

Linear Circle Actions on Complex Space

Extending a result from orbifolds

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Introduction

Circles, spheres and tori are examples of “nice” spaces - what these spaces have in common is their smoothness, i.e. the spaces have no corners or cusps. This allows us to perform basic calculus on the objects with a simple extension of undergraduate analysis. However, consider spaces with corners and or/cusps, or other strange artifacts, such as a cone in 3-space. A large class of these singular spaces are known as **orbifolds**, defined as spaces which locally take the appearance of a Euclidean space modulo a finite group action. Calculus on these surfaces can be defined; it is a prior result that the set of infinitely differentiable (or smooth) functions on the orbifolds forms a ring which can detect the corners, boundaries and cusps of the object and distinguish between different orbifolds [4].

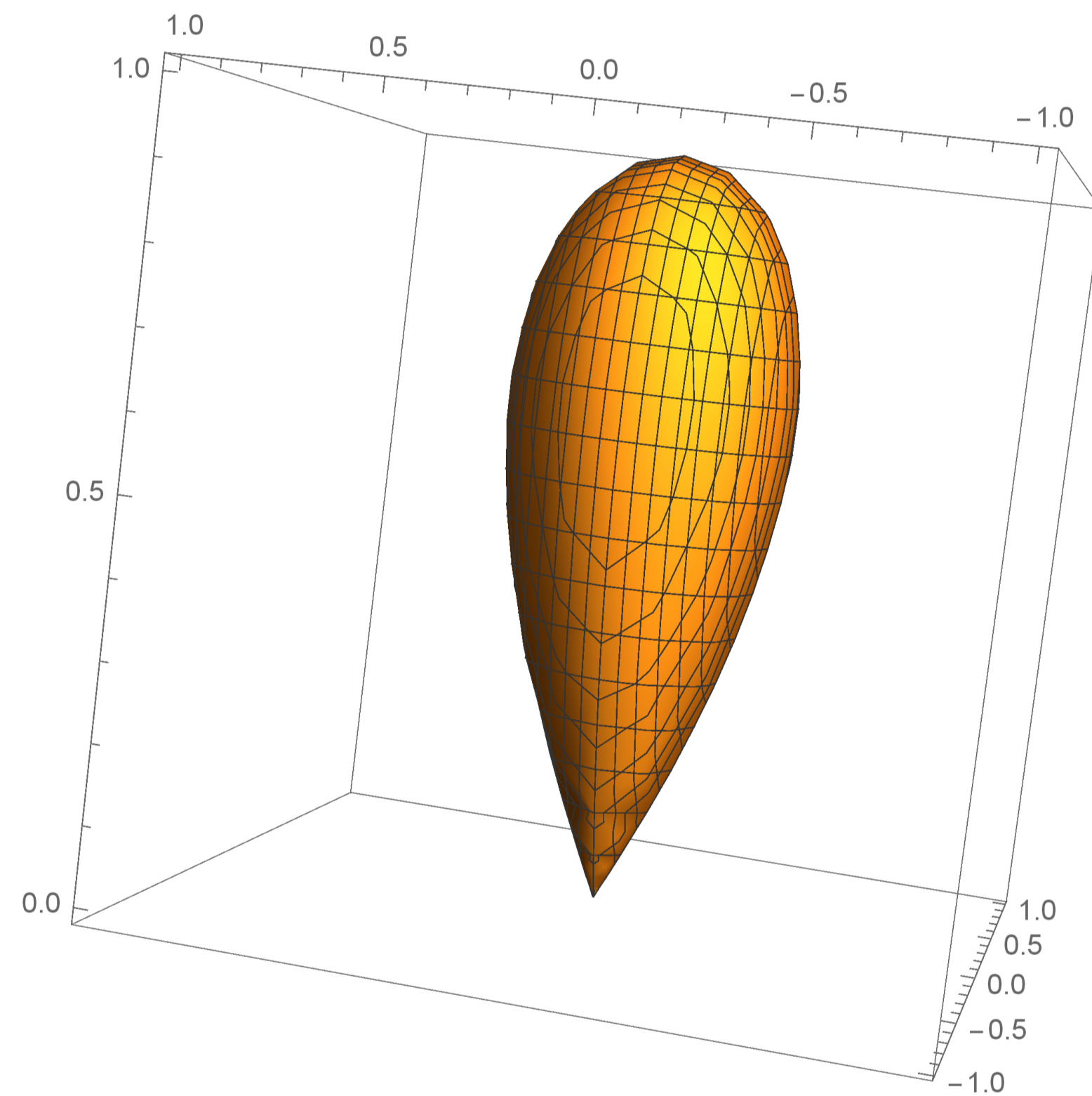


Figure 1: A cross-section of the quotient space produced by $S^1 \curvearrowright \mathbb{C}^2$ with weights 1 and 2

Our project examines analogous situations - but rather than orbifolds, we consider spaces which look like an Euclidean space modulo a circle action, one of the simplest cases of an infinite group action. Considering the ring of smooth functions, can we distinguish between different spaces to the same degree as with orbifolds?

Main Objectives

1. Understand the circle action on \mathbb{C}^2
2. Generalize to \mathbb{C}^n for all n .
3. Look at generalizing to other spaces and group actions, such as the torus action.

Materials and Methods

Naiche Downey wrote Mathematica software to determine the invariant polynomials needed to distinguish between spaces generated by the circle action along with the relations between them. This allowed us to determine when the spaces \mathbb{C}^n/S^1 were considered distinct via the ring of smooth functions. To check our results the number of polynomials generated was compared to the output of another Mathematica program designed to determine how many invariant polynomials exist at each degree [2].

Example : $S^1 \curvearrowright \mathbb{C}^2$

Define the action of S^1 on \mathbb{C}^2 by $e^{2\pi i\theta} \cdot (z_1, z_2) = (e^{2\pi i a_1 \theta} z_1, e^{2\pi i a_2 \theta} z_2)$ with $\theta \in [0, 1]$. Assuming the action is effective requires that the weights, a_1, a_2 be relatively prime. Then the invariant polynomials are

$$P_1 = |z_1|^2 \quad (1)$$

$$P_2 = |z_2|^2 \quad (2)$$

$$P_3 = \operatorname{Re}(z_1^{a_2} * z_2^{a_1}) \quad (3)$$

$$P_4 = \operatorname{Im}(z_1^{a_2} * z_2^{a_1}) \quad (4)$$

These are further restricted with relationships; that $P_1, P_2 \geq 0$ and:

$$P_3^2 + P_4^2 = P_1^{a_2} * P_2^{a_1} \quad (5)$$

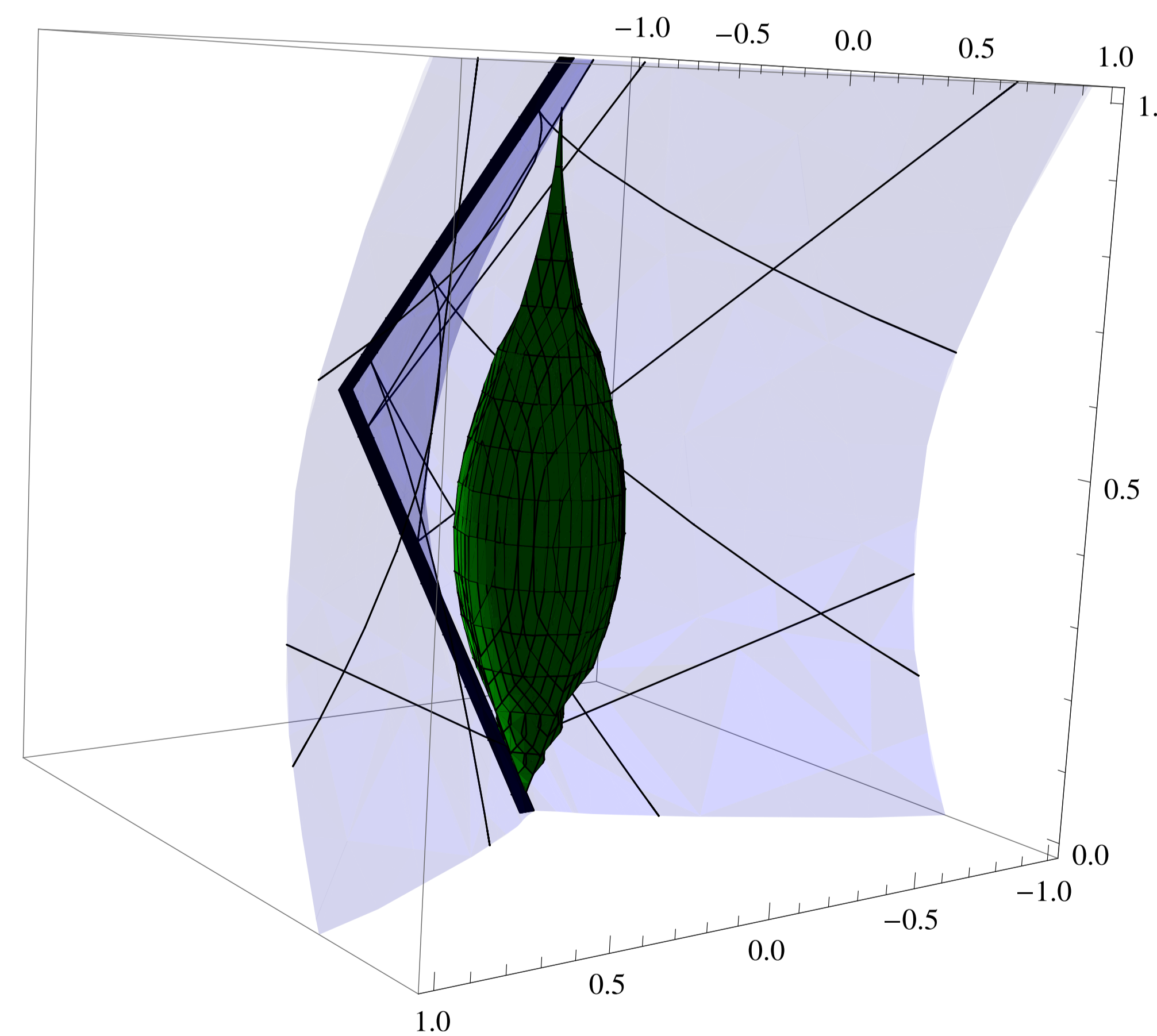


Figure 2: Stratification of $S^1 \curvearrowright \mathbb{C}^2$ with weights 3 and 2

Results

Theorems

Theorem 1. Let S^1 act linearly and effectively on \mathbb{R}^n . Then the orbit space \mathbb{R}^n/S^1 , equipped with its smooth structure $C^\infty(\mathbb{R}^n/S^1)$, contains enough invariants such that the linear action of S^1 on \mathbb{R}^n can be recovered (up to S^1 equivariant diffeomorphism).

Theorem 2. Let S^1 act effectively on a connected manifold M .

1. If M/S^1 has no codimension-1 strata in its orbit-type stratification, then the circle action can be recovered from the differential space $(M/S^1, C^\infty(M/S^1))$ up to a mild form of equivalence.
2. If M/S^1 does contain codimension-1 strata in its orbit-type stratification, append to each codimension-1 stratum S an integer label $n = \dim \Gamma$ where Γ is the isotropy group at any point of S . Then the circle action can be recovered from the differential space $(M/S^1, C^\infty(M/S^1))$ with these integer labels up to a mild form of equivalence.

Example: The Codimension-1 Case

When codimension-1 strata are present, we need integer labels to tell the difference between orbit spaces of circle actions. For example, consider the circle acting on the 2-sphere by rotation. The orbit space is diffeomorphic to a compact connected interval $[0, 1]$. Both endpoints have isotropy groups equal to the circle itself.

By identifying antipodal points on 2-sphere one obtains the real projective plane, and the circle action descends to this. The orbit space is again diffeomorphic to $[0, 1]$ but one endpoint has the circle as an isotropy group, while the other has the cyclic group \mathbb{Z}_2 .

Forthcoming Research

We hope to continue work with the torus action and submit a paper.

References

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Acknowledgements

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