g(x_0 + h) - g(x_0) - Dg|_{x_0} h). \end{aligned}$$

Let the three terms at the end of the equalities above be referred to as $T_1(h)$, $T_2(h)$, and $T_3(h)$, resp. It suffices to show that $T_i(h)/|h| \to 0$ as $h \to 0$; we do so for $T_2(h)/|h|$, referring to [**Wad18**] for the other two.

$$\begin{aligned} |T_2(h)| &= \Big| (Df|_{x_0} h) \cdot (g(x_0 + h) - g(x_0)) \Big| \\ &\leq \Big| (Df|_{x_0} h \Big| |g(x_0 + h) - g(x_0)| \\ &\leq \Big| Df|_{x_0} \Big| |h| |g(x_0 + h) - g(x_0)|. \end{aligned}$$

Above, we applied the Cauchy-Schwarz Inequality twice, the second time using the **operator norm** for a linear transformation:

$$|T| := \sup \left\{ T(x) \middle| \; |x| = 1 \right\}.$$

Since g is differentiable at x_0, it is continuous there by Item **1**, and so $g(x_0 + h) \to g(x_0)$ as $h \to 0$. Thus $|T_2(h)|/|h| \to 0$ as $h \to 0$. $\square$

Remark 3.5. The converse to Item **2** of Proposition **3.4** is generally not true: just because all partial derivatives of a function exist at a point does not imply that the total derivative exists there; one also requires the partial derivatives to exist in an open neighbourhood of x_0 and to be continuous at x_0 to obtain the existence of $Df|_{x_0}$. See [**GO64**] for examples of such functions, and Definition **3.9** which indicates how to fix the issue. ⌟

The notation used here, a cross between calculus and linear algebra, can be confusing at first. Let us break down the proof above for $m = n = 2$ to illustrate it even further.

Example 3.6 (Dot Product Proof)**.** Let $f, g\colon \mathbb{R}^2 \to \mathbb{R}^2$, in which case we write $f = (f_1, f_2)$ and $g = (g_1, g_2)$, or in matrix notation,

$$f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

and similarly for g. Then, recalling that $f \cdot g = f_1 g_1 + f_2 g_2 = f^{\mathrm{t}} g$,

$$\begin{aligned} D(f \cdot g) &= \begin{bmatrix} \frac{\partial(f_1 g_1)}{\partial x_1} + \frac{\partial(f_2 g_2)}{\partial x_1} & \frac{\partial(f_1 g_1)}{\partial x_1} + \frac{\partial(f_2 g_2)}{\partial x_2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial f_1}{\partial x_1} g_1 + \frac{\partial g_1}{\partial x_1} f_1 + \frac{\partial f_2}{\partial x_1} g_2 + \frac{\partial g_2}{\partial x_1} f_2 & \frac{\partial f_1}{\partial x_2} g_1 + \frac{\partial g_1}{\partial x_2} f_1 + \frac{\partial f_2}{\partial x_2} g_2 + \frac{\partial g_2}{\partial x_2} f_2 \end{bmatrix}. \end{aligned}$$

On the other hand,

$$\begin{aligned} g^{\mathrm{t}} Df + f^{\mathrm{t}} Dg &= \begin{bmatrix} g_1 & g_2 \end{bmatrix} \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} + \begin{bmatrix} f_1 & f_2 \end{bmatrix} \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{bmatrix} \\ &= \begin{bmatrix} g_1 \frac{\partial f_1}{\partial x_1} + g_2 \frac{\partial f_2}{\partial x_1} & g_1 \frac{\partial f_1}{\partial x_2} + g_2 \frac{\partial f_2}{\partial x_2} \end{bmatrix} + \begin{bmatrix} f_1 \frac{\partial g_1}{\partial x_1} + f_2 \frac{\partial g_2}{\partial x_1} & f_1 \frac{\partial g_1}{\partial x_2} + f_2 \frac{\partial g_2}{\partial x_2} \end{bmatrix}. \end{aligned}$$

The two results are equal. //

The following example is a great "sanity check".

Example 3.7 (The Norm Squared Function)**.** Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be given by

$$x \mapsto |x|^2 = x_1^2 + \cdots + x_n^2.$$

Then

$$Df|_x = 2 \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}.$$

Now, if we instead use Item **5** of Proposition **3.4**, noting that $f(x) = x \cdot x = \mathrm{id}_{\mathbb{R}^n}(x) \cdot \mathrm{id}_{\mathbb{R}^n}(x)$, then

$$\begin{aligned} Df|_x &= 2(\mathrm{id}_{\mathbb{R}^n}(x)^{\mathrm{t}}) D(\mathrm{id}_{\mathbb{R}^n})|_x \\ &= 2 \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \mathrm{id}_{\mathbb{R}^n}|_x \\ &= 2 \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}. \end{aligned}$$

Here, we are using the fact that $\mathrm{id}_{\mathbb{R}^n}$ is linear, and so by Example **3.2**, its derivative evaluated at x is $\mathrm{id}_{\mathbb{R}^n}$, which can be thought of as the $n \times n$ identity matrix. The two results match, as expected. //

Example 3.8 (Computation Using Chain Rule)**.** Let $f(x,y) = \sin(xy) + xy^2$, $x(s,t) = s^2t$, and $y(s,t) = e^{st}$. We want to find $\frac{\partial f}{\partial s}$. Note that $\frac{\partial f}{\partial s}$ is equal to $D(f(x(s,t), y(s,t))) \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, where the total derivative D is with respect to s and t. Let $g(s,t) = (x(s,t), y(s,t))$; then we are looking for $D(f \circ g)$. By the chain rule, this is $Df(g(s,t))Dg(s,t)$. Thus,

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\Big|_{g(s,t)} \frac{\partial g_1}{\partial s} + \frac{\partial f}{\partial y}\Big|_{g(s,t)} \frac{\partial g_2}{\partial s}.$$

(One often replace "g_1" and "g_2" with "x" and "y", resp.) Computing:

$$\begin{aligned} \frac{\partial f}{\partial s} &= (y\cos(xy) + y^2)\big|_{g(s,t)} \cdot 2st + (x\cos(xy) + 2xy)\big|_{g(s,t)} \cdot te^{st} \\ &= 2st(e^{st}\cos(s^2te^{st}) + e^{2st}) + te^{st}(s^2t\cos(s^2te^{st}) + 2s^2te^{st}) \end{aligned}$$

One should be warned here that in different contexts, especially in applications, the notation can change. In particular, given a function $f(x,y,t)$ and $x = x(t)$ and $y = y(t)$, then f depends *explicitly* on t, as well as *implicitly* on t via x and y. Sometimes the notation $\frac{\partial f}{\partial t}$ is used in this case for the derivative of f with respect to the *explicit* variable t, and $\frac{df}{dt}$ for derivative of f with respect to *all* instances of t, explicit and implicit. In the latter case,

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial t}.$$

We will not need this for our purposes, however. //

Often times we want the derivative of a function to be continuous. We have a special notation and name for such functions.

Definition 3.9 (Continuously Differentiable Function)**.** Let $U \subseteq \mathbb{R}^m$ be open and $f\colon U \to \mathbb{R}^n$ a function. Then f is **continuously differentiable**, or C^1, if its derivative $Df\colon U \times \mathbb{R}^m \to \mathbb{R}^n$ exists and is continuous; equivalently, all first order partial derivatives of f exist and are continuous on U. Denote the set of such functions by $C^1(U, \mathbb{R}^n)$, or just $C^1(U)$ if $n = 1$. ◇

One can now ask whether Df is (continuously) differentiable. And so on.

Definition 3.10 (k-Times Continuously Differentiable Function)**.** Let $U \subseteq \mathbb{R}^m$ be open and $f\colon U \to \mathbb{R}^n$ a function. The second derivative of f, denoted D^2f, is the derivative of $Df\colon U \times \mathbb{R}^m \to \mathbb{R}^n$. Similarly, the kth derivative of f, denoted D^kf, is the derivative of $D^{k-1}f$, defined recursively. We say that f is **k-times continuously differentiable**, or C^k, if D^kf exists and is continuous; equivalently, all partial derivatives of f of all orders up to and including k exist and are continuous on U. Denote the set of such functions by $C^k(U, \mathbb{R}^n)$, or just $C^k(U)$ if $n = 1$. ◇

Note that if $f \in C^k(U, \mathbb{R}^n)$, then D^if is continuous automatically for $i = 0, \ldots, k-1$ (where $D^0f := f$). One can further ask f to be in $C^k(U, \mathbb{R}^n)$ for all $k \in \mathbb{N}$.

Definition 3.11 (Smooth Function)**.** Let $U \subseteq \mathbb{R}^m$ be open and $f\colon U \to \mathbb{R}^n$ a function. Then f is **smooth**, or C^∞, if it is in $C^k(U, \mathbb{R}^n)$ for all $k \in \mathbb{N}$; equivalently, all partial derivatives of all orders exist (and hence are continuous) on U. Denote the set of smooth functions on U by $C^\infty(U, \mathbb{R}^n)$, or just $C^\infty(U)$ if $n = 1$. ◇

Examples 3.12 (Examples of Smooth Functions)**.** All real-valued polynomials with real coefficients of a finite number of variables are smooth. Similarly, rational functions are smooth on their domains. Any analytic function is smooth; recall that a function $f\colon U \to \mathbb{R}^n$ is **analytic** if for any $x \in U$, there is an open neighbourhood $V \subseteq U$ of x such that the restriction $f|_V$ of f to V is equal to the Taylor series of f at x. //

★**Exercise 3.13** (A Smooth Non-Analytic Function)**.** Show that the following function is smooth, but not analytic:

$$f(x) = \begin{cases} e^{-1/x} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

The function in the previous exercise is extremely important. It allows us to construct so-called **smooth bump functions**.

Example 3.14 (Smooth Bump Function)**.** Let $a < b < c < d$ be real numbers. Consider the function $\varphi(x)$ defined by

$$\varphi(x) = \frac{f(x-a)}{f(x-a) + f(b-x)} \frac{f(d-x)}{f(d-x) + f(x-c)}.$$

This function is smooth, equal to 0 on $(-\infty, a) \cup (d, \infty)$, equal to 1 on (b, c), strictly increasing on (a, b), and strictly decreasing on (c, d). //

We end this subsection with a definition of "isomorphism" for open subsets of Cartesian spaces with smooth maps.

Definition 3.15 (Diffeomorphism)**.** Let $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ be open subsets, and $f\colon U \to V$ a function. Then f is a **diffeomorphism** if it is a smooth bijection with a smooth inverse. ◇

Exercise 3.16.

(1) Are an open annulus and an open punctured disk diffeomorphic?

(2) Is the map $\mathbb{R} \to \mathbb{R}\colon x \mapsto x^3$ a diffeomorphism?

Remark 3.17 (Invariance of Domain)**.** It is a non-trivial result that homeomorphisms (and hence diffeomorphisms) preserve the dimension of open sets of Cartesian spaces. And so, in Definition **3.15**, it is automatic that $m = n$. ⌟

Smooth Manifolds. Differentiability is very important: we use it to do calculus. However, we often need to do calculus on spaces that are not Cartesian space, and so we need to come up with a way of pushing calculus beyond $\mathbb{R}^n$. The first place we arrive in such a generalisation is topological manifolds; however, here we need to be careful. For instance, as topological manifolds, the 2-sphere is homeomorphic to any (hollow) Platonic solid, which has cusps and ridges. As you may recall, calculus misbehaves at such points, and so we need to eliminate this sort of ambiguity: that the calculus you do is not preserved by homeomorphism. We fix this problem by introducing a "smooth structure" on a manifold.

Definition 3.18 (Smooth Manifold)**.** Let M be a topological n-manifold, and let $\mathcal{A}$ be a topological atlas for it. We say that two charts $\varphi\colon U \to \widetilde{U} \subseteq \mathbb{R}^n$ and $\psi\colon V \to \widetilde{V} \subseteq \mathbb{R}^n$ are **smoothly compatible** if $U \cap V = \emptyset$, or the map F from $\varphi(U \cap V) \subseteq \widetilde{U}$ to $\psi(U \cap V) \subseteq \widetilde{V}$ given by $F := \psi \circ \varphi^{-1}|_{\varphi(U \cap V)}$ is a diffeomorphism; we call such a map a **transition function**.

$$\begin{tikzcd}
 & U \arrow[dl, "\varphi"'] & & U \cap V \arrow[ll, hook'] \arrow[rr, hook] \arrow[d, phantom, "\circlearrowleft"] & & V \arrow[dr, "\psi"] & \\
\widetilde{U} & & \varphi(U \cap V) \arrow[ll, hook'] \arrow[rr, "F"'] & {} & \psi(U \cap V) \arrow[rr, hook] & & \widetilde{V}
\end{tikzcd}$$

If all charts in an atlas are smoothly compatible with each other, then we call the atlas a **smooth atlas** and its charts **smooth charts**. A topological manifold equipped with a smooth atlas is called a **smooth manifold**.

We say that two smooth atlases on a topological manifold are **equivalent** if their union is also a smooth atlas (that is, all charts from either atlas are smoothly compatible with each other). In this way we can construct a **maximal atlas**: the union of all smooth atlases that are equivalent. In theory, it is often convenient to work with a maximal atlas; in practice, it is typically sufficient to work with a non-maximal one. $\diamond$

Exercise 3.19. Confirm that the atlas $\{\varphi_1, \varphi_2\}$ constructed for the circle $\mathbb{S}^1$ (see Example **2.26**) is in fact a smooth atlas.

Example 3.20 (The n-Sphere $\mathbb{S}^n$)**.** Similar to $\mathbb{S}^1$, the atlas for $\mathbb{S}^n$ whose charts are stereographic projections (see Exercise **2.28**) is a smooth atlas. //

Exercise 3.21. Confirm that the product of two smooth manifolds is a smooth manifold.

Example 3.22. Any open subsets of a smooth manifold is itself a smooth manifold. //

Week 04: Smooth Maps Between Manifolds

In this section, we look at that "arrows" in the "category" of smooth manifolds: smooth maps. Throughout this section, unless stated otherwise, all manifolds are smooth.

The Definition of a Smooth Map.

Definition 4.1 (Smooth Map Between Manifolds)**.** Let M and N be (smooth) manifolds of dimensions m and n, resp. A function $F\colon M \to N$ is **smooth** if for any $x \in M$ there exists a (smooth) chart $\varphi\colon U \to \widetilde{U} \subseteq \mathbb{R}^m$ about x, a (smooth) chart $\psi\colon V \to \widetilde{V} \subseteq \mathbb{R}^n$ about $F(x)$ such that $F(U) \subseteq V$, and a smooth map $\widetilde{F}\colon \widetilde{U} \to \widetilde{V}$ such that the restriction $F|_U$ satisfies

$$F|_U = \psi^{-1} \circ \widetilde{F} \circ \varphi;$$

that is, the following diagram commutes:

$$\begin{tikzcd}
U \arrow[r, "\varphi"] \arrow[d, "F|_U"'] \arrow[dr, phantom, "\circlearrowleft"] & \widetilde{U} \arrow[r, hook] \arrow[d, "\widetilde{F}"] & \mathbb{R}^m \\
V \arrow[r, "\psi"'] & \widetilde{V} \arrow[r, hook] & \mathbb{R}^n.
\end{tikzcd}$$

$\diamond$

Proposition 4.2 (Smooth Maps are Continuous)**.** *Smooth maps are continuous.*

Proof. Let $F\colon M \to N$ be smooth. Fix an open set $W \subseteq N$. It suffices to show that $F^{-1}(W)$ is open in M; that is, any point $x \in F^{-1}(W)$ has an open neighbourhood $U \subseteq F^{-1}(W)$. Fix $x \in F^{-1}(W)$. Since F is smooth, there are charts $\varphi\colon U \to \widetilde{U}$ and $\psi\colon V \to \widetilde{V}$ about x and $F(x)$, resp., and a smooth map $\widetilde{F}\colon \widetilde{U} \to \widetilde{V}$ such that $F|_U = \psi^{-1} \circ \widetilde{F} \circ \varphi$. Smooth maps between open subsets of Cartesian spaces are continuous, and φ and ψ are homeomorphisms; thus

$$F^{-1}(W \cap V) = \varphi^{-1}\left(\widetilde{F}^{-1}(\psi(W \cap V))\right)$$

is open in M, and is an open neighbourhood of x contained in $F^{-1}(W)$. $\square$

Exercise 4.3. Show that smoothness of a function $F\colon M \to N$ between manifolds is a local property; that is, F is smooth if and only if for any $x \in M$ there exists an open neighbourhood U of x such that $F|_U$ is smooth.

Examples 4.4.

(1) A constant map $F\colon M \to N$ is smooth. (Why?)

(2) The identity map $\mathrm{id}_M\colon M \to M$ is smooth. (Why?)

(3) Let $U \subseteq M$ be an open set. Then the inclusion map $i\colon U \to M$ is smooth.

(4) Given manifolds M and N, the **projection maps** $\mathrm{pr}_1\colon M \times N \to M$ and $\mathrm{pr}_2\colon M \times N \to N$ are smooth. //

Exercise 4.5. Prove the last example: that the projection maps $\mathrm{pr}_1\colon M \times N \to M$ and $\mathrm{pr}_2\colon M \times N \to N$ are smooth.

Proposition 4.6 (Composition of Smooth Maps)**.** *The composition of two smooth maps is again smooth.*

Proof. Let $F\colon M \to N$ be smooth and $G\colon N \to P$ be smooth. Fix $x \in M$. There exist charts $\varphi\colon U \to \widetilde{U}$, $\psi\colon V \to \widetilde{V}$, and $\chi\colon W \to \widetilde{W}$ about x, $F(x)$, and $G \circ F(x)$, resp., such that $F(U) \subseteq V$ and $G(V) \subseteq W$, and smooth maps $\widetilde{F}\colon \widetilde{U} \to \widetilde{V}$ and $\widetilde{G}\colon \widetilde{V} \to \widetilde{W}$ such that

$$F|_U = \psi^{-1} \circ \widetilde{F} \circ \varphi$$
$$G|_V = \chi^{-1} \circ \widetilde{G} \circ \psi.$$

Thus,

$$G \circ F|_U = (\chi^{-1} \circ \widetilde{G} \circ \psi) \circ (\psi^{-1} \circ \widetilde{F} \circ \varphi) = \chi^{-1}\widetilde{G} \circ \widetilde{F} \circ \varphi.$$

Since $\widetilde{G} \circ \widetilde{F}$ is smooth and x is arbitrary, this shows that $G \circ F$ is smooth. □

Example 4.7 (The Exponential Map)**.** The **exponential map** $\exp\colon \mathbb{R} \to \mathbb{S}^1\colon \theta \mapsto e^{i\theta} = (\cos\theta, \sin\theta)$ is smooth. (Why?) //

The previous example generalises.

Proposition 4.8 (Smooth Maps to a Product Manifold)**.** *Given manifolds M, $N_1, \ldots, N_k$, a map $F\colon M \to \prod_{i=1}^k N_i$ is smooth if and only if $\mathrm{pr}_i \circ F\colon M \to N_i$ is smooth for $i = 1, \ldots, k$.*

★Exercise 4.9. Prove Proposition **4.8**.

We now define the "isomorphisms" for smooth manifolds.

Definition 4.10 (Diffeomorphism)**.** Let M and N be manifolds and $F\colon M \to N$ a function. Then F is a **diffeomorphism** if it is a smooth bijection with smooth inverse. ◇

Example 4.11 ($\mathbb{R}^n$ and the n-Ball)**.** The following functions serve as diffeomorphisms from the unit ball centred at the origin in $\mathbb{R}^n$ to $\mathbb{R}^n$ itself:

$$f_1(x) = \frac{x}{\sqrt{1 - |x|^2}},$$
$$f_2(x) = \tan\left(\frac{\pi}{2}|x|^2\right) x.$$

//

Exercise 4.12. Proving that f_1 and f_2 of Example **4.11** are indeed diffeomorphisms is a very good exercise. You have the technology to do this exercise now, although the next section may also be useful.

Exercise 4.13. Let $F\colon M \to N$ be smooth. The **graph** of F is the set $\Gamma_F := \{(x, y) \in M \times N \mid y = F(x)\}$. Construct a smooth atlas on Γ_F such that the resulting smooth manifold is diffeomorphic to M.

The Implicit Function Theorem. The Implicit Function Theorem is one of the most powerful theorems to come out of the study of differentiable maps between Cartesian spaces. In this section we use it to prove the Inverse Function Theorem, as well as to prove that levels sets of regular values of smooth functions are smooth manifolds.

Definition 4.14 (Partial Jacobian)**.** Let $U \subseteq \mathbb{R}^m$ be open and $F\colon \mathbb{R}^m \to \mathbb{R}^n$ a C^1 map with $m \geq n$. Define the **partial Jacobian** of F by

$$\frac{\partial(F_1, \ldots, F_n)}{\partial(x_{k_1}, \ldots, x_{k_n})} := \det \left[\frac{\partial F_i}{\partial x_{k_j}} \right]_{i,j=1}^n .$$

Here, typically $k_1, \ldots, k_n$ is some increasing sequence of integers from the set $\{1, \ldots, m\}$. The matrix which we take the determinant of is often called the **partial Jacobian matrix**. $\diamond$

Theorem 4.15 (The Implicit Function Theorem)**.** *Let $U \subseteq \mathbb{R}^{m+n}$ be open and $F\colon U \to \mathbb{R}^n$ be C^k where $k \in (\mathbb{N} \smallsetminus \{0\}) \cup \{\infty\}$. If $(x_0, y_0) \in \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$ such that $F(x_0, y_0) = 0$ and*

$$\frac{\partial(F_1, \ldots, F_n)}{\partial(y_1, \ldots, y_n)}\bigg|_{(x_0,y_0)} \neq 0,$$

then there exist an open neighbourhood $V \subseteq \mathbb{R}^m$ of x_0 and a unique C^k function $\phi\colon V \to \mathbb{R}^n$ such that $\phi(x_0) = y_0$ and $F(x, \phi(x)) = 0$ for all $x \in V$.

Presentation 1. Prove the Implicit Function Theorem (without using the Inverse Function Theorem below).

Theorem 4.16 (The Inverse Function Theorem)**.** *Let $U \subseteq \mathbb{R}^m$ be open and $F\colon U \to \mathbb{R}^m$ a C^k where $k \in (\mathbb{N} \smallsetminus \{0\}) \cup \{\infty\}$. If $x_0 \in U$ and $\det(Df|_{x_0}) \neq 0$, then there exists an open neighbourhood V of x_0 on which $F|_V$ is a diffeomorphism onto its image.*

Presentation 2. Show that the Implicit Function Theorem and the Inverse Function Theorem are equivalent; that is, if you assume one, you can prove the other.

Definition 4.17 (Regular Points and Values)**.** Let $U \subseteq \mathbb{R}^m$ be open, and $F\colon U \to \mathbb{R}^n$ be smooth. A point $x \in U$ is **regular point of** F if $DF|_x$ is surjective; otherwise, it is a critical point. A point $y \in \mathbb{R}^n$ is a **regular value of** F if every $x \in F^{-1}(y)$ is a regular point; otherwise, y is a **critical value of** F. $\diamond$

★Exercise 4.18. Let $U \subseteq \mathbb{R}^m$ be open, and $F\colon U \to \mathbb{R}^n$ be smooth. Given a regular value $y_0 \in \mathbb{R}^n$, show that the preimage $F^{-1}(y)$ admits a smooth manifold structure in a very natural way. (Hint: Use the Implicit Function Theorem and Exercise **4.13**.)

We will study precisely what the "natural way" mentioned in the previous exercise above is later, but this is not needed for the construction of the smooth atlas.

Example 4.19 (The n-Sphere as a Level Set)**.** By the previous exercise, $\mathbb{S}^n$ is a smooth manifold, since it is the level set of $F(x_1, \ldots, x_{n+1}) := x_1^2 + \cdots + x_{n+1}^2$ at the regular value 1. //

Remark 4.20. Everything in this section can be generalised from open subsets of Euclidean spaces to smooth manifolds, using charts. ⌟

Week 05: Lie Groups

Topological groups combine the algebraic theory of groups with topology. We go further and combine groups with smooth manifolds, yielding Lie groups.

Definition 5.1 (Lie Group)**.** A **Lie group** is a topological group G equipped with a smooth atlas making it into a smooth manifold such that the multiplication map $m\colon G \times G \to G\colon (g,h) \mapsto gh$ is smooth. $\diamond$

★**Exercise 5.2.** Given a Lie group G, prove that the inversion map $\operatorname{inv}\colon G \to G\colon g \mapsto g^{-1}$ is smooth.

Example 5.3 (Finite Group)**.** Any finite group is a Lie group. (It is a 0-dimensional manifold.) //

Example 5.4 (General Linear Group)**.** We saw that $\mathrm{Mat}(n;\mathbb{R})$, the $n \times n$ matrices with real entries, form a space homeomorphic to $\mathbb{R}^{n^2}$; in fact, this is a diffeomorphism. We also saw that $\mathrm{GL}(n;\mathbb{R})$ is an open subset of this, which makes it a smooth manifold by Example **3.22**. Since matrix multiplication is smooth (why?) and inverse is smooth (why?), this makes $\mathrm{GL}(n;\mathbb{R})$ into a Lie group. //

Example 5.5 (Special Linear Group)**.** The **special linear group**, denoted $\mathrm{SL}(n;\mathbb{R})$ is the subgroup of $\mathrm{GL}(n;\mathbb{R})$ given by

$$\mathrm{SL}(n;\mathbb{R}) := \{A \in \mathrm{GL}(n;\mathbb{R}) \mid \det A = 1\}.$$

It is a closed subgroup of $\mathrm{GL}(n;\mathbb{R})$, and is a "Lie subgroup" of it as well, meaning that it is also a Lie group whose smooth structure is obtained from that of $\mathrm{GL}(n;\mathbb{R})$. //

Exercise 5.6. Let $x_1, \dots, x_n \in \mathbb{R}^n$ be linearly independent vectors. Let P be the parallelopiped determined by $\{x_1, \dots, x_n\}$. Show that the volume of P is equal to the volume of AP (the parallelopiped determined by $\{Ax_1, \dots, Ax_n\}$) for any $A \in \mathrm{SL}(n;\mathbb{R})$. Then show the converse: any matrix $A \in \mathrm{Mat}(n;\mathbb{R})$ that preserves the volume of all parallelopipeds P as described above is in $\mathrm{SL}(n;\mathbb{R})$.

Example 5.7 (Orthogonal Group)**.** The **orthogonal group**, denoted $\mathrm{O}(n)$, is the subgroup of $\mathrm{GL}(n;\mathbb{R})$ given by

$$\mathrm{O}(n) := \{A \in \mathrm{GL}(n;\mathbb{R}) \mid AA^{\mathrm{t}} = I\}$$

where I is the identity matrix (equal to $\mathrm{id}_{\mathbb{R}^n}$). This is a closed subgroup of $\mathrm{GL}(n;\mathbb{R})$, and thus is a Lie subgroup. //

Exercise 5.8. Let $\langle x, y\rangle$ be the standard inner product (or dot product) for $x, y \in \mathbb{R}^n$, and let $A \in \mathrm{O}(n)$. Show that $\langle Ax, Ay\rangle = \langle x, y\rangle$ for all $x, y \in \mathbb{R}^n$. Use this to show that the columns of A form an orthonormal basis of $\mathbb{R}^n$. Then show the converses, obtaining that $A \in \mathrm{O}(n)$ if and only if $\langle Ax, Ay\rangle = \langle x, y\rangle$ for all $x, y \in \mathbb{R}^n$ if and only if the columns of A form an orthonormal basis of $\mathbb{R}^n$.

Example 5.9 (Unitary Group)**.** We can define $\mathrm{GL}(n;\mathbb{C})$ analogously to $\mathrm{GL}(n;\mathbb{R})$: invertible $(n\times n)$-matrices with complex entries. This is a Lie group, in the same way that $\mathrm{GL}(n;\mathbb{R})$ is (it is a subspace of $\mathrm{Mat}(n;\mathbb{C})$, which is identified with $\mathbb{C}^{n^2}\cong\mathbb{R}^{2n^2}$). Recall that the **adjoint** of a matrix $A\in\mathrm{Mat}(n;\mathbb{C})$ is the conjugate transpose A^*: if $A=[a_{ij}]$, then $A^*=[\overline{a_{ji}}]$. The **unitary group**, denoted $\mathrm{U}(n)$, is the subgroup of $\mathrm{GL}(n;\mathbb{C})$ given by

$$\mathrm{U}(n):=\{A\in\mathrm{GL}(n;\mathbb{C})\mid AA^*=I\}.$$

This is a closed subgroup of $\mathrm{GL}(n;\mathbb{C})$, and thus is a Lie subgroup. //

Exercise 5.10. Let $\langle z,w\rangle_{\mathbb{C}}$ be the **Hermitian inner product** on $\mathbb{C}^n$; that is, for $z,w\in\mathbb{C}^n$,

$$\langle z,w\rangle_{\mathbb{C}}:=\sum_{i=1}^{n}\overline{z}_i w_i.$$

Show that $\langle Az,Aw\rangle_{\mathbb{C}}=\langle z,w\rangle_{\mathbb{C}}$ for any $z,w\in\mathbb{C}^n$ and any $A\in\mathrm{U}(n)$. Use this to show that the columns of A form a "unitary basis" of $\mathbb{C}^n$ as a complex vector space equipped with the Hermitian inner product. (A **unitary basis** has the same definition as an orthonormal basis, but with respect to the Hermitian inner product.) Then show the converses, obtaining that $A\in\mathrm{U}(n)$ if and only if $\langle Az,Aw\rangle_{\mathbb{C}}=\langle z,w\rangle_{\mathbb{C}}$ for all $z,w\in\mathbb{C}^n$ if and only if the columns of A form a unitary basis of $\mathbb{C}^n$ with respect to the Hermitian inner product.

Exercise 5.11. We often identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$. What happens to the Hermitian inner product under this identification? Show that if $z=(z_1,\dots,z_n)$ and $w=(w_1,\dots,w_n)$ in $\mathbb{C}^n$ and each $z_j=x_j+iy_j$ and $w_j=u_j+iv_j$ for $j=1,\dots,n$, then

$$\langle z,w\rangle_{\mathbb{C}}=\langle (x,y),(u,v)\rangle+i(x,y)^{\mathrm{t}}\Omega(u,v)$$

where by (x,y) we mean $(x_1,\dots,x_n,y_1,\dots,y_n)$, which is also identified with the corresponding column vector in $\mathbb{R}^{2n}$, and Ω is the **symplectic form** given in block form by

$$\Omega=\begin{bmatrix}0 & I_n\\ -I_n & 0\end{bmatrix}.$$

(Here, I_n is the $(n\times n)$-identity matrix.)

Example 5.12. The **symplectic group**, denoted $\mathrm{Sp}(2n,\mathbb{R})$, is the subgroup of $\mathrm{GL}(2n;\mathbb{R})$ given by

$$\mathrm{Sp}(2n;\mathbb{R}):=\{A\in\mathrm{GL}(2n;\mathbb{R})\mid A^{\mathrm{t}}\Omega A=\Omega\}$$

where Ω is the symplectic form from Exercise **5.11**. This is a closed subgroup of $\mathrm{GL}(2n;\mathbb{R})$, and hence a Lie subgroup. //

Examples 5.13. Define $\mathrm{SO}(n):=\mathrm{O}(n)\cap\mathrm{SL}(n;\mathbb{R})$ and $\mathrm{SU}(n):=\mathrm{U}(n)\cap\mathrm{SL}(n;\mathbb{C})$ (where $\mathrm{SL}(n;\mathbb{C})$ is defined as all $(n\times n)$-matrices with complex entries and determinant 1). These are called the **special orthogonal group** and **special unitary group**, resp. //

Exercise 5.14. Prove that $\mathrm{SO}(2)$ and $\mathrm{U}(1)$ are each diffeomorphic to $\mathbb{S}^1$; moreover, the resulting diffeomorphism between $\mathrm{SO}(2)$ and $\mathrm{U}(1)$ can be chosen to be a group isomorphism. (This is called a **Lie group isomorphism**.)

Week 06: Lie Group Actions

In this section we begin to examine Lie group actions. We will immediately find that we need to cover more material on manifolds and general topology to make sense of what is happening.

Basics of Lie Group Actions.

Definition 6.1 (Lie Group Action)**.** Let M be a manifold and G a Lie group. A **(left) Lie group action** of G on M is a group action $G \circlearrowright M$ such that the action map $\alpha \colon G \times M \to M \colon (g, x) \mapsto g \cdot x$ is smooth. $\diamond$

Example 6.2 (Matrix Group Action)**.** Let G be a Lie subgroup of $\mathrm{GL}(n;\mathbb{R})$. Then G acts on $\mathbb{R}^n$ via matrix multiplication. //

Proposition 6.3 (Stabilisers are Lie Subgroups)**.** *Let G be a Lie group acting on a manifold M. For each $x \in M$, the stabiliser $\mathrm{Stab}(x)$ is a closed, hence Lie, subgroup of G.*

Proof. Fix $x \in M$. Then $\mathrm{Stab}(x) := \{g \in G \mid g \cdot x = x\}$, which we showed is a subgroup of G (see Exercise **1.8**). Now the action map $\alpha \colon G \times M \to M$ is smooth, and restricts to a smooth map $G \times \{x\} \cong G \to M$ sending g to $g \cdot x$. The pre-image of $\{x\}$ is a closed set since α is continuous, and is exactly the stabiliser of x in G. $\square$

Proposition 6.4 (Actions as Representations)**.** *An action of a Lie group G on a manifold M induces a group homomorphism $\rho \colon G \to \mathrm{Diff}(M)$, the group of all diffeomorphisms of M. Conversely, any such homomorphism induces an action.*

Proof. Let G act on M. Then any $g \in G$ induces a function $M \to M \colon x \mapsto g \cdot x$, which is smooth since the action map is smooth. Moreover, this is invertible (with inverse g^{-1}), and so g induces a diffeomorphism on M. That we obtain a group homomorphism $G \to \mathrm{Diff}(M)$ now follows from the definition of an action.

Conversely, given a group homomorphism $\rho \colon G \to \mathrm{Diff}(M)$, define an action of G on M as follows: $g \cdot x := \rho(g)(x)$. That this is an action follows from the definition of a group homomorphism. $\square$

We also would like to talk about the structure of the orbits, as well as the orbit space. We, in fact, start with the orbit space. Since, these can be complicated, we first need to introduce some conditions on a group action to "tame" the orbit space. This, requires introducing what we mean by a "compact manifold".

Definition 6.5 (Compact Space)**.** A topological space X is **compact** if any open cover of it has a finite subcover. $\diamond$

If the manifold can be "embedded" into Cartesian space as a closed and bounded subset, then it follows from Heine-Borel that the manifold is compact. This proves to be a useful way of telling when a manifold is compact instead of using open covers.

Examples 6.6 (Examples of Compact and Non-compact Spaces)**.**

(1) The manifolds $\mathbb{S}^n$ and $\mathbb{T}^n$ are compact.

(2) Any finite set is a compact manifold.

(3) $\mathbb{R}^n$ is not compact.

(4) The cylinder $\mathbb{S}^1 \times \mathbb{R}$ is not compact.

(5) $\mathrm{GL}(n;\mathbb{R})$ (for $n > 0$) and $\mathrm{SL}(n;\mathbb{R})$ (for $n > 1$) are not compact. (Why?) //

Recall that a continuous map sends compact sets to compact sets. There is another type of map that does the opposite: requiring the pre-image of a compact set to be compact.

Definition 6.7 (Proper Map)**.** Let $f\colon X \to Y$ be a continuous map between topological spaces. Then f is **proper** if for any compact $C \subseteq Y$, the pre-image $f^{-1}(C)$ is compact in X. ◇

★**Exercise 6.8.** Any polynomial $p \in P(\mathbb{R})$ is proper. However, polynomials in $P(\mathbb{R}^n)$ for $n > 1$ are generally not proper.

We will use the following theorem without proof.

Theorem 6.9 (Bolzano-Weierstrass)**.** *Let M be a manifold and $C \subseteq M$. Then C is compact if and only if for any sequence (x_n) in C, there is a convergent subsequence (x_{n_i}) whose limit is contained in C.*

★**Exercise 6.10.** Prove Theorem **6.9**.

Definition 6.11 (Types of Actions)**.** Let G be a Lie group acting on a manifold M. Denote by $\chi\colon G \times M \to M \times M$ the smooth map sending (g, x) to $(x, g \cdot x)$.

(1) The action is **free** if all of the stabilisers are trivial. Equivalently, χ is injective.

(2) The action is **effective** if $\bigcap_{x \in M} \mathrm{Stab}(x) = \{1_G\}$. Equivalently, the corresponding representation $\rho\colon G \to \mathrm{Diff}(M)$ is injective.

(3) The action is **transitive** if M is the only orbit; that is, for any $x, y \in M$, there exists $g \in G$ such that $y = g \cdot x$. Equivalently, χ is surjective.

(4) The action is **proper** if χ is proper. ◇

Example 6.12 (Group Multiplication)**.** A Lie group G acts on itself via left and right multiplication. These actions are free and transitive. //

Example 6.13 (A Non-Effective Action)**.** Any action of a non-trivial group on a point is non-effective. For a more complicated example, $\mathrm{O}(n)$ acts on $\mathbb{R}$ by $(g, x) \mapsto \det(g)x$; this action is not effective for $n > 1$. //

Example 6.14 (Rotations & Reflections)**.** $\mathrm{O}(n)$ acts on $\mathbb{R}^n$ by rotations and reflections: given $A \in \mathrm{O}(n)$, this matrix sends the standard orthonormal basis of $\mathbb{R}^n$ to the columns of A, which recall is an orthonormal basis. If $A \in \mathrm{SO}(n)$, then A is a rotation (why?). In either case, any $(n-1)$-sphere centred at the origin is an orbit of this action. Thus, restricting the action to such a sphere yields a transitive action. //

Proposition 6.15 (Actions of Compact Groups are Proper)**.** *Any action of a compact Lie group is proper.*

Proof. Let G be a compact Lie group acting on a manifold M. We need to show that the map $\Phi\colon G \times M \to M \times M$ sending (g, x) to $(x, g \cdot x)$ is proper. Fix a compact set $C \subseteq M \times M$. We need to show that $\Phi^{-1}(C)$ is compact. By Bolzano-Weierstrass, it is enough to show that any sequence $(g_n, x_n) \in \Phi^{-1}(C)$ has a convergent subsequence (g_{n_i}, x_{n_i}) with limit contained in $\Phi^{-1}(C)$. Fix a sequence $(g_n, x_n) \in \Phi^{-1}(C)$. Then $\Phi(g_n, x_n) \in C$; in particular, $(x_n) \in \mathrm{pr}_1(C) \in M$. Since continuous maps send compact sets to compact sets, $\mathrm{pr}_1(C)$ is compact. By Bolzano-Weierstrass applied to $\mathrm{pr}_1(C)$, the sequence (x_n) has a convergent subsequence (x_{n_i}) that converges to some $x \in \mathrm{pr}_1(C)$, and similarly since G is compact, the sequence (g_{n_i}) has a convergent subsequence $(g_{n_{i_j}})$ that converges to some $g \in G$. Thus $(g_{n_{i_j}}, x_{n_{i_j}})$ converges to $(g, x) \in G \times M$; we only need to show that this is in $\Phi^{-1}(C)$. However, Φ is continuous and C closed, and so $\Phi^{-1}(C)$ is closed. Since $(g_{n_{i_j}}, x_{n_{i_j}})$ is contained in $\Phi^{-1}(C)$, its limit is also. □

***Quotient Topologies*.** We next want to consider the orbit space of a Lie group action. These are naturally topological spaces, however, we need to describe how.

Definition 6.16 (Quotient Topology)**.** Let X be a topological space, and $\sim$ an equivalence relation on it. We define a topology on $X/\sim$, the quotient set, as follows: let $\pi\colon X \to X/\sim$ be the quotient map sending x to its equivalence class $[x]$. A subset $U \subseteq X/\sim$ is open if the preimage $\pi^{-1}(U)$ is open in X. We call this the **quotient topology**. ◇

Exercise 6.17. Prove that the quotient topology is in fact a topology.

From now on, we can assume that the orbit space of a Lie group action is equipped with the quotient topology.

Example 6.18 ($\mathrm{GL}(n;\mathbb{R})$)**.** Consider $\mathrm{GL}(n;\mathbb{R})$ acting on $\mathbb{R}^n$ be matrix multiplication. Its orbit space consists of two points (why?). The topology, however, is non-Hausdorff: the open sets consist of the empty set, the complement of $[0]$, and the whole quotient. //

When is an orbit space Hausdorff? To answer this, we first obtain a crucial property of proper maps.

Definition 6.19 (Open and Closed Maps)**.** Let $f\colon X \to Y$ be a continuous map. Then f is **open/closed** if the image of every open/closed set is open/closed. ◇

Proposition 6.20 (Proper Maps are Closed). *Proper continuous maps between manifolds are closed.*

Proof. Fix a proper continuous map $f\colon M \to N$, and a closed set $C \subseteq M$. To show that $f(C)$ is closed, we show that every convergent sequence in $f(C)$ achieves its limit in $f(C)$. Fix a sequence (y_n) in $f(C)$ that converges to $y \in N$. Then y is in the closure of $f(C)$. Let K be a compact neighbourhood of y. Then it follows from the definitions that there exists $n_0 \in \mathbb{N}$ such that K contains all y_n for $n \geq n_0$; without loss of generality, assume that K contains all y_n. There exists a sequence (x_n) in $C \cap f^{-1}(K)$ such that $f(x_n) = y_n$ for each n. Moreover, since f is proper, $f^{-1}(K)$ is compact, and since C is closed, so is $C \cap f^{-1}(K)$. Thus, by Bolzano-Weierstrass, there is a subsequence (x_{n_i}) of (x_n) that converges to $x \in C \cap f^{-1}(K)$. By continuity, $y_{n_i} = f(x_{n_i})$ converges to $f(x)$, and since limits of sequences in Hausdorff spaces are unique, it follows that $f(x) = y$. But $f(x) \in f(C)$, which completes the proof. ☐

Lemma 6.21. *The quotient map of a Lie group action is open.*

Proof. Let G be a Lie group acting on a manifold M, and let $\pi\colon M \to G\backslash M$ be the quotient map. Fix an open set $U \subseteq M$. To show that $\pi(U)$ is open, we must show that $\pi^{-1}(\pi(U))$ is open. However,

$$\pi^{-1}(\pi(U)) = \bigcup_{g \in G} g \cdot U$$

where $g \cdot U := \{g \cdot x \in M \mid g \in G,\ x \in U\}$. By Proposition **6.4**, each g acts by diffeomorphisms, and so $g \cdot U$ is open for each g, and thus $\pi^{-1}(\pi(U))$ is a union of open sets, and hence open. ☐

Exercise 6.22. A topological space X is Hausdorff if and only if the **diagonal map** $\Delta\colon X \to X \times X\colon x \mapsto (x, x)$ has closed image. (We typically call this image the **diagonal**.)

Proposition 6.23 (Hausdorff Orbit Space). *A proper Lie group action has a Hausdorff orbit space.*

Proof. Let a Lie group G act on a manifold M properly, and let $\pi\colon M \to G\backslash M$ be the quotient map. By Proposition **6.20**, the image of the map $\chi\colon G \times M \to M \times M$ sending (g, x) to $(x, g \cdot x)$ is closed (since $G \times M$ is closed). Define $\Pi\colon M \times M \to \left(G\backslash M\right) \times \left(G\backslash M\right)$ by $\Pi(x, y) := (\pi(x), \pi(y))$. By Lemma **6.21**, π is open, from which it follows that Π is open (check this!). Thus, Π sends the complement of the image of χ to an open set W in $\left(G\backslash M\right) \times \left(G\backslash M\right)$; we claim this is the complement of the diagonal of $\left(G\backslash M\right) \times \left(G\backslash M\right)$, which would complete the proof.

A point $([x], [y])$ is in W if and only if $(x, y) \notin \operatorname{im}(\chi)$ if and only if there does not exist $g \in G$ such that $y = g \cdot x$ if and only if $[x] \neq [y]$ if and only if $([x], [y])$ is not in the diagonal. This completes the proof. ☐

Exercise 6.24 (Second Countable Orbit Space)**.** Show that the orbit space of a proper Lie group action is second-countable.

We now want to go one step further and see when we actually get a smooth manifold as an orbit space.

Theorem 6.25 (Quotient Manifold Theorem)**.** *The orbit space of a proper and free Lie group action obtains a unique smooth manifold structure induced by the original manifold.*

Presentation 3. Prove Theorem **6.25**.

We will unravel what is meant by "obtains" and "induces" soon.

Exercise 6.26. Let G be a Lie group with Lie subgroup H. Show that ${}_{H}\diagdown^{G}$ is a smooth manifold. (These manifolds are called **homogeneous manifolds**; they are important, showing up in many different places in geometry.)

You may remember from calculus that tangent lines/planes to graphs of functions are intimately connected to their derivatives. How can we generalise this idea?

Tangent Vectors.

Definition 7.1 (Tangent Vectors)**.** Let M be a manifold and $x \in M$. A **derivation at** x of $C^\infty(M)$ (recall that this is the ring of smooth real-valued functions) is a linear map $v\colon C^\infty(M) \to \mathbb{R}$ that satisfies Leibniz' Rule (*i.e.* the product rule):

$$v(fg) = f(x)v(g) = g(x)v(f).$$

Denote the set of all derivations at x of $C^\infty(M)$ by T_xM, called the **tangent space** of M at x. ◇

Exercise 7.2. Show that T_xM is a vector space.

To explore what these tangent vectors really look like, we will need some further definitions.

Definition 7.3 (Pullbacks and Pushforwards)**.** Let M and N be manifolds, fix $x \in M$, and let $F\colon M \to N$ be smooth. For any $f \in C^\infty(N)$, define the **pullback of** f **by** F to be the function $F^*f \in C^\infty(M)$ given by

$$F^*f(x) := f \circ F(x).$$

For any $v \in T_xM$, define the **pushforward of** v **by** F to be $F_*v \in T_{F(x)}N$ defined for $f \in C^\infty(N)$ by

$$F_*v(f) := v(F^*f).$$ ◇

Exercise 7.4. Show that F_*v in Definition **7.3** is indeed a tangent vector in $T_{F(x)}N$.

Remark 7.5. The pushforward F_* is also denoted dF and TF in the literature. ⌟

Exercise 7.6. Let $F\colon M \to N$ be a smooth map between manifolds, $x \in M$, $\varphi\colon U \to \widetilde{U} \subseteq \mathbb{R}^m$ a chart about x, $\psi\colon V \to \widetilde{V} \subseteq \mathbb{R}^n$ a chart about $F(x)$, and $\widetilde{F} = \psi \circ F \circ \varphi^{-1}$ the associated representative of F using charts. Then on T_xM,

$$F_* = \psi_* \circ D\widetilde{F}|_{\varphi(x)} \circ \varphi_*^{-1}.$$

Now let M be a manifold, $x \in M$, $v \in T_xM$, and let $\varphi\colon U \to \widetilde{U} \subseteq \mathbb{R}^m$ be a chart about x. Then φ is a diffeomorphism (why?), and it suffices to explore φ_*v. To this end, we need some lemmas.

Lemma 7.7. *Let $x_0 \in \mathbb{R}^m$ and $v \in T_{x_0}\mathbb{R}^m$. If $c\colon \mathbb{R}^m \to \mathbb{R}$ is constant, then $v(c) = 0$.*

Proof. Since v is linear, it suffices to show this for $c = 1$. In this case,

$$v(1) = v(1 \cdot 1) = 1 \cdot v(1) + 1 \cdot v(1) = 2v(1),$$

from the claim follows. □

Notation 7.8. On $\mathbb{R}^m$, denote by ∂_i the ith partial derivative operator $\frac{\partial}{\partial q^i}$ where q^i is the ith coordinate function (sending $x = (x_1, \ldots, x_m)$ to $x_i \in \mathbb{R}$). In differential geometry, it is customary to use superscripts for coordinate functions in this way. $\diamond$

Lemma 7.9. *Let $x_0 \in \mathbb{R}^m$ and $v \in T_{x_0}(\mathbb{R}^m)$. Then*

$$v = \sum_{i=1}^{m} v^i \partial_i|_{x_0}$$

where $v^i := v(q^i)$.

Proof. Fix $f \in C^\infty(\mathbb{R}^m)$. Applying the right-hand side to f yields

$$\sum_{i=1}^{m} v(q^i)\partial_i|_{x_0} f = \sum_{i=1}^{m} v^i(\partial_i f)(x_0). \tag{2}$$

On the other hand, by Taylor's Theorem (see the appendices of [Lee13]), there exist $g_i \in C^\infty(M)$ such that $g_i(x_0) = 0$ for each i and

$$f(x) = f(x_0) + \sum_{i=1}^{m} \partial_i f(x_0)(q^i(x) - q^i(x_0)) + \sum_{i=1}^{m} g_i(x)(q^i(x) - q^i(x_0)).$$

Applying v:

$$\begin{aligned} v(f) &= 0 + \sum_{i=1}^{m} \partial_i f(x_0)(v(q^i) - v(q^i(x_0))) + \sum_{i=1}^{m} v(g_i \cdot (q_i - q^i(x_0))) \\ &= \sum_{i=1}^{m} v(q^i)\partial_i|_{x_0} f + \sum_{i=1}^{m} (g_i(x_0)v(q^i) + v(g_i) \cdot (q^i(x_0) - q^i(x_0))) \end{aligned}$$

where the last term vanishes, leaving the right-hand side of Equation (2). $\square$

This is a good opportunity to introduce Einstein notation.

Notation 7.10 (Einstein Notation)**.** Instead of writing something like $v = \sum_{i=1}^{m} v^i \partial_i|_{x_0}$, we will drop the sum and understand that when a term contains an index, such as i, as a superscript and as a subscript, then we sum over this index (the index set being understood from the context). Thus, in the statement of Lemma **7.9**, we could write $v = v^i \partial_i|_{x_0}$ instead of using the sum. As another example, if $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, the dot product of x and y can be written $x^i y_i$. Thus we can view raising the index i from a subscript to a superscript as taking the transpose of the column matrix representing x. $\diamond$

★Exercise 7.11. Let M be a manifold and fix $x_0 \in M$. Denote by $(v^i_\varphi \partial_i\Big|_{\varphi(x_0)})$ the family of derivations of $C^\infty(\mathbb{R}^m)$ at $\varphi(x_0)$ as φ runs over all charts about x_0, and such that if $\varphi\colon U \to \widetilde{U}$ and $\psi\colon V \to \widetilde{V}$ are two charts about x_0 with transition function $\widetilde{F}\colon \widetilde{U} \to \widetilde{V}$, then

$$D\widetilde{F}|_{\varphi(x_0)}(v^i_\varphi \partial_i|_{\varphi(x_0)}) = v^i_\psi \partial_i|_{\psi(x_0)}.$$

Show that these form a linear space, denoted $\mathcal{V}_{x_0}$, which is naturally linearly isomorphic to $T_{x_0}M$. Consequently, if $v \in T_{x_0}M$ and φ and ψ are two charts about x_0 with transition function $\widetilde{F}$, then $\psi_* v = D\widetilde{F}|_{\psi(x_0)}(\varphi_* v)$.

Corollary 7.12. *If M is a manifold, $x \in M$, and φ a chart about x, then φ_* is a linear isomorphism from T_xM to $T_{\varphi(x)}\mathbb{R}^m$. In particular, T_xM is m-dimensional.*

Exercise 7.13. Prove Corollary **7.12**.

There is a more geometric way of defining T_xM.

Definition 7.14 (Tangent Vector II)**.** Let M be a manifold and fix $x \in M$. Let $\mathcal{C}_x$ be the family of all smooth curves $\gamma\colon \mathbb{R} \to M$ such that $\gamma(0) = x$. Define an equivalence relation on $\mathcal{C}_x$ as follows: $\gamma_1 \sim \gamma_2$ if there exists a chart $\varphi\colon U \to \widetilde{U}$ about x such that

$$\frac{d}{dt}\Big|_{t=0}(\varphi \circ \gamma_1) = \frac{d}{dt}\Big|_{t=0}(\varphi \circ \gamma_2). \qquad \diamond$$

★**Exercise 7.15.** Show that $\sim$ in Definition **7.14** is indeed an equivalence relation, and independent of the chart φ chosen (you may want to look at ★Exercise **7.11**). Then show that there is a natural bijection from $\mathcal{C}/\sim$ to T_xM.

Definition 7.16 (Tangent Bundle)**.** Given a manifold M, define the **tangent bundle** TM as a set to be the disjoint union

$$\coprod_{x \in M} T_xM.$$

Let $\tau\colon TM \to M\colon v \mapsto x$ for $v \in T_xM$ be the projection sending a vector to the point above which it is defined. $\diamond$

Proposition 7.17 (Tangent Bundles are Manifolds)**.** *Let M be an m-manifold. Then TM is a $2m$-manifold.*

★**Exercise 7.18.** Prove Proposition **7.17**.

Example 7.19 (Tangent Bundle to $\mathbb{S}^1$)**.** $T\mathbb{S}^1$ is diffeomorphic to the cylinder $\mathbb{S}^1 \times \mathbb{R}$. //

Not every manifold has this nice product structure (globally).

Example 7.20 (Tangent Bundle to $\mathbb{S}^2$)**.** The tangent bundle to the 2-sphere is *not* diffeomorphic $\mathbb{S}^2 \times \mathbb{R}^2$. //

Exercise 7.21 (Tangent Bundle to a Product)**.** If $M_1, \dots M_k$ are manifolds, show

$$T(M_1 \times \cdots \times M_k) \cong TM_1 \times \dots TM_k.$$

Proposition 7.22 (Properties of the Pushforward)**.** *Let $F\colon M \to N$ and $G\colon N \to P$ be smooth maps of manifolds. Then*

(1) $(G \circ F)_* = G_* \circ F_*$,

(2) $(\mathrm{id}_M)_* = \mathrm{id}_{TM}$,

(3) *if F is a diffeomorphism, then F_* is invertible and $(F_*)^{-1} = (F^{-1})_*$, and*

(4) *$F_* \colon TM \to TN$ is smooth. Thus if F is a diffeomorphism, so is F_*.*

★**Exercise 7.23.** Prove Proposition **7.22**.

Week 08: Global Derivations

One benefit of tangent bundles is that it allows us to discuss derivatives of smooth maps between manifolds in a coordinate-free way. A second benefit is it provides a natural method of discussing vector fields on manifolds.

Definition 8.1 (Global Derivations)**.** Let M be a smooth manifold. A **global derivation of** $C^\infty(M)$ is a linear map $X\colon C^\infty(M) \to C^\infty(M)$ satisfying Leibniz' Rule: for all $f, g \in C^\infty(M)$,

$$X(fg) = fX(g) + gX(f).$$

Denote the set of all global derivations by $\mathrm{Der}C^\infty(M)$. For $x \in M$ and $X \in \mathrm{Der}C^\infty(M)$, denote by $X|_x$ the map $C^\infty(M) \to \mathbb{R}$ sending f to $(Xf)(x)$. ◇

Exercise 8.2. $\mathrm{Der}C^\infty(M)$ is a (left) $C^\infty(M)$-module: given a global derivation X and $f \in C^\infty(M)$, the product

$$fX\colon g \mapsto fXg\colon x \mapsto f(x)(Xg)(x)$$

is a global derivation.

Exercise 8.3. For any global derivation X and $x \in M$, $X|_x \in T_xM$.

Example 8.4 (Coordinate Global Derivation)**.** On $\mathbb{R}^m$, the differential operator $\partial_i\colon C^\infty(\mathbb{R}^m) \to C^\infty(\mathbb{R}^m)\colon f \mapsto \partial_i f$, sending f to its ith partial derivative, is a global derivation, called a **coordinate global derivation**. //

Definition 8.5 (Pushforward of a Global Derivation by a Diffeomorphism)**.** Let M and N be manifolds, $X \in \mathrm{Der}C^\infty(M)$, and $F\colon M \to N$ a diffeomorphism. The **pushforward of** X **by** F is the global derivation F_*X defined by $F_*X(f) := (F^{-1})^*\left(X(F^*f)\right)$. ◇

Note that we needed F to be a diffeomorphism in Definition 8.5. While it is possible to push forward global derivations by certain types of smooth maps more general than diffeomorphisms, it remains not well-defined for an arbitrary smooth map.

Exercise 8.6. Confirm that F_*X in Definition 8.5 is indeed a global derivation.

Proposition 8.7. *For any $v \in T_xM$, there exists $X \in \mathrm{Der}C^\infty(M)$ such that $v = X|_x$.*

Proof. Fix $v \in T_xM$, and let $\varphi\colon U \to \widetilde{U} \subseteq \mathbb{R}^m$ be a chart about x. By Lemma 7.9,

$$\varphi_*v = v^i\partial_i|_{\varphi(x)}.$$

Let $b\colon M \to \mathbb{R}$ be a smooth bump function equal to 1 on an open neighbourhood V of x such that $\overline{V} \subseteq U$, with **support** $\mathrm{supp}(b) := \overline{\{x \in M \mid b(x) \neq 0\}}$ contained in U. Define $X := b\varphi_*^{-1}(v^i\partial_i)$. For any $f \in C^\infty(M)$, we have

$$Xf = b\varphi^*(v^i\partial_i((\varphi^{-1})^*f).$$

This is linear in f, smooth on M (why?), and so we only need to confirm Leibniz' Rule. Let g also be in $C^\infty(M)$.

$$X(fg) = v^i b\varphi^* \left((\varphi^{-1})^* f \partial_i((\varphi^{-1})^* g) + (\varphi^{-1})^* g \partial_i((\varphi^{-1})^* f)\right) = fXg + gXf.$$

Thus, $X \in \mathrm{Der} C^\infty(M)$.

Finally, for any $f \in C^\infty(M)$,

$$\begin{aligned} X|_x f &= b(x)(v^i \partial_i((\varphi^{-1})^* f)(\varphi(x))) \\ &= v^i \partial_i|_{\varphi(x)}((\varphi^{-1})^* f) \\ &= \varphi_* v((\varphi^{-1})^* f) \\ &= vf. \end{aligned}$$

Thus, $X|_x = v$. □

Definition 8.8 (Smooth Section)**.** A **smooth section** of a tangent bundle TM is a map $\sigma\colon M \to TM$ so that $\tau \circ \sigma = \mathrm{id}_M$. ◇

Theorem 8.9 (Global Derivations are Smooth Sections)**.** $\mathrm{Der} C^\infty(M)$ *is exactly the set of smooth sections of* TM*, where* $X \in \mathrm{Der} C^\infty(M)$ *is defined as a section via* $x \mapsto X|_x$.

★Exercise 8.10. Prove Theorem **8.9**.

Remark 8.11 (Vector Fields)**.** A **vector field** is a global derivation X that admits a **local flow**; that is, for every point $x \in M$ there is a curve $\gamma\colon (-\varepsilon, \varepsilon) \to M$ for some $\varepsilon > 0$ such that $\gamma(0) = x$ and $\frac{d\gamma}{dt} = X|_x$, and these so-called **integral curves** fit together into a smooth family. It turns out that *every* global derivation is a vector field. This follows from the existence and uniqueness theorem of first order differential equations. For this reason, we will often call global derivations on a manifold vector fields. However, on a space such as $[0, \infty)$, where the boundary point 0 does not allow for non-trivial integral curves at that point, it is no longer true that every global derivation is a vector field. ⌟

Lie Brackets. In this subsection we will focus on vector fields (*i.e.* global derivations) on a manifold M; denote vector fields on M by $\mathrm{vect}(M)$. In general, given vector fields $X, Y \in \mathrm{vect}(M)$, the map $f \mapsto X(Yf)$ is *not* a vector field:

Example 9.1 (XY is not a Vector Field)**.** Consider $\frac{d}{dx}\frac{d}{dx}x^5 = 20x^3$. Using the fact that $x^5 = x^2 \cdot x^3$, we can test Leibniz' Rule: if $\frac{d}{dx}\frac{d}{dx}$ were a vector field, then $20x^3$ would be equal to $x^2\frac{d}{dx}\frac{d}{dx}x^3 + x^3\frac{d}{dx}\frac{d}{dx}x^2 = 6x^3 + 2x^3 = 8x^3$, which it is not. //

However, it turns out there is an easy fix to this.

Definition 9.2 (Lie Bracket)**.** Let $X, Y \in \mathrm{vect}(M)$. The **Lie bracket** (of vector fields) is the assignment to each $f \in C^\infty(M)$ the smooth function $X(Y(f)) - Y(X(f))$. ⋄

Lemma 9.3. *The Lie bracket of two vector fields is a vector field.*

Proof. Fix $X, Y \in \mathrm{vect}(M)$ and $f, g \in C^\infty(M)$. It is straightforward to check that $[X, Y]$ is a linear operator from $C^\infty(M)$ to itself, so we only need to check Leibniz' Rule.

$$\begin{aligned}
[X, Y](fg) &= X(Y(fg)) - Y(X(fg)) \\
&= X(fYg + gYf) - Y(fXg + gXf) \\
&= XfYg + fXYg + XgYf + gXYf - YfXg - fYXg - YgXf - gYXf \\
&= f[X, Y]g + g[X, Y]f.
\end{aligned}$$

□

Proposition 9.4 (Properties of the Lie Bracket of Vector Fields)**.** *Let $X, Y, Z \in \mathrm{vect}(M)$.*

(1) ***(Bilinearity)*** *$[X, Y]$ is linear in both X and Y.*

(2) ***(Antisymmetry)*** *$[X, Y] = -[Y, X]$.*

(3) ***(Jacobi Identity)*** *$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.*

Exercise 9.5. Prove Proposition 9.4.

Example 9.6 (Coordinate Computations for Lie Brackets)**.** On $\mathbb{R}^m$, we have that $[\partial_i, \partial_j] = 0$ for all i, j; this is a consequence of Clairaut's Theorem. This does not hold for more general vector fields, however, such as $X = y\partial_1$ and $Y = -x\partial_2$. Indeed, for $f \in C^\infty(\mathbb{R}^m)$, the Lie bracket $[X, Y]f$ is $(-y\partial_2 + x\partial_1)f$. Notice that this is equal to $(X(-x)\partial_2 - Y(y)\partial_1)f$. This suggests the following formula for computing the Lie bracket in coordinates: let $X = X^i\partial_i$ and $Y = Y^j\partial_j$ where $X^i, Y^j \in C^\infty(M)$. Then

$$[X, Y] = X^i\partial_i(Y^j)\partial_j - Y^j\partial_j(X^i)\partial_i. \tag{3}$$

This allows us to do explicit local computations of the Lie bracket on manifolds via charts. //

Exercise 9.7. Prove Equation (3).

The Lie bracket gives a type of "anti-commutative multiplicative" structure to the set $\mathrm{vect}(M)$. This type of structure is called a "Lie algebra".

Definition 9.8 (Lie Algebra)**.** A **(real) Lie algebra** A is an $\mathbb{R}$-vector space equipped with a multiplication $A \times A \to A\colon (a,b) \mapsto [a,b]$, called the **Lie bracket** of A, in which $[\cdot,\cdot]$ is bilinear (linear in each slot), is antisymmetric ($[a,b] = -[b,a]$), and satisfies the Jacobi Identity (see Item **3** of Proposition **9.4**). $\diamond$

Example 9.9 (The Lie Algebra of Vector Fields)**.** Given a manifold M, vector fields $\mathrm{vect}(M)$ form a Lie algebra with the Lie bracket. //

Exercise 9.10. Show that the vector space $\mathbb{R}^3$ equipped with the cross product $\times$ is a Lie algebra.

It turns out that we can put a natural Lie algebra structure onto the tangent space at the identity of a Lie group. To prove this, we need a definition and some results.

Definition 9.11 (Left-Invariant Vector Field)**.** Let G be a Lie group, and let left multiplication by a fixed $h \in G$ be denoted $L_h\colon G \to G\colon g \mapsto hg$. A vector field $X \in \mathrm{vect}(G)$ is **left-invariant** if $((L_h)_*X)|_g = X|_g$ for all $g, h \in G$. $\diamond$

Proposition 9.12 (Left-Invariant Vector Fields)**.** *Let G be a Lie group. Denote by $\mathfrak{g}$ the set of all left-invariant vector fields on G. Then $\mathfrak{g}$ is a Lie algebra with respect to the Lie bracket on vector fields.*

★**Exercise 9.13.** Prove Proposition **9.12**.

Corollary 9.14. *The tangent space at the identity of a Lie group has a natural Lie algebra structure.*

Proof. Let G be a Lie group and let $\xi \in T_{1_G}G$. Then ξ induces a left-invariant vector field X^ξ by defining $X^\xi|_g := (L_g)_*\xi$ (convince yourself that this is actually a vector field). Conversely, any left-invariant vector field is completely determined by its value at 1_G. Thus we have a linear isomorphism E from $\mathfrak{g}$ to $T_{1_G}G$. We define a Lie bracket on $T_{1_G}G$ as follows: given $\xi, \zeta \in T_{1_G}G$,

$$[\xi, \zeta] := [X^\xi, X^\zeta]|_{1_G}.$$

The required bilinearity, anti-symmetry, and satisfaction of the Jacobi identity now follow from Propositions **9.4** and **9.12**. $\square$

In practice, we often identify $\mathfrak{g}$ with $T_{1_G}G$.

Corollary 9.15. *If G is a Lie group, then TG is diffeomorphic to $G \times \mathfrak{g}$.*

Proof. The map $TG \to G \times \mathfrak{g}\colon v \mapsto (\tau(v), (L_{\tau(v)^{-1}})_*v)$ is smooth, and has inverse $(g, \xi) \mapsto (L_g)_*\xi$, which is also smooth. $\square$

The benefit of Corollary **9.14** is that we now have an easy way to find the Lie algebras of many of our matrix Lie group examples.

Examples 9.16 (Examples of Matrix Lie Algebras)**.**

(1) Let $[a_{ij}(t)]$ be a curve in $\mathrm{GL}(n;\mathbb{R})$ such that $[a_{ij}(0)] = I_n$. Its derivative at $t = 0$ is exactly $\left[\frac{d}{dt}\Big|_0 a_{ij}\right]$, which is in $\mathrm{Mat}(n;\mathbb{R})$. Thus the Lie algebra of $\mathrm{GL}(n;\mathbb{R})$, denoted $\mathfrak{gl}(n;\mathbb{R})$, is a linear subspace of $\mathrm{Mat}(n;\mathbb{R})$. But for any $B \in \mathrm{Mat}(n;\mathbb{R})$, the curve $I_n + tB$ is in $\mathrm{GL}(n;\mathbb{R})$ for small t by continuity of the determinant, and it satisfies $\frac{d}{dt}\Big|_{t=0}(I_n + tB) = B$. Since $\dim \mathfrak{gl}(n;\mathbb{R}) = n^2 = \dim \mathrm{Mat}(n;\mathbb{R})$, we conclude that $\mathfrak{gl}(n;\mathbb{R}) = \mathrm{Mat}(n;\mathbb{R})$. In fact, the Lie bracket is exactly the **commutator** of matrices $[A, B] := AB - BA$.

(2) Since $\mathrm{SL}(n;\mathbb{R})$ is a Lie subgroup of $\mathrm{GL}(n;\mathbb{R})$, its Lie algebra will be a Lie subalgebra of $\mathfrak{gl}(n;\mathbb{R})$, denoted $\mathfrak{sl}(n;\mathbb{R})$. Suppose $A \in \mathfrak{sl}(n;\mathbb{R})$. Then for small t, the curve $I_n + tA$ is in $\mathrm{SL}(n;\mathbb{R})$ provided $\det(I_n + tA) = 1$. Differentiating,

$$\begin{aligned} 0 &= \frac{d}{dt}\Big|_{t=0} \det(I_n + tA) \\ &= \frac{d}{dt}\Big|_{t=0} \left(t^n \det\left(\frac{1}{t} I_n + A\right)\right) \\ &= \frac{d}{dt}\Big|_{t=0} (t^n p(1/t)) \end{aligned}$$

where p is the characteristic polynomial of $-A$, given by $c_n\lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$, where $c_n = 1$, c_{n-1} is the trace $\mathrm{tr}(-A)$, etc. In particular, the derivative above is equal to $\mathrm{tr}(-A)$, and so elements of $\mathfrak{sl}(n;\mathbb{R})$ must be traceless (*i.e.* have trace equal to 0). We will soon have the ability to show that $\mathfrak{sl}(n;\mathbb{R})$ are exactly the traceless matrices.

(3) Again, since $\mathrm{O}(n)$ is a Lie subgroup of $\mathrm{GL}(n;\mathbb{R})$, its Lie algebra will be a Lie subalgebra of $\mathfrak{gl}(n;\mathbb{R})$. Note that elements of $\mathrm{O}(n)$ have determinant ± 1, and by continuity of the determinant, we can conclude that $\mathrm{SO}(n)$ is just the collection of connected components of $\mathrm{O}(n)$ whose determinant is 1. In particular, both $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ have the same Lie algebra, denoted $\mathfrak{so}(n)$. We know that if $A \in \mathfrak{so}(n)$, then for small t, $I_n + tA \in \mathrm{O}(n)$. In particular,

$$I_n = (I_n + tA)(I_n + tA)^{\mathrm{t}}.$$

Applying $\frac{d}{dt}\Big|_{t=0}$, it follows that $A^{\mathrm{t}} = -A$; that is, A is anti-symmetric. Again, we will soon have the ability to show that $\mathfrak{so}(n)$ are exactly the anti-symmetric matrices.

//

Exercise 9.17.

(1) Show that the Lie algebra of $\mathrm{U}(n)$, denoted $\mathfrak{u}(n)$, is made up of **skew-Hermitian** matrices; that is, matrices $[a_{ij}]$ in which $a_{ij} \in \mathbb{C}$ and $a_{ij} = -\overline{a_{ji}}$ for all i and j.

(2) Show that the Lie algebra of $\mathrm{SU}(n)$, denoted $\mathfrak{su}(n)$, is made up of traceless skew-Hermitian matrices.

Rank.

Definition 9.18 (Rank)**.** Let $F\colon M \to N$ be a smooth map, and fix $x \in M$. The **rank** of F at x is the rank of $F_*|_x\colon T_xM \to T_{F(x)}N$ as a linear map. $\diamond$

Remark 9.19. Given a smooth map $F\colon M \to N$, and a fixed $x \in M$ with chart $\varphi\colon U \to \widetilde{U} \subseteq \mathbb{R}^m$ about x and $\psi\colon V \to \widetilde{V} \subseteq \mathbb{R}^n$ about $F(x)$, let $\widetilde{F} := \psi \circ F \circ \varphi^{-1}$. Then the rank of F at x is exactly the rank $\mathrm{rk}\Big(D\widetilde{F}|_x\Big)$. $\lrcorner$

Recall that a linear transformation $G\colon V \to W$ has **maximal rank** if

$$\mathrm{rk}(G) = \min\{\dim V, \dim W\}.$$

Definition 9.20 (Immersions and Submersion)**.** Let $F\colon M \to N$ be a smooth map such that it has maximal rank at every point. If $\dim M \leq \dim N$, then F is an **immersion**. If $\dim M \geq \dim N$, then F is a **submersion**. $\diamond$

Proposition 9.21. *Let $F\colon M \to N$ be smooth, and fix $x \in M$. If $F_*|_x$ is injective (resp. surjective), then there is an open neighbourhood W of x such that $F|_W$ is an immersion (resp. submersion).*

Proof. Let $\varphi\colon U \to \widetilde{U} \subseteq \mathbb{R}^m$ and $\psi\colon V \to \widetilde{V} \subseteq \mathbb{R}^n$ be charts about x and $F(x)$, resp., and let $\widetilde{F} = \psi \circ F \circ \varphi^{-1}$. By Remark **9.19**, $\mathrm{rk}\left(D\widetilde{F}|_{\varphi(x)}\right)$ is the rank of F at x. Suppose $D\widetilde{F}|_{\varphi(x)}$ has maximal rank (which is equivalent to it being injective if $\dim M \leq \dim N$, or surjective if $\dim M \geq \dim N$). It suffices to show that having maximal rank is an "open condition"; that is, $D\widetilde{F}$ has maximal rank at each point in an open neighbourhood of $\varphi(x)$.

Let A be an $n \times m$ matrix with rank k. There is a $k \times k$ submatrix, B, of A such that $\det B \neq 0$. Indeed, since $\mathrm{rk}(A) = k$, there exist k linearly independent columns of A; let C be the corresponding $n \times k$ submatrix of A made up of these columns. Then $\mathrm{rk}(C) = k$, and so C must have k linearly independent rows; let B be the $k \times k$ submatrix of C (and hence A) made up of these rows. Then $\mathrm{rk}(B) = k$, and so it is non-degenerate. The determinant of B is non-zero, and since determinants are continuous, along with the appropriate projection maps, there is an open neighbourhood of A in $\mathbb{R}^{mn}$ on which the $k \times k$ formed used in the same columns and rows of as C remains non-degenerate; that is, on this open neighbourhood, all $n \times m$ matrices have rank at least k.

If A has maximal rank $k = \min\{m, n\}$, then all matrices in this open neighbourhood have rank at most k, and so we conclude that the rank of all of them is k (which is maximal for all of them).

Let $D\colon \widetilde{U} \to \mathbb{R}^{mn}$ be the map sending $y \in \widetilde{U}$ to $D\widetilde{F}|_y$. By hypothesis, $D(\varphi(x))$ has maximal rank, and so there is an open neighbourhood W' of $D(\varphi(x))$ in $\mathbb{R}^{mn}$ whose elements all have maximal rank. Since D is continuous, $W := D^{-1}(W')$ is open in $\widetilde{U}$, completing the proof. $\square$

Examples 9.22 (Examples of Immersions and Submersions)**.** Let M be a manifold. Then $\tau\colon TM \to M$ is a surjective submersion. If $M_1, \ldots, M_k$ are smooth manifolds, then the projection maps $\mathrm{pr}_i\colon \prod_j M_j \to M_i$ is a surjective submersion. If $\gamma\colon \mathbb{R} \to M$ is a smooth curve such that $\frac{d}{dt}\Big|_{t=t_0} \gamma \neq 0$ for each $t_0 \in \mathbb{R}$, then γ is an immersion (not necessarily injective). In particular, the image of γ has no corners nor cusps. //

We can now a consequence of the Inverse Function Theorem for smooth manifolds.

Exercise 9.23. Let M and N be manifolds of the same dimension. If $F\colon M \to N$ has maximal rank at each point (*i.e.* is nonsingular), show that for each $x \in M$ there is an open neighbourhood U of x such that $F|_U$ is a diffeomorphism onto its image. We call such maps **local diffeomorphisms**.

We can also restate the Quotient Manifold Theorem (see Theorem **6.25**).

Theorem 9.24 (Quotient Manifold Theorem)**.** *Let G be a Lie group acting freely and properly on a manifold M. There exists a unique smooth structure on the orbit space ${}_{G}\backslash^{M}$ making it into a manifold of dimension* $\dim M - \dim G$ *such that the quotient map $\pi\colon M \to {}_{G}\backslash^{M}$ is a surjective submersion.*

Submanifolds, Dimensions of Lie Groups, and the Topology of Lie Groups. Subobjects are mathematical entities that show up all of the time in all areas of mathematics: subsets, subgroups, subrings and ideals, subspaces. For manifolds, the same is true, except these "submanifolds" can have slightly different features that one needs to be careful of.

Definition 10.1 (Submanifolds)**.** Let $F\colon M \to N$ be an immersion. We call the image of F an **immersed submanifold** of N. If F is also a homeomorphism onto its image, then we call the image of F an **embedded submanifold** of N. If furthermore F is a proper map, then we call it a **proper embedding** and its image a **properly embedded submanifold** of N. $\diamond$

Examples 10.2 (Examples of Submanifolds)**.** A smooth curve $\gamma\colon \mathbb{R} \to \mathbb{R}^2$ such that $\left.\frac{d\gamma}{dt}\right|_{t=t_0} \neq 0$ for each t_0 is an immersion, and thus an immersed submanifold, even if the curve is self-intersecting. Thus, the topology on the submanifold is *not* necessarily equal to the subspace topology coming from $\mathbb{R}^2$; instead, the topology comes from the domain. Even if the map is injective, the two different topologies may not match: consider a figure-8 parametrised in a non-intersecting way.

Another famous example, which is relevant for this class, is the irrationally-sloped line in $\mathbb{T}^2$. One can construct $\mathbb{T}^2$ as the quotient space $\mathbb{R}^2/\mathbb{Z}^2$, where an element $(a,b) \in \mathbb{Z}^2$ acts on an element $(x,y) \in \mathbb{R}^2$ by $(a,b)\cdot(x,y) := (x+a, y+b)$. Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, and consider the line $y = \alpha x$ in $\mathbb{R}^2$. This, in fact, is a subgroup of $\mathbb{R}^2$ under addition. It descends to a subgroup of $\mathbb{T}^2$ via the quotient map $\pi\colon \mathbb{R}^2 \to \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, which is dense in $\mathbb{T}^2$. More precisely, let $f\colon \mathbb{R} \to \mathbb{T}^2$ be the curve $t \mapsto \pi(\alpha t)$. Then $\overline{\operatorname{im}(f)} = \mathbb{T}^2$. //

Presentation 4. Prove the statement in Examples **10.2** that $\operatorname{im}(f)$ is dense, injectively-immersed submanifold of $\mathbb{T}^2$ that is also a subgroup, and show that $\mathbb{T}^2/\operatorname{im}(f)$ has the trivial topology. (See [**Lee13**, Example 4.20].)

An important case of submanifolds are level sets of smooth functions.

Definition 10.3 (Regular and Critical Points and Values)**.** Let $F\colon M \to N$ be a smooth map, and fix $x \in M$. If $F_*|_x$ is surjective, then x is a **regular point**; otherwise, it is a **critical point**. Given $y \in N$, if the **level set** $F^{-1}(y)$ comprises only regular points, then we say that y is a **regular value** and $F^{-1}(y)$ is a **regular level set**; otherwise, y is a **critical value**. $\diamond$

Theorem 10.4 (Regular Level Sets are Submanifolds)**.** *Let $F\colon M \to N$ be smooth, and let $y \in N$ be a regular value. The regular level set $F^{-1}(y)$ is a properly embedded submanifold of M of dimension $k = \dim M - \dim N$.*

Presentation 5. Prove Theorem **10.4**.

Let us now go back to finding the Lie algebras of some of the classical Lie groups.

Examples 10.5 (Examples of Matrix Lie Algebras II)**.** Recall that $\mathfrak{sl}(n;\mathbb{R})$ is a linear subspace of all traceless matrices of $\mathfrak{gl}(n;\mathbb{R})$. It has the same dimension as $\mathrm{SL}(n;\mathbb{R}) = \det^{-1}(1)$ where $\det\colon \mathrm{GL}(n;\mathbb{R}) \cong \mathbb{R}^{n^2} \to \mathbb{R}$. By Exercise **10.6** below, the critical points of det are all in its zero set. Thus 1 is a regular value of det, and so by Theorem **10.4**, $SL(n;\mathbb{R})$ has dimension n^2-1. Similarly, since trace is a linear map $\mathrm{Mat}(n;\mathbb{R}) \to \mathbb{R}$, and traceless matrices are the zero set of the trace map, it follows that traceless matrices form a linear subspace of dimension n^2-1 in $\mathrm{Mat}(n;\mathbb{R})$. By Examples **9.16**, we conclude that $\mathfrak{sl}(n;\mathbb{R})$ are exactly the traceless matrices:

$$\mathfrak{sl}(n;\mathbb{R}) = \{A \in \mathrm{Mat}(n;\mathbb{R}) \mid \mathrm{tr}A = 0\}.$$

Turning to $\mathfrak{so}(n)$, the linear map $\mathrm{GL}(n;\mathbb{R}) \to \mathrm{GL}(n;\mathbb{R})^2$ sending A to (A, A^{t}) has maximal rank. The multiplication map $\mathrm{Mat}(n;\mathbb{R})^2 \to \mathrm{Mat}(n;\mathbb{R})$, by Exercise **10.7**, has no critical points in $\mathrm{GL}(n;\mathbb{R})^2$. So the identity matrix I_n is a regular value of the composition $A \mapsto (A, A^{\mathrm{t}}) \mapsto AA^{\mathrm{t}}$. Moreover, by ★Exercise **10.8** below, this composition has image exactly the positive-definite symmetric matrices, which is a submanifold of $\mathrm{Mat}(n;\mathbb{R})$ of dimension $\frac{1}{2}(n^2+n)$. By Theorem **10.4**, the level set of I_n by the smooth map $A \mapsto AA^{\mathrm{t}}$ has dimension $n^2 - \left(\frac{1}{2}(n^2+n)\right) = \frac{1}{2}(n^2-n)$. Thus, $\mathrm{SO}(n)$ and $\mathrm{O}(n)$ have dimension $\frac{1}{2}(n^2-n)$. But this is exactly the dimension of the $n\times n$ anti-symmetric matrices. By Examples **9.16**, we conclude that $\mathfrak{so}(n)$ is exactly the anti-symmetric matrices:

$$\mathfrak{so}(n) = \{A \in \mathrm{Mat}(n;\mathbb{R}) \mid A^{\mathrm{t}} = -A\}. \qquad /\!\!/$$

Exercise 10.6. Show that a square $n\times n$ matrix A is a critical point of det if and only if its rank is less than $n-1$.

Exercise 10.7. Find the critical points of the matrix multiplication map.

★Exercise 10.8. Show that the image of the map $\mathrm{GL}(n;\mathbb{R}) \to \mathrm{GL}(n;\mathbb{R})$ sending A to AA^{t} is the subset of all $n\times n$ positive-definite symmetric matrices. This is in fact a submanifold of dimension $\frac{1}{2}(n^2+n)$.

Exercise 10.9. Complete Exercise **9.17**, showing that $\mathfrak{u}(n)$ comprises exactly the skew-Hermitian matrices, and $\mathfrak{su}(n)$ comprises exactly the traceless skew-Hermitian matrices.

Before moving onto the next subsection, we end this one with some facts about the topology of some of the classical Lie groups.

Presentation 6 (Topology of $\mathrm{O}(n)$, $\mathrm{SO}(n)$, $\mathrm{U}(n)$, and $\mathrm{SU}(n)$)**.** Show that $\mathrm{O}(n)$, $\mathrm{SO}(n)$, $\mathrm{U}(n)$, and $\mathrm{SU}(n)$ are compact, that $\mathrm{SO}(n)$, $\mathrm{U}(n)$, and $\mathrm{SU}(n)$ are connected, and that $\mathrm{O}(n)$ has exactly two connected components in which the identity component is equal to $\mathrm{SO}(n)$.

Proposition 10.10 (Connectivity of $\mathrm{GL}(n;\mathbb{R})$)**.** *The Lie group* $\mathrm{GL}(n;\mathbb{R})$ *has exactly two connected components: the identity component* $\mathrm{GL}(n;\mathbb{R})^+$ *consisting of positive-determinant matrices, and the second component* $\mathrm{GL}(n;\mathbb{R})^-$ *consisting of negative-determinant matrices.*

★Exercise 10.11. Prove Proposition **10.10**.

Smooth Maps with Constant Rank. One of the most important theorems in differential geometry is the following.

Theorem 10.12 (Rank Theorem)**.** *Let $F\colon M \to N$ be smooth, where $\dim M = m$ and $\dim N = n$, and F has a constant rank of r at all points of M. For any $x \in M$, there exist charts $\varphi\colon U \to \widetilde{U}$ about x and $\psi\colon V \to \widetilde{V}$ about $F(x)$ such that*

(1) *$F(U) \subseteq V$,*

(2) *$\varphi(x) = 0$ and $\psi(F(x)) = 0$, and*

(3) *$\widetilde{F} = \psi \circ F \circ \varphi^{-1}$ satisfies*

$$\widetilde{F}(x^1, \dots, x^r, x^{r+1}, \dots, x^m) = (x^1, \dots, x^r, \underbrace{0, \dots, 0}_{n-r}) \in \mathbb{R}^n$$

where $(x^1, \dots, x^m)$ are coordinates in $\mathbb{R}^m$.

Presentation 7. Prove Theorem **10.12**.

Example 10.13. It follows from the proof of the Quotient Manifold Theorem that the quotient map $\pi\colon M \to {}_G\backslash^{M}$ for a Lie group action $G \circlearrowright M$ is a surjective submersion (and hence has constant rank) if the action is free and proper. In particular, if $H \leq G$ is a closed subgroup, then the quotient map $G \to {}_H\backslash^{G}$ has constant rank. //

As an immediate corollary, submersions locally look like projection maps between cartesian spaces, and immersions locally look like injection maps between cartesian spaces. Here is another immediate corollary.

Corollary 10.14 (Equivariant Rank Theorem)**.** *Let G be a Lie group acting on manifolds M and N, and let $F\colon M \to N$ be a smooth equivariant map (recall that a map F is equivariant if $F(g \cdot x) = g \cdot F(x)$ for all $g \in G$ and $x \in M$). Suppose the G-action on M is transitive. Then F has constant rank. In particular, if F is a bijective equivariant map, then it is a diffeomorphism.*

Proof. It suffices to show that for arbitrarily chosen $x_1, x_2 \in M$, the ranks of F at x_1 and x_2 are equal. Fix $x_1, x_2 \in M$. Since the action of G is transitive on M, there exists $g \in G$ such that $x_2 = g \cdot x_1$. Let ρ_M be the representation of the G-action on M, and ρ_N be the representation of the G-action on N. We have the following commutative diagram:

$$\begin{tikzcd} T_{x_1}M \arrow[r, "F_*|_{x_1}"] \arrow[d, "\rho_M(g)_*|_{x_1}"'] \arrow[dr, phantom, "\circlearrowright"] & T_{F(x_1)}N \arrow[d, "\rho_N(g)_*|_{F(x_1)}"] \\ T_{x_2}M \arrow[r, "F_*|_{x_2}"'] & T_{F(x_2)}N \end{tikzcd}$$

Since $\rho_M(g)$ and $\rho_N(g)$ are diffeomorphisms, they have ranks m and n, resp. Thus the rank of $F_*|_{x_2}$ is equal to the rank of $F_*|_{x_1}$.

If F is additionally a bijection, then by the Rank Theorem, there exist local coordinates so that it looks like the identity map between cartesian spaces. It follows from the Inverse Function Theorem that F is a diffeomorphism. $\square$

Example 10.15. Let G be a Lie group acting on a manifold M, and fix $x \in M$. The map $\alpha_x : G \to M$ sending g to $g \cdot x$ has constant rank. Indeed, G acts on itself by left multiplication, which is transitive. Since α_x is smooth and equivariant, the Equivariant Rank Theorem implies that α_x has constant rank. //

Proposition 10.16. *Let $G \circlearrowright M$ be a Lie group action, and fix $x \in M$. The orbit $G \cdot x$ is an immersed submanifold of M.*

To prove this, we require a lemma.

Lemma 10.17. *Let M, N, and P be smooth manifolds, and let $\pi : M \to N$ be a surjective submersion. Given maps $F : M \to P$ and $F' : N \to P$ satisfying $F = F' \circ \pi$, we have that F is smooth if and only if F' is.*

Exercise 10.18. Prove Lemma **10.17**.

*Proof of Proposition **10.16**.* Let $H := \mathrm{Stab}(x)$. By Example **10.13**, the quotient map $\pi : G \to G/H$ is a surjective submersion. Let $\beta : G/H \to M$ be the map sending gH to $g \cdot x$. By the Orbit Stabiliser Theorem, this map is a well-defined injection with image $G \cdot x$, and by Lemma **10.17**, it is smooth. It follows from Example **10.15** that β has constant rank. By the Rank Theorem, β must be an immersion. $\square$

Presentation 8 (Orbit Stabiliser Theorem for Lie Group Actions)**.** Let $G \circlearrowright M$ be a proper Lie group action. Show that the orbits of the action are properly embedded submanifolds. In fact, for any $x \in M$ with $H = \mathrm{Stab}(x)$, we have a diffeomorphism $\beta : G/H \to G \cdot x$.

As we have seen, a nice collection of group actions is given by matrix Lie groups acting on a cartesian space. The goal of this section is to show that about a so-called "fixed point" of a compact Lie group action on a manifold, the action is equivariantly diffeomorphic to a linear one, and so in the effective case, this is precisely one of the linear matrix Lie group actions (see Exercise **1.14**). To get to this result, called Bochner's Linearisation Theorem, we must take several steps. The presentation here is expanded on that of [**DK00**, Section 2.2].

Invariant Neighbourhoods. We begin with a simple observation about compact spaces, which is extremely important. For this purpose, define an **open neighbourhood** of a subset S of a topological space X to be any open subset U of X that contains S.

Lemma 11.1 (Tube Lemma)**.** *Let X and Y be topological spaces with Y compact. Suppose $x_0 \in X$, and U is an open neighbourhood of the subset $\{x_0\} \times Y \subseteq X \times Y$. There exists an open neighbourhood V of x_0 such that $\{x_0\} \times Y \subseteq V \times Y \subseteq U$.*

Proof. The map $Y \to X \times Y$ sending $y \in Y$ to (x_0, y) is continuous, and since Y is compact, the image $\{x_0\} \times Y$ is compact. Since U is open, every point $(x_0, y) \in \{x_0\} \times Y$ is contained in a basic open set $V_y \times W_y \subseteq U$. Since $\{x_0\} \times Y$ is compact, there is a finite subcover $\{V_{y_i} \times W_{y_i}\}_{i=1}^k$ of $\{x_0\} \times Y$. The intersection $V := \bigcap_{i=1}^k V_{y_i}$ is open in X, and satisfies the requirements of the lemma. □

We use this to show that for a compact Lie group action $G \circlearrowright M$, there are lots of G-invariant open neighbourhoods about fixed points. A subset S of M is **G-invariant** if for any $x \in S$ and $g \in G$, the point $g \cdot x \in S$. A point $x \in M$ is a **fixed point of the G-action** if $\mathrm{Stab}(x) = G$.

Lemma 11.2 (Invariant Neighbourhoods)**.** *Let $G \circlearrowright M$ be a compact Lie group action, let $x \in M$ be a fixed point, and let U be an open neighbourhood of x. There is a G-invariant neighbourhood V of x contained in U.*

Proof. Since that action map $\alpha\colon G \times M \to M\colon (g, x) \mapsto g \cdot x$ is smooth, it is continuous, and so $\alpha^{-1}(U)$ is open in $G \times M$. Since x is a fixed point, $G \times \{x\} \subseteq \alpha^{-1}(U)$. By Lemma **11.1**, there is an open neighbourhood W of x such that $G \times \{x\} \subseteq G \times W \subseteq \alpha^{-1}(U)$. Define $V := G \cdot W = \alpha(G \times W) \subseteq U$. Then $V = \bigcup_{g \in G} g \cdot W$, and since G acts by diffeomorphisms, each $g \cdot W$ is open. Thus V is an open neighbourhood of V contained in U. Let $y \in V$ and $g \in G$. Then $g \cdot y = g(g')^{-1}(g') \cdot y$ where $g' \cdot y \in W$. Then $g \cdot y \in g(g')^{-1} \cdot W \subseteq V$. Thus V is G-invariant. □

Induced Actions on Tangent Bundles. Let G be a Lie group acting on a manifold M, and let $\alpha\colon G \times M \to M$ be the action map. Let g_t be a curve in G such that $g_0 = 1_G$ and $\frac{d}{dt}\big|_{t=0} g_t = \xi \in \mathfrak{g}$. For a fixed $x \in M$, we have the curve $g_t \cdot x$ in M whose derivative at $t = 0$ is

$$\frac{d}{dt}\Big|_{t=0} g_t \cdot x = \frac{d}{dt}\Big|_{t=0} (\alpha(g_t, x)) = \alpha_*(\xi, 0) =: \xi_M|_x.$$

It follows from the definition of a tangent vector (using curves) that $\xi_M|_x$ is independent of the curve g_t used to define it (provided the velocity of g_t at $t = 0$ is ξ). Moreover, this is smooth in x, and so defines a vector field on M. We call it the **induced vector field on M by ξ**. This gives us a linear map $\mathfrak{g} \to \mathrm{vect}(M)$, often called the **infinitesimal action induced by G on M**.

Now let x_t be a smooth curve in M through a point x_0 such that $v = \frac{d}{dt}\Big|_{t=0} x_t \in T_{x_0}M$, and fix $g \in G$. Then $g \cdot x_t$ is a smooth curve in M, and

$$\frac{d}{dt}\Big|_{t=0} g \cdot x_t = g_* v;$$

here, the notation g_* is shorthand for $\rho(g)_*$ where $\rho\colon G \to \mathrm{Diff}(M)$ is the corresponding representation to the action.

We thus have two actions: G acts on TM by $g \cdot v := g_* v$ and $\mathfrak{g}$ "acts" on TV where $\xi \in \mathfrak{g}$ sends $v \in T_xM$ to $v + \xi_M|_x$. Together, given a smooth curve (g_t, x_t) pairing the two curves above in $G \times M$, we have

$$\frac{d}{dt}\Big|_{t=0} \alpha(g_t, x_t) = g_* v + g_*(\xi_M|_x).$$

We end this subsection by coming up with a formula for $g_*(\xi_M|_x)$.

★**Exercise 11.3** (Adjoint Action)**.**

(1) Let G be a Lie group. Show that G acts on itself by conjugation: for a fixed $g \in G$, we have a left action $\alpha(h, g) = C_h(g) := hgh^{-1}$ for any $h \in G$.

(2) Let g_t be a smooth curve in G so that $g_0 = 1_G$ and $\xi = \frac{d}{dt}\Big|_{t=0} g_t \in \mathfrak{g}$. Define the **adjoint action of G on $\mathfrak{g}$** to be for $h \in G$:

$$\mathrm{Ad}_h(\xi) := \frac{d}{dt}\Big|_{t=0} C_h(g_t) = (L_h)_*(R_{h^{-1}})_*\xi;$$

here, $R_{h^{-1}}$ is right multiplication by h^{-1}. Note that left multiplication and right multiplication operators commute: $L_h \circ R_{h'} = R_{h'} \circ L_h$. Show that the adjoint action is a left action of G.

(3) Let h_t be a smooth curve in G such that $h_0 = 1_G$ and $\frac{d}{dt}\Big|_{t=0} h_t = \zeta \in \mathfrak{g}$. Define the **infinitesimal adjoint action of $\mathfrak{g}$** to be the "action"

$$\mathrm{ad}(\zeta, \xi) := \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}_{h_t}(\xi).$$

Show that $\mathrm{ad}(\zeta, \xi) = [\zeta, \xi]$.

Lemma 11.4. *Let $G \circlearrowright M$ be a Lie group action, $\xi \in \mathfrak{g}$, and $h \in G$. Then $h_*(\xi_M) = \mathrm{Ad}_h(\xi)_M$.*

Proof. Let $f \in C^\infty(M)$ and $x \in M$.

$$\begin{aligned}(h_*(\xi_M))(f)(x) &= (h^{-1})^*(\xi_M(h^*f))(x)\\ &= \xi_M(h^*f)(h^{-1}\cdot x)\\ &= \xi_M|_{h^{-1}\cdot x}(h^*f).\end{aligned}$$

On the other hand, if $\frac{d}{dt}\Big|_{t=0} g_t = \xi$, then

$$\begin{aligned}\mathrm{Ad}_h(\xi)_M|_x(f) &= \left(\frac{d}{dt}\Big|_{t=0}(hg_th^{-1})\right)_M\Big|_x f\\ &= \frac{d}{dt}\Big|_{t=0} f(hg_th^{-1}\cdot x)\\ &= \frac{d}{dt}\Big|_{t=0} h^*f(g_th^{-1}\cdot x)\\ &= \xi_M|_{h^{-1}\cdot x}(h^*f).\end{aligned}$$

Since x and f are arbitrary, this proves the desired equality. □

Exercise 11.5. Let $x \in M$ be a fixed point of a Lie group action $G \circlearrowright M$. Then G acts on T_xM via linear isomorphisms by $g \cdot v = g_*v$ for all $g \in G$ and $v \in T_xM$.

Averaging over G**.** The next step is to find a way to make a function G-invariant. Recall that a function $f\colon M \to \mathbb{R}$ is G**-invariant** if $f(g\cdot x) = f(x)$ for all $g \in G$ and all $x \in M$.

Example 11.6 (Invariant Function for a Finite Group Action)**.** Let a finite group Γ act on a manifold M. Then Γ acts on $C^\infty(M)$ as follows: given $f \in C^\infty(M)$ and $g \in \Gamma$, denote by $g \cdot f$ the function sending x to $f(g^{-1}\cdot x)$. Note that this is a left action:

$$g'\cdot(g\cdot f)(x) = (g\cdot f)((g')^{-1}\cdot x) = f(g^{-1}\cdot((g')^{-1}\cdot x)) = f((g'g)^{-1}\cdot x) = (g'g\cdot f)(x).$$

For a fixed $f \in C^\infty(M)$, define

$$\overline{f} := \sum_{g\in\Gamma} g\cdot f\colon x \mapsto \sum_{g\in\Gamma} f(g^{-1}\cdot x).$$

Then $\overline{f}$ is Γ-invariant. Indeed, for a fixed $x \in M$ and $g' \in \Gamma$,

$$\overline{f}(g'\cdot x) = \sum_{g\in\Gamma} f(g^{-1}g'\cdot x) = \sum_{g\in\Gamma} f(g^{-1}\cdot x) = \overline{f}(x),$$

since Γ acts on itself transitively by right multiplication. We call $\overline{f}$ the **average of** f **over** Γ. //

We would like to extend this notion of averaging a function over a finite group to infinite groups. To accomplish this, we need to use some measure theory, but this would take us far outside the scope of this course. So we will use the following result without proof. (For details, see [**DK00**, Section 3.13].)

Theorem 11.7 (Haar Measure)**.** *Given a compact Lie group G, there is a measure μ, called the (normalised) Haar measure, such that $\int_G 1\, d\mu = 1$ and $\int_G f(g)\, d\mu = \int_G f(g'g)\, d\mu$ for all $g' \in G$ and $f \in C^\infty(G)$. (We think of g here as the variable being integrated.)*

We will typically denote integration with respect to the Haar measure by $\int_G \cdot\, dg$. What the last statement in the theorem means is that we can translate functions around via multiplication on G and not change the result of the integration.

Given an action of the compact Lie group G on M and a function $f \in C^\infty(M)$, define the **average of f over G**, denoted $\overline{f}$, by

$$\overline{f}(x) := \int_G f(g \cdot x)\, dg.$$

This is a smooth G-invariant function on M (the proof of this will also be omitted). Of course, now that we have the ability to average a real-valued function over G, we can do the same for any vector-valued function by integrating one component at a time.

Exercise 11.8. Given $x \in M$, there is an open neighbourhood U of x, an open neighbourhood V of $0 \in T_xM$, and a diffeomorphism $\lambda \colon U \to V$ such that $\lambda(x_0) = 0$ and $\lambda_*|_x = \mathrm{id}_{T_xM}$; here, we identify $T_0(T_xM)$ with T_xM itself.

Suppose that $G \circlearrowright M$ is a compact Lie group action and $x_0 \in M$ is a fixed point. By Exercise **11.8**, there is an open neighbourhood U of x_0, an open neighbourhood V of $0 \in T_xM$, and a diffeomorphism $\lambda \colon U \to V$ such that $\lambda(x_0) = 0$ and $\lambda_*|_{x_0} = \mathrm{id}_{T_{x_0}M}$. By Lemma **11.2**, we may assume without loss of generality that U is G-invariant. Define $\overline{\lambda} \colon U \to V$ by

$$\overline{\lambda}(x) := \int_G g_*\lambda(g^{-1} \cdot x)\, dg.$$

It turns out that, due to its definition, $\overline{\lambda}$ is *not* invariant; however, it is equivariant: for a fixed $g' \in G$:

$$\begin{aligned}
\overline{\lambda}(g' \cdot x) &= \int_G g_*\lambda(g^{-1}g' \cdot x)\, dg \\
&= \int_G (g'g)_*\lambda(g^{-1} \cdot x)\, dg \\
&= (g')_* \left(\int_G g_*\lambda(g^{-1} \cdot x)\, dg \right) \\
&= (g')_*\overline{\lambda}(x).
\end{aligned}$$

Now, for $v \in T_{x_0}M$,

$$\overline{\lambda}_* v = \int_G g_*\lambda_*|_{g^{-1}\cdot x_0}(g_*^{-1}v)\, dg$$

and since x_0 is a fixed point, $g^{-1} \cdot x_0 = x_0$, and so the right-hand side above becomes $\int_G g_*\lambda_*|_{x_0}(g_*^{-1}v)\, dg$. But by Exercise **11.8**, $\lambda_*|_{x_0} = \mathrm{id}_{T_{x_0}M}$, and so this reduces further to $\int_G g_*g_*^{-1}v\, dg = v\int_G 1\, dg = v$. That is, $\overline{\lambda}_*|_{x_0} = \mathrm{id}_{T_{x_0}M}$ as well. By the Inverse Function Theorem and Lemma **11.2**, we may shrink U and V so that U remains invariant and so that

$\overline{\lambda}$ is a diffeomorphism from U to V. Combining this with Exercise **11.5**, we have just proven the following.

Theorem 11.9 (Bochner Linearisation Theorem)**.** *Let $G \circlearrowright M$ be a compact Lie group action, and let $x_0 \in M$ be a fixed point. There is an equivariant diffeomorphism $\overline{\lambda}$ from an invariant open neighbourhood U of x_0 to an invariant open neighbourhood V of $0 \in T_{x_0}M$ equipped with the induced action $(g, v) \mapsto g_* v$ for $g \in G$ and $v \in T_{x_0}M$.*

We now want to push Bochner's Linearisation Theorem beyond fixed points. The goal is to show that for a compact Lie group action, there is an invariant neighbourhood of any orbit that *locally* looks like the orbit crossed with one of the linear actions given by Bochner's theorem.

To this end, we begin by averaging inner products.

Invariant Inner Products.

Lemma 12.1. *Let V be a vector space and $\langle\cdot,\cdot\rangle$ an inner product on it. Let G be a compact Lie group acting on V linearly. Then $\langle\cdot,\cdot\rangle_G$ is a G-invariant inner product, called the* ***average of*** $\langle\cdot,\cdot\rangle$ ***over*** G*, where*

$$\langle v, w\rangle := \int_G \langle g\cdot v, g\cdot w\rangle\, dg.$$

Exercise 12.2. Prove Lemma **12.1**.

Lemma 12.3. *Let $\langle\cdot,\cdot\rangle$ be an inner product on $\mathbb{R}^n$. There is a linear change of coordinates $\varphi\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $\langle x,y\rangle = \langle\varphi(x),\varphi(y)\rangle_{\mathrm{Euc}}$, where $\langle\cdot,\cdot\rangle_{\mathrm{Euc}}$ is the standard Euclidean inner product.*

Proof. There exists a positive definite symmetric matrix A such that $\langle x,y\rangle = x^{\mathrm{t}}Ay$ for all $x,y\in\mathbb{R}^n$. By the spectral theorem, A is diagonalisable with real eigenvalues: $A = PDP^{\mathrm{t}}$ where $P\in \mathrm{O}(n)$ and D is a diagonal matrix whose entries are the eigenvalues of A. Since A is positive definite, these eigenvalues are positive. Thus

$$\sqrt{A} := P\sqrt{D}P^{\mathrm{t}}$$

is well-defined and symmetric, where $\sqrt{D}$ is the diagonal matrix whose entries are the square roots of the entries of D. Define $\varphi(x) := \sqrt{A}x$. Then

$$\langle\varphi(x),\varphi(y)\rangle_{\mathrm{Euc}} = x^{\mathrm{t}}\sqrt{A}^{\mathrm{t}}\sqrt{A}y = x^{\mathrm{t}}Ay = \langle x,y\rangle,$$

where the second-last equality follows from the fact that $\sqrt{A}$ is symmetric. □

What this means is that any inner product on $\mathbb{R}^n$ is the Euclidean one after a linear change of coordinates.

Corollary 12.4. *Let V be an n-dimensional vector space and $\langle\cdot,\cdot\rangle$ an inner product on it, and let G be a compact Lie group acting on V linearly. There exist coordinates on V (*i.e.* a basis) such that V becomes identified with $\mathbb{R}^n$ and G acts orthogonally; that is, the representation $\rho\colon G\to \mathrm{GL}(V) = \mathrm{GL}(n;\mathbb{R})$ has image in $\mathrm{O}(n)$. In particular, if the action is effective, then G is (isomorphic to) a closed subgroup of $\mathrm{O}(n)$.*

Proof. Let $\langle\cdot,\cdot\rangle_G$ be the average of $\langle\cdot,\cdot\rangle$ over G, which is an invariant inner product by Lemma **12.1**. By Lemma **12.3**, there exist coordinates such that V can be identified with $\mathbb{R}^n$ and $\langle\cdot,\cdot\rangle_G$ can be identified with $\langle\cdot,\cdot\rangle_{\mathrm{Euc}}$. In particular, $\langle g\cdot x, g\cdot y\rangle = \langle x,y\rangle$ for all $x,y\in\mathbb{R}^n$ and $g\in G$; that is, G acts orthogonally. □

Exercise 12.5. Let G act orthogonally on $\mathbb{R}^n$ with respect to an invariant inner product, and let V be a G-invariant linear subspace of $\mathbb{R}^n$; that is, for any $v \in V$ and $g \in G$, we have $g \cdot v \in V$. Show that the orthogonal complement $V^\perp$ is also G-invariant.

Slices. We make the following observation about orbits.

Lemma 12.6. *Let $G \circlearrowright M$ be a Lie group action and fix $x_0 \in M$. The linear map $\alpha_*|_{(1_G,x_0)} \colon \mathfrak{g} \times \{0\} \to T_{x_0}M$ has image $T_{x_0}(G \cdot x_0)$ and kernel $\mathfrak{h} \times \{0\}$, where $\mathfrak{h}$ is the Lie algebra of $H := \mathrm{Stab}(x_0)$.*

Proof. Let $F \colon \mathfrak{g} \to T_{x_0}(G \cdot x_0)$ be the linear map $\xi \mapsto \alpha_*|_{(1_G,x_0)}(\xi, 0)$. Let $\xi \in \mathfrak{g}$ and g_t be a smooth curve such that $g_0 = 1_G$ and $\frac{d}{dt}\Big|_{t=0} g_t = \xi$. Then

$$\frac{d}{dt}\Big|_{t=0} g_t \cdot x_0 = \xi_M|_{x_0} \in T_{x_0}M.$$

However, the curve $g_t \cdot x_0$ is contained in the orbit $G \cdot x_0$, and so $\xi_M|_{x_0} \in T_{x_0}(G \cdot x_0)$. Thus $F(\mathfrak{g})$ is a linear subspace of $T_{x_0}(G \cdot x_0)$.

On the other hand,

$$\dim T_{x_0}(G \cdot x_0) = \dim G \cdot x_0 = \dim G/H$$

by Proposition **10.16** and the Orbit Stabiliser Theorem. By the Quotient Manifold Theorem,

$$\dim(G/H) = \dim(G) - \dim(H).$$

But

$$\dim G = \dim \mathfrak{g} = \dim(\ker F) + \dim(\operatorname{im} F).$$

If we can show that $\ker F = \mathfrak{h}$, then

$$\dim H = \dim \mathfrak{h} = \dim(\ker F),$$

and we would conclude that $\dim T_{x_0}(G \cdot x_0) = \dim \operatorname{im} F$, and we would have the desired equality.

Let $\zeta \in \mathfrak{h}$, and let h_t be a path in H such that $h_0 = 1_H = 1_G$ and $\frac{d}{dt}\Big|_{t=0} h_t = \zeta$. Then

$$F(\zeta) = \frac{d}{dt}\Big|_{t=0} h_t \cdot x = 0$$

since $h_t \in H$. Thus $\mathfrak{h} \leq \ker F$. In the other direction, let $\zeta \in \ker F$; hence $\zeta_M|_{x_0} = 0$. Recall that ζ can be identified with a left-invariant vector field X^ζ on G. Let g_t be an integral curve of X^ζ through 1_G defined on some open interval $(-\varepsilon, \varepsilon)$; that is, for all $t_0 \in (-\varepsilon, \varepsilon)$,

$$\frac{d}{dt}\Big|_{t=t_0} g_t = X^\zeta|_{g_{t_0}}.$$

Identifying TG with $G \times \mathfrak{g}$, the above equation is equivalent to

$$\frac{d}{dt}\Big|_{t=t_0} g_t = (g_{t_0}, \zeta).$$

Consider the curve $g_t \cdot x_0$. Differentiating:

$$\begin{aligned}\frac{d}{dt}\Big|_{t=t_0} g_t \cdot x_0 &= \alpha_*|_{(1_G,x_0)}((g_{t_0},\zeta),0) \\ &= (g_{t_0})_*(\zeta_M|_{x_0}) + 0 \\ &= (g_{t_0})_* 0 \\ &= 0.\end{aligned}$$

Since t_0 is arbitrary, $g_t \cdot x_0$ is a constant curve. That is, $g_t \in H$ for all $t \in (-\varepsilon, \varepsilon)$, and hence $\zeta \in \mathfrak{h}$. □

We now define some special submanifolds that are "complementary" to orbits.

Definition 12.7 (Slices)**.** Let $G \circlearrowright M$ be a Lie group action, and let $x_0 \in M$. A **slice** for the action through x_0 is an embedded submanifold S satisfying the following conditions.

(1) $x_0 \in S$,

(2) $T_{x_0}M = T_{x_0}(G \cdot x_0) \oplus T_{x_0}S$,

(3) $T_xM = T_x(G \cdot x) + T_xS$ for all $x \in S$,

(4) S is $\mathrm{Stab}(x_0)$-invariant: $h \cdot x \in S$ for all $x \in S$ and $h \in \mathrm{Stab}(x_0)$, and

(5) if $g \in G$ and $x \in S$ such that $g \cdot x \in S$, then $g \in \mathrm{Stab}(x_0)$. ⋄

Example 12.8. Consider the action of $\mathbb{S}^1$ on $\mathbb{R}^2$ by rotations about the origin. Any open ball B centred at the origin is a slice for the action through the origin. Given $x_0 \neq 0$, let $\varepsilon \in (0,1)$ and define $S = \{tx_0 \mid t \in (1-\varepsilon, 1+\varepsilon)\}$. Then S is a slice for the action through x_0. //

Given a slice for an action through a point, we can start to construct a "local model" of the action about that point. This means, the action takes on a very particular form around that point. The following is adapted from [**DK00**, Lemma 2.1.1].

Proposition 12.9. *Let $G \circlearrowright M$ be a Lie group action and fix $x_0 \in M$. Let S be a slice for the action through x_0.*

(1) *There is a submanifold N of G such that $1_G \in N$ and $\mathfrak{g} = T_{1_G}N \oplus \mathfrak{h}$, where $\mathfrak{h}$ is the Lie algebra of* $\mathrm{Stab}(x_0)$*.*

(2) *There are open neighbourhoods V in N of 1_G, W in S of x_0, and U of x_0 in M such that the action map $\alpha\colon G \times M \to M$ restricts to a diffeomorphism $V \times W \to U$.*

Proof. Item **1** is a consequence of Exercise **11.8**. To prove Item **2**, by the Inverse Function Theorem, it suffices to show that α restricted to $N \times S$ has bijective pushforward at $(1_G, x_0)$. Let F be the restriction of $\alpha_*|_{(1_G,x_0)}$ to $T_{1_G}N \times T_{x_0}S$. Suppose $F(\xi, v) = 0$. Then $\xi_M|_{x_0} + v = 0$, or $\xi_M|_{x_0} = -v \in T_{x_0}S$. By the second condition in the definition of a slice and Lemma **12.6**, this implies that $\xi_M|_{x_0} = 0$ (and so $v = 0$ as well). Again by Lemma **12.6**, $\xi \in \mathfrak{h}$. By definition of N, however, $\xi = 0$. Thus F is injective. On the other hand, letting $H = \mathrm{Stab}(x_0)$, and applying the Orbit Stabiliser Theorem and the Quotient Manifold Theorem,

$$\begin{aligned}\dim N \times S &= (\dim \mathfrak{g} - \dim \mathfrak{h}) + (\dim M - \dim(G \cdot x_0)) \\ &= (\dim G - \dim H) + (\dim M - \dim G + \dim H) \\ &= \dim M.\end{aligned}$$

Thus F is surjective. □

Remark 12.10. In the proof of Proposition **12.9**, we only used the first two conditions from the definition of a slice, and so the proposition generalises to any submanifold S of M containing x_0 such that $T_{x_0}M = T_{x_0}(G \cdot x_0) \oplus T_{x_0}S$. This is important for the proof of the Slice Theorem below. ⌟

Example 12.11. Let G be a Lie group and H a closed subgroup, which acts freely on G by right multiplication. Let N be a submanifold of G such that $1_G \in N$ and $\mathfrak{g} = T_{1_G}N \oplus \mathfrak{h}$, where $\mathfrak{h}$ is the Lie algebra of H. By Proposition **12.9** and Remark **12.10**, there are open neighbourhoods W of 1_G in H, V of 1_G in N, and U of 1_G in G, such that the multiplication map $m\colon G \times H \to G$ restricts to a diffeomorphism $V \times W \to U$. //

We still need to discuss the existence of slices. In this case, we will assume that the action is proper, and make the following observation.

Exercise 12.12. Let G be a Lie group.

(1) Suppose $G \circlearrowright M$ is a Lie group action. Show that the action is proper if and only if for any sequence (g_i, x_i) in $G \times M$ such that $(x_i, g_i \cdot x_i)$ converges to (x, x') in $M \times M$, there is a subsequence (g_{i_j}) of (g_i) converging to some $g \in G$. In this case, $g \cdot x = x'$.

(2) Let H be a closed subgroup of G. Then left (or right) multiplication $H \times G \to G\colon (h, g) \mapsto hg$ is a proper action of H on G.

(3) If $G \circlearrowright M$ is a proper Lie group action and $x \in M$, then $\mathrm{Stab}(x)$ is compact.

Theorem 12.13 (Slice Theorem)**.** *Let $G \circlearrowright M$ be a proper Lie group action. Then every point admits a slice for the action.*

Proof. Fix $x_0 \in M$. By Item **3** of Exercise **12.12**, $H := \mathrm{Stab}(x_0)$ is compact. Thus, if we restrict to the H-action (*i.e.* only allow elements of H to act), then x_0 is a fixed point of this action. By Bochner's Linearisation Theorem there is an H-equivariant diffeomorphism λ from an H-invariant open neighbourhood U of x_0 to an H-invariant open neighbourhood V of $0 \in T_{x_0}M$, where the action of H on $T_{x_0}M$ is the induced linear action. By Corollary **12.4**, we may assume that H acts orthogonally on $T_{x_0}M \cong \mathbb{R}^m$ with respect to the Euclidean inner product, and thus $V = B_\varepsilon(0)$, an open ball of radius $\varepsilon > 0$ centred at 0.

Let $v \in T_{x_0}(G \cdot x_0) \leq T_{x_0}M$. By Lemma **12.6**, there exists $\xi \in \mathfrak{g}$ such that $v = \xi_M|_{x_0}$. Thus, for any $h \in H$, we have $h_* v = h_* \xi_M|_{x_0} = \mathrm{Ad}_h(\xi)_M|_{x_0}$. Again by Lemma **12.6**, this means that $h_* v \in T_{x_0}(G \cdot x_0)$, and so the linear subspace $T_{x_0}(G \cdot x_0)$ is invariant by the

H-action. By Exercise **12.5**, the orthogonal complement $T_{x_0}(G\cdot x_0)^\perp$ is H-invariant as well. Define $S_\varepsilon := \lambda^{-1}(V\cap T_{x_0}(G\cdot x_0)^\perp)$; since λ is equivariant, it follows that S_ε is H-invariant as well.

Recall that we showed in $\mathrm{Mat}_{m\times n}(\mathbb{R})$, the matrices of rank at least $k\leq n$ form an open set. It follows from this and Lemma **12.6** that nearby orbits to $G\cdot x_0$ have dimension greater than or equal to that of $G\cdot x_0$. In other words, since $T_{x_0}M = T_{x_0}(G\cdot x_0)\oplus T_{x_0}S$, after shrinking ε,

$$T_xM = T_x(G\cdot x) + T_xS_\varepsilon.$$

By definition, S_ε is H-invariant, and so we only need to show the last condition of Definition **12.7** to complete the proof.

Suppose the last condition of Definition **12.7** does not hold for any S_η where $0<\eta\leq\varepsilon$. For sufficiently large K, we have that for each $k\geq K$, there exists $x_k\in S_{1/k}$ and $g_k\in G$ such that $g_k\cdot x_k\in S_{1/k}\subseteq S_\varepsilon$, but $g_k\notin H$. Then $x_k\to x_0$ and $g_k\cdot x_k\to x_0$, and since the action is proper, there exists a subsequence of (g_k) that converges to some $g\in G$ such that $g\cdot x_0 = \lim g_k\cdot x_k = x_0$. Then $g\in H$. Without loss of generality, assume this subsequence is (g_k), and since $g^{-1}g_k\to 1_G$ but $g^{-1}g_k\notin H$ and $g^{-1}g_k\cdot x_0\in S_{1/k}$ for each k, we also assume that $g=1_G$.

By Example **12.11**, there is a submanifold N of G such that $1_G\in N$ and $\mathfrak{g} = T_{1_G}N\oplus\mathfrak{h}$, and there are open neighbourhoods W of 1_G in H, V of 1_G in N, and U of 1_G in G such that the multiplication map $m\colon G\times H\to G$ restricts to a diffeomorphism $\mu\colon V\times W\to U$. Let $(a_k,h_k) := \mu^{-1}(g_k)$. Then the sequence $((a_k,h_k))$ converges to $(1_G,1_G)$, and $a_kh_k = g_k$ for each k. Since $g_k = a_kh_k\notin H$ for each k, we have $a_k\neq 1_G$ for each k.

By Proposition **12.9** and Remark **12.10**, there is an open neighbourhood V' of 1_G in N, some $\varepsilon'\in(0,\varepsilon)$, and an open neighbourhood U' of x_0 such that the restriction of the action map to $V'\times S_{\varepsilon'}$ is a diffeomorphism λ onto U'. For sufficiently large k, $a_k\in V'$ and $h_k\cdot x_k\in S_{\varepsilon'}$, and since $a_kh_k\cdot x_k\in S_{\varepsilon'}$ for these k, we have $\lambda^{-1}(a_kh_k\cdot x_k) = (a_k, h_k\cdot x_k)$. But $\lambda^{-1}(a_kh_k\cdot x_k)$ is also equal to $(1_G, a_kh_k\cdot x_k)$, and so by bijectivity of λ, we must have $a_k = 1_G$ for sufficiently large k, a contradiction. □

Remark 12.14 (Isotropy Action)**.** It follows from the proof above that the slice S is G-equivariantly diffeomorphic to $T_{x_0}(G\cdot x_0)^\perp$. The latter can be ($H$-equivariantly) identified with $T_{x_0}M/T_{x_0}(G\cdot x_0)$. Here, the H-action on $T_{x_0}M$ (called the **isotropy action of H on $T_{x_0}M$**) descends to the quotient linear space $T_{x_0}M/T_{x_0}(G\cdot x_0)$, and this action is sometimes called the **isotropy action** as well. ⌟

In Section **12**, we proved the Slice Theorem, which indicated that through any point of a proper Lie group action, there is a slice for that action. However, our goal was to show that there is an open neighbourhood of any orbit that has a very particular form, involving a linear action. We are halfway there. We only need to describe how this "particular form" will look like.

Before we push the rest of the way, we make some observations about proper actions. Let H be a Lie group acting on two manifolds M and N. Then H acts on $M \times N$ diagonally:

$$h \cdot (x, y) = (h \cdot x, h \cdot y).$$

Lemma 13.1. *Suppose H is a Lie group acting on manifolds M and N, and H acts on M freely and properly. Then H acts freely and properly on $M \times N$ via the diagonal action, and hence the quotient space $M \times_H N := {}_H\backslash(M \times N)$ admits a unique manifold structure for which the quotient map is a surjective submersion.*

Proof. To check freeness, suppose $h \cdot (x, y) = (x, y)$. Then $(h \cdot x, h \cdot y) = (x, y)$, and so in particular $h \cdot x = x$. Since the action of H on M is free, we have $h = 1_H$. We conclude that the H action on $M \times N$ is free.

To check properness, suppose $((h_i, (x_i, y_i)))$ is a sequence in $H \times (M \times N)$ such that $((x_i, y_i), h_i \cdot (x_i, y_i))$ converges to $((x, y), (x', y'))$ in $(M \times N)^2$. Then, in particular, $(x_i, h \cdot x_i)$ converges to (x, x') in M^2. Since the action of H on M is proper, there exists a subsequence (h_{i_j}) of (h_i) that converges to some $h \in H$. Then $((h_{i_j}, x_{i_j}))$ converges to (h, x) in $H \times M$, and by continuity, we have that $x' = h \cdot x$. Similarly, we have $((h_{i_j}, y_{i_j}))$ converges to (h, y) in $H \times N$, and so by continuity, we have that $y' = h \cdot y$. We conclude that $((h_{i_j}, (x_{i_j}, y_{i_j})))$ converges to $(h, (x, y))$, and $h \cdot (x, y) = (x', y')$, and conclude that the H-action on $M \times N$ is proper. The last statement now follows from the Quotient Manifold Theorem. □

Lemma 13.2. *Let G be a Lie group, let H be a closed subgroup, and let $H \circlearrowright M$ be a Lie group action. Then $G \times_H M$ is a manifold such that the quotient map $G \times M \to G \times_H M$ is a surjective submersion, and there is a G-action on $G \times_H M$ such that the quotient map is G-equivariant with respect to the G-action $g' \cdot (g, x) := (g'g, x)$ on $G \times M$.*

Remark 13.3. There are two actions on $G \times M$: the H-action and the G-action. To distinguish them, we take the H-action to be the **anti-diagonal action**: $h \cdot (g, x) := (gh^{-1}, h \cdot x)$. This has the advantage that (gh, x) and $(g, h \cdot x)$ are in the same orbit, and so represent the same element in $G \times_H M$. ⌟

Proof of Lemma 13.2. The first claim follows from Exercise **12.12**, Lemma **13.1**, and the fact that group multiplication (on the left or right) is a free action. Define a G-action on $G \times_H M$ by $g \cdot [g_0, x_0] = [gg_0, x_0]$. This is well-defined: if $[g_1, x_1] = [g_0, x_0]$, then there exists $h \in H$ such that $(g_1, x_1) = (g_0 h^{-1}, h \cdot x_0)$, in which case $(gg_1, x_1) = (gg_0 h^{-1}, h \cdot x_0)$; that is, $g \cdot [g_0, x_0] = g \cdot [g_1, x_1]$. It is immediate that the quotient map $G \times M \to G \times_H M$ is G-equivariant. Since the action $G \times (G \times M) \to G \times M$ is smooth, as is the quotient map to

$G \times_H M$, their composition is smooth: thus the action $G \times (G \times_H M) \to G \times_H M$ is smooth by Lemma **10.17**. This completes the proof. $\square$

One more quick observation about submersions, and we will be ready.

Exercise 13.4. Let $f\colon M \to N$ be a submersive smooth map. Show that f is open.

We are now prepared to obtain our "local model" of a compact Lie group action about one of the orbits. We follow the presentation of [**DK00**, Theorem 2.4.1].

Theorem 13.5 (Equivariant Tubular Neighbourhood Theorem)**.** *Let $G \circlearrowleft M$ be a compact Lie group action, and let $x_0 \in M$. There exists a G-invariant open neighbourhood U of $G \cdot x_0$ and a G-equivariant diffeomorphism $F\colon U \to G \times_H V$ where $H := \mathrm{Stab}(x_0)$ and $V = T_{x_0}M/T_{x_0}(G \cdot x_0)$ equipped with the isotropy H-action.*

Proof. By the Slice Theorem, there is a slice S for the G-action through x_0. Consider the restriction of the action $\alpha\colon G \times M \to M$ to the product $G \times S$; denote this by $\widetilde{F}$. Since $T_xM = T_x(G \cdot x) + T_xS$ for each $x \in S$, by Lemma **12.6**, $\widetilde{F}$ is submersive at each $(1_G, x)$. For any $x \in M$, $g \in G$, and $(\xi, v) \in T_{(1_G,x)}(G \times M)$,

$$\alpha_*((g, \xi), v) = g_*\alpha_*(\xi, v).$$

Since $g_* = \rho(g)_*$ is a linear isomorphism from T_xM to $T_{g\cdot x}M$, it follows that $\widetilde{F}$ is submersive at all $(g, x) \in G \times S$. By Exercise **13.4**, the image $U := \mathrm{im}\big(\widetilde{F}\big)$ is open.

Suppose that (g_1, x_1) and (g_2, x_2) satisfy $[g_1, x_1] = [g_2, x_2]$ in $G \times_H S$. There is some $h \in H$ such that $(g_2, x_2) = (g_1h^{-1}, h \cdot x_1)$ Then

$$\widetilde{F}(g_2, x_2) = g_2 \cdot x_2 = g_1h^{-1} \cdot h \cdot x_1 = g_1 \cdot x_1 = \widetilde{F}(g_1, x_1).$$

This shows that $\widetilde{F}$ descends to a well-defined function $F\colon G \times_H S \to U$ sending $[g, x]$ to $g \cdot x$.

Suppose $(g_1, x_1), (g_2, x_2) \in G \times S$ such that $\widetilde{F}(g_1, x_1) = \widetilde{F}(g_2, x_2)$. Then $g_1 \cdot x_1 = g_2 \cdot x_2$, or $x_2 = g_2^{-1}g_1 \cdot x_1$. Since $x_1, x_2 \in S$, by definition of a slice, $g_2^{-1}g_1 =: h \in H$. Thus

$$[g_2, x_2] = [g_2, h \cdot x_1] = [g_2h, x_1] = [g_1, x_1]$$

in $G \times_H S$. It follows that F is injective. Moreover, since $\widetilde{F}$ is surjective, we conclude that F is bijective. Since the quotient map $G \times S \to G \times_H S$ is a surjective submersion by the Quotient Manifold Theorem, and $\widetilde{F}$ is smooth, we conclude that F is smooth by Lemma **10.17**. Since $\widetilde{F}$ is submersive at each point, so is F, and so we conclude that F is as diffeomorphism. In fact, for $g' \in G$ and $[g, x] \in G \times_H S$,

$$F(g' \cdot [g, x]) = F([g'g, x]) = g'g \cdot x = g' \cdot F[g, x],$$

and so F is a G-equivariant diffeomorphism. The remainder of the proof follows from Remark **12.14**. $\square$

Remark 13.6. An explicit way of writing down the G-invariant neighbourhood U in Theorem **13.5** is as $G \cdot S$. ⌟

Week 14: The Orbit Type Stratification

The moral of the Equivariant Tubular Neighbourhood Theorem is that a compact Lie group action locally about a point looks like an orbit cross a linear action of the stabiliser of that point. How the picture generalises to the entire orbit requires knowledge of fibre bundles, which is outside the scope of this course. However, we can look more closely at the structure of the linear action, and how this structure gets "dragged out" along the orbits by the action.

It turns out the way a compact Lie group action behaves about a point is highly influenced by how it behaves in a neighbourhood of that point; in particular, the stabiliser of a given point is related to that of nearby points. For instance, suppose (x_i) is a sequence in a manifold M with a compact Lie group action of G on it such that for each i, a closed subgroup H is a subgroup of $\mathrm{Stab}(x_i)$. Suppose also that (x_i) converges to $x \in M$. For any $h \in H$, we have

$$\alpha(h, x) = \lim \alpha(h, x_i) = \lim h \cdot x_i = \lim x_i = x$$

by continuity of the action. In other words, we have the following lemma.

Lemma 14.1. *Let $G \circlearrowright M$ be a compact Lie group action, and let M^H be the set of all points that are fixed by a closed subgroup H. Then M^H is a closed subset of M.*

So, if we restrict to the corresponding H-action on M, and fix $x \in M^H$, then x is a fixed point of the H-action. By Bochner's Linearisation Theorem, there are H-invariant open neighbourhoods U of x and V of $0 \in T_xM$ and an H-equivariant diffeomorphism $\lambda\colon U \to V$; that is, we can approximate the H-action about x by a linear H-action.

Exercise 14.2. Let H be a Lie group acting linearly on a vector space W. Then W^H is a linear subspace of W.

Combining Lemma **14.1** and Exercise **14.2** yields the following proposition.

Proposition 14.3. *Let $G \circlearrowright M$ be a compact Lie group action. For any closed subgroup $H \leq G$, the connected components of M^H are closed submanifolds of M.*

Proof. Fix a closed subgroup $H \leq G$. If $M^H = \emptyset$, then we are done. Otherwise, suppose $x \in M^H$. By Exercise **14.2** and the argument above, there are H-invariant open neighbourhoods U of x, V of $0 \in T_xM$, and an H-equivariant diffeomorphism $\lambda\colon U \to V$. Since λ is H-equivariant, it follows that $M^H \cap U = \lambda^{-1}((T_xM)^H \cap V)$; that is, $M^H \cap U$ is diffeomorphic to an open subset of a linear subspace of a vector space. Since x is arbitrary, it follows that each connected component of M^H has this feature, and so each connected component is a submanifold of M. Closedness follows from Lemma **14.1**. □

Remark 14.4. We have to take connected components in the statement of Proposition **14.3**, as the dimensions of these components do not have to be equal. ⌟

We now want to consider going along an orbit.

Exercise 14.5. Let $G \circlearrowright M$ be a Lie group action, and fix $x \in M$. If $H = \mathrm{Stab}(x)$, then $gHg^{-1} = \mathrm{Stab}(g \cdot x)$. In particular, stabilisers of points in the same orbit are isomorphic as groups.

Definition 14.6 (Orbit Types)**.** Let $G \circlearrowright M$ be a Lie group action. Define an equivalence relation $\sim$ on M by: $x \sim y$ if there is a G-equivariant bijection between $G \cdot x$ and $G \cdot y$. The equivalence classes are called **orbit types** of the action. ⋄

Exercise 14.7. Verify that $\sim$ in Definition **14.6** is an equivalence relation.

Lemma 14.8. *Given a Lie group action $G \circlearrowright M$ and $x, y \in M$, we have $x \sim y$ if and only if* $\mathrm{Stab}(x)$ *and* $\mathrm{Stab}(y)$ *are conjugate subgroups of G.*

Proof. ($\Rightarrow$) Suppose $x \sim y$. Then there is a G-equivariant bijection $\varphi\colon G \cdot x \to G \cdot y$. In particular, there is some $g_\varphi \in G$ so that $\varphi(x) = g_\varphi \cdot y$, and thus $\varphi(g \cdot x) = gg_\varphi \cdot y$ for all $g \in G$.

Let $z = \varphi(x) = g_\varphi \cdot y$. If $h \in \mathrm{Stab}(x)$, then

$$h \cdot z = h \cdot \varphi(x) = \varphi(h \cdot x) = \varphi(x) = z,$$

and so $h \in \mathrm{Stab}(z)$. On the other hand, since φ is a bijection, if $k \in \mathrm{Stab}(z)$, then

$$k \cdot x = k \cdot \varphi^{-1}(z) = \varphi^{-1}(k \cdot z) = \varphi^{-1}(z) = x.$$

Thus we can conclude that $\mathrm{Stab}(x) = \mathrm{Stab}(z)$. Exercise **14.5** now finishes the proof of this direction.

($\Leftarrow$) Suppose $\mathrm{Stab}(y) = g\mathrm{Stab}(x)g^{-1}$ for some $g \in G$. Then the map $\mathrm{Stab}(x)\backslash G \to \mathrm{Stab}(y)\backslash G$ sending $\mathrm{Stab}(x)g'$ to $\mathrm{Stab}(y)g'$ is a well-defined diffeomorphism (why?). By Proposition **10.16**, this implies that the orbit $G \cdot x$ and $G \cdot y$ are in bijection, and so $x \sim y$. ☐

Notation 14.9. Given a Lie group action $G \circlearrowright M$, denote by $M_{(H)}$ the set

$$\{x \in M \mid \mathrm{Stab}(x) \text{ is conjugate to } H\}.$$

⋄

Note that $M_{(H)} \subset G \cdot M^H$ for any subgroup H of G, and that $M_{(H)}$ is G-invariant.

Corollary 14.10. *Given a Lie group action $G \circlearrowright M$ and $x \in M$, the orbit type of x is equal to $M_{(H)}$ where $H = \mathrm{Stab}(x)$.*

Proof. If $y \in M_{(H)}$, then there is some $g \in G$ such that $g \cdot y \in M^H$. Thus the stabiliser of y is $g^{-1}Hg$, and we conclude that $x \sim y$ by Lemma **14.8**.

Conversely, if $x \sim y$, then $\mathrm{Stab}(y) = gHg^{-1}$ for some $g \in G$ by Lemma **14.8**, in which case $y \in M_{(H)}$ by definition. ☐

Given a manifold M, a subset $S \subseteq M$ is **locally closed** if it is equal to the union of a closed subset of M and an open subset of M. More intrinsically, for manifolds, this is equivalent to S being **locally compact** in the subspace topology of M; that is, every point of S had a compact neighbourhood in the subspace topology. Examples of subspaces of $\mathbb{R}^2$ that are *not* locally closed include $\mathbb{R}^2 \smallsetminus \{(0, y) \mid y \neq 0\}$ and the open disk with one of its boundary points. The irrationally-sloped line in $\mathbb{T}^2$ is an example of a submanifold that is not locally closed. Any embedded submanifold, however, is locally closed. We are now on the verge of introducing spaces that are no longer manifolds, but spaces that have nice partitions into manifolds.

Definition 14.11 (Stratified Space)**.** Let X be a topological space. A **stratification** of X is a locally finite partition $\mathcal{S}$ (*i.e.* for every $x \in X$ there is an open neighbourhood which intersects only finitely-many pieces of the partition) satisfying the following two conditions:

(1) **(Manifold Condition)** Each piece of $\mathcal{S}$, called a **stratum**, is a locally closed smooth manifold, and

(2) **(Frontier Condition)** if $S_1, S_2 \in \mathcal{S}$ such that $S_1 \cap \overline{S_2} \neq \emptyset$, then S_1 is in the boundary of S_2. ◇

Example 14.12. Let $p\colon \mathbb{R}^m \to \mathbb{R}^n$ be a surjective polynomial. Any level set of p comes equipped with a stratification, and so is a stratified space. For instance, consider the level set $xy = 1$, the union of the axes of $\mathbb{R}^2$. This has two strata: the origin, and the complement (which is the union of four rays). //

It turns out that given a proper Lie group action $G \circlearrowleft M$, the orbit types induce a stratification of M, which in turn induce a stratification of ${}_{G}\backslash^{M}$. This is important, since without freeness of the action, the space ${}_{G}\backslash^{M}$ is generally not a manifold; the stratification gives us a way of understanding its structure.

To prove this result, we need another definition and a series of lemmas.

Definition 14.13 (Local Action Types)**.** Let $G \circlearrowleft M$ be a Lie group action. Define an equivalence relation $\approx$ on M as follows: $x \approx y$ if for any G-invariant open neighbourhoods U and V of x and y, resp., after possibly shrinking U and V, there exists a G-equivariant diffeomorphism $\Phi\colon U \to V$ sending $G \cdot x$ onto $G \cdot y$. The equivalence classes are called **local action types**. ◇

Remark 14.14. This is a modification of [**DK00**, Definition 2.6.5], which is slightly erroneous: in their definition, one only requires for there to exist U and V and a diffeomorphism Φ. However, in this case, one can take $U = V = G$ and $\Phi = \mathrm{id}_M$. I believe Definition **14.13** is more along the lines of what the authors intended. ⌟

Exercise 14.15. Verify that $\approx$ in Definition **14.13** is an equivalence relation. Also, show that a local action type is contained within an orbit type, and that local orbit types are G-invariant.

Lemma 14.16. *Let $G \circlearrowright M$ be a proper Lie group action, let $x_0 \in M$ with stabiliser H, and let S be a slice for the action through x_0. Any two points in S^H are in the same local action type. Consequently, any two points in $G \cdot S^H$ are in the same local action type.*

Proof. By the Equivariant Tubular Neighbourhood Theorem and Remark **12.14**, it suffices to show that for any $v, w \in V^H$, the two points $[1_G, v]$ and $[1_G, w]$ in $G \times_H V$ are in the same local action type for the standard G-action on $G \times_H V$, where $V = T_{x_0}M/T_{x_0}(G \cdot x_0)$. By Exercise **14.2**, V^H is a linear subspace of V, on which the H-action can be assumed to be orthogonal by Corollary **12.4** and Item **3** of Exercise **12.12** with respect to an invariant inner product on V. Choose $\varepsilon > 0$ sufficiently small so that the open balls $B_\varepsilon^{V^H}(v)$ and $B_\varepsilon^{V^H}(w)$ in V^H of radius ε centred at v and w, resp., are disjoint. Let $B_\varepsilon^{(V^H)^\perp}(v)$ and $B_\varepsilon^{(V^H)^\perp}(w)$ be open balls in $v + (V^H)^\perp$ and $w + (V^H)^\perp$, resp., each of radius ε centred at v and w, resp. Denote by $B_v := B_\varepsilon^{V^H}(v) \times B_\varepsilon^{(V^H)^\perp}(v)$ and $B_w := B_\varepsilon^{V^H}(w) \times B_\varepsilon^{(V^H)^\perp}(w)$; these are disjoint product open neighbourhoods of v and w, resp., in V. Moreover, there is an H-equivariant diffeomorphism $\varphi\colon B_v \to B_w$ sending $u \in B_v$ to $u - v + w \in B_w$.

In $G \times V$, the sets $G \times B_v$ and $G \times B_w$ are G-invariant open neighbourhoods of $(1_G, v)$ and $(1_G, w)$, resp., with respect to the diagonal action of G on $G \times V$, in which G acts by left multiplication on G and trivially on V. The quotient map $G \times V \to G \times_H V$ is open (see Lemma **6.21**), and so $G \times B_v$ and $G \times B_w$ descend to open neighbourhoods U_v and U_w of $[1_G, v]$ and $[1_G, w]$, resp. We claim that $\mathrm{id}_G \times \varphi$ descends to a G-equivariant diffeomorphism $\Phi\colon U_v \to U_w$ sending $[g, v']$ to $[g, \varphi(v')]$. Since $\mathrm{id}_G \times \varphi$ is H-equivariant with respect to the anti-diagonal action of H on $G \times V$, Φ is a well-defined bijection. Smoothness in both directions follow from Lemma **10.17**. Finally, G-equivariance of Φ follows from that of the quotient map $G \times V \to G \times_H V$.

The last statement follows from the fact that the action is via diffeomorphisms, which themselves are homeomorphisms. □

Lemma 14.17. *Let $G \circlearrowright M$ be a proper Lie group action, let $x_0 \in M$ with stabiliser H, and let S be a slice for the action through x_0.*

$$M_{(H)} \cap G \cdot S = G \cdot S^H.$$

Proof. Let $x \in M_{(H)} \cap G \cdot S$. There is some $g \in G$ such that $y := g \cdot x \in S$, and some $g' \in G$ such that $\mathrm{Stab}(x) = g'H(g')^{-1}$. Hence, $gg'h(g')^{-1}g^{-1} \cdot y \in S$ for any $h \in H$. By definition of a slice, $gg'h(g')^{-1}g^{-1} \in H$ for all $h \in H$; that is, the conjugate $gg'H(g')^{-1}g^{-1} \leq H$. Since conjugation by gg' is a diffeomorphism on G and H is compact, it follows that $gg'H(g')^{-1}g^{-1} = H$. Since $\mathrm{Stab}(y) = gg'H(g')^{-1}g^{-1} = H$, we have $y \in S^H$, and so we conclude that $x \in G \cdot S^H$.

In the other direction, let $x \in G \cdot S^H$. There exists $g \in G$ such that $y := g \cdot x \in S^H$, and so $\mathrm{Stab}(y) \geq H$. But by definition of a slice, $\mathrm{Stab}(y)$ sends an element of S (namely, y) to S, and so $\mathrm{Stab}(y) \leq H$. Thus $\mathrm{Stab}(y) = H$, and we conclude that $x \in M_{(H)}$. □

Corollary 14.18. *Let $G \circlearrowright M$ be a proper Lie group action, let $x_0 \in M$, and let $H =$ $\mathrm{Stab}(x_0)$. Then the connected components of $M_{(H)}$ are locally closed submanifolds of M.*

Proof. Since $G \cdot S^H$ via the Equivariant Tubular Neighbourhood Theorem can be identified with the subspace $G \times_H V^H \cong \left(H \backslash G \right) \times V^H$, where $V = T_{x_0}M/T_{x_0}(G \cdot x_0)$, it is a submanifold of M. Moreover, V^H is a closed submanifold of $G \times_H V^H$, and hence S^H is a closed submanifold of $G \cdot S^H$, and hence is locally compact. By Lemma **14.17**, $M_{(H)} \cap G \cdot S$ is a locally compact submanifold of M, and thus so are connected components of $M_{(H)}$. ☐

Corollary 14.19. *Let $G \circlearrowright M$ be a proper Lie group action, and let $x_0 \in M$. The local action type of x_0 is an open subset of the orbit type of x_0.*

Proof. By Exercise **14.15**, the local action type of x_0 is contained in the orbit type of x_0. Let $H = \mathrm{Stab}(x_0)$, and so the orbit type of x_0 is $M_{(H)}$. Thus we only need to show that there is an open neighbourhood of x_0 in $M_{(H)}$ that is contained in the local action type of x_0.

Let S be a slice for the action through x_0. By Lemma **14.17**, the G-invariant open neighbourhood $M_{(H)} \cap G \cdot S$ of x_0 in $M_{(H)}$ is equal to $G \cdot S^H$. However, by Lemma **14.16**, elements of S^H are in the same local action type, and since local action types are G-invariant by Exercise **14.15**, $G \cdot S^H$ is contained in the same local action type. This serves as our open neighbourhood of x_0. ☐

★Exercise 14.20. Let X be a locally path-connected topological space; this means that about any $x \in X$, given an open neighbourhood U of x, there exists an open neighbourhood $V \subseteq U$ of x that is path-connected (*i.e.* every point in V is in the image of a continuous map $c\colon [0,1] \to V$). Let $C \subseteq X$ be non-empty, open, and closed. Show that C is a union of path-connected components of X. In particular, if C is path-connected, then it is an entire path-connected component of X.

Corollary 14.21. *Let $G \circlearrowright M$ be a proper Lie group action, and let $x_0 \in M$. The local action type of x_0 is a union of (path-)connected components of the orbit type of x_0. Consequently, the local action types of x_0 are locally closed submanifolds of M.*

Proof. By Corollary **14.19**, the local action type of x_0 is an open subset of the orbit type of x_0. But the orbit type is a union of (disjoint) local action types. Thus the complement of the local action type in the orbit type is a union of local action types, all of which are open. Thus, the complement of a local action type is open, and hence a local action type is also closed. By ★Exercise **14.20**, the first statement follows. The second statement follows from Corollary **14.18** and the fact that connected components of local action types have the same dimension; in particular, the dimension $\dim G + \dim V^H - \dim H$ using the notation set above in the previous corollaries and lemmas. ☐

The connected components of the local action types will serve as the strata of a stratification of M. So far, the Manifold Condition is shown by Corollary **14.21**. We still need to show local finiteness of the partition into the local action types, as well as the Frontier Condition. We prove local finiteness by induction on the dimension of M.

If $\dim M = 0$, then the result is immediate. Suppose the local finiteness result is true for all manifold of dimension less than $m > 0$. Let $G \circlearrowright M$ is a proper Lie group action and $\dim M = m$. Fix $x_0 \in M$. By the Equivariant Tubular Neighbourhood Theorem, we can identify an open G-invariant neighbourhood of x_0 with $G \times_H V$ where $H = \mathrm{Stab}(x_0)$ and $V = T_{x_0}M/T_{x_0}(G \cdot x_0)$; it suffices to prove local finiteness for the G-action on $G \times_H V$. A point $[g, v] \in G \times_H V$ has the same local action type as $[1_G, v]$, and so it suffices to prove the result for V itself. If $\dim V < m$, then we are done. Suppose otherwise; the proof is now reduced to showing this for an orthogonal action of a compact Lie group H on an m-dimensional vector space V with an invariant inner product.

The linear subspace $V_{(H)} = V^H$ is the linear action type of 0. The orthgonal subspace $W := V_{(H)}^{\perp}$ is H-invariant. For a non-zero point $w \in W$, its linear action type is the same as that of $w/|w|$ in the unit sphere Σ of W. In particular, $W \setminus \{0\}$ is H-equivariantly diffeomorphic to $\Sigma \times \mathbb{R}^+$; the linear action types in $W \setminus \{0\}$ are precisely $T \times \mathbb{R}^+$ under this identification, where T runs over the local action types of Σ. Since Σ is a manifold of dimension at most $m - 1$, by the induction hypothesis, there are only finitely-many such T. We have just proven the following lemma.

Lemma 14.22. *Given a proper Lie group action on a manifold, the partition of the manifold into connected components of local action types is locally finite.*

Returning to the linear situation in the paragraph above, note that the linear action types of V are precisely V^H and $V^H + T \times \mathbb{R}^+$ as T runs over the local action types of Σ. In particular, these satisfy the Frontier Condition: V^H is in the boundary of all connected components of local action types. This property now extends to connected components of local action types of a proper Lie group action via the Equivariant Tubular Neighbourhood Theorem. By Corollary **14.21** we now have the following theorem.

Theorem 14.23. *Let $G \circlearrowright M$ be a proper Lie group action. The connected components of the orbit types form a stratification of M, called the **orbit type stratification**.*

It turns out that the orbit type stratification of M descends via the quotient map to a stratification of $G \backslash M$, also called the **orbit type stratification**. In the notation above, $G \times_H V$ descends to $G \backslash G \times_H V$, which can be identified with $H \backslash V$, in which V^H remains present in $H \backslash V$ as a stratum (proving the Manifold Condition). The rest of the condition for a stratification follow from those of the orbit type stratification on M. However, we will not have time to cover this.

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