Tame Circle Actions

CMS Winter Meeting 2015

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Motivation

Question: Does there exist a symplectic non-Hamiltonian circle action on a closed connected symplectic manifold admitting isolated fixed points?

- Frankel showed that in the (closed) Kähler case, a Kähler circle action is Hamiltonian if and only if it admits fixed points.

- McDuff showed that any symplectic circle action on a 2- or 4-dimensional closed connected symplectic manifold is Hamiltonian if and only if it admits fixed points.

- However, she also constructed an example of a six-dimensional closed symplectic manifold with a symplectic non-Hamiltonian circle action admitting tori of fixed points.

The Answer

Susan Tolman recently answered the question in the affirmative by constructing a six-dimensional such manifold.

We briefly describe this construction.

Piece I

Tolman begins with an auxiliary symplectic manifold (M', ω') admitting a free Hamiltonian circle action.

The symplectic quotient at each value of the momentum map is a tame $K3$ surface whose symplectic form satisfies certain properties.

Using the $K3$ surfaces is key to making this construction work. (See Tolman's paper for details.)

Pieces II and III

Next, introduce two more symplectic complex manifolds (M_+, ω_+) and (M_-, ω_-) admitting locally free Hamiltonian circle actions.

For each of these spaces, the symplectic quotient at each value of the momentum map is a symplectic Kummer surface whose symplectic form again satisfies certain properties.

The **Kummer surface** is a complex orbifold with 16 isolated $\mathbb{Z}_2$-singularities (and is smooth otherwise). If you blow up each singularity (in the complex category), then you obtain a $K3$ surface.

Pieces IV and V

Finally, do certain modifications to (M_+, ω_+) and (M_-, ω_-) to obtain two new symplectic complex manifolds $(\widetilde{M}_+, \widetilde{\omega}_+)$ and $(\widetilde{M}_-, \widetilde{\omega}_-)$ admitting Hamiltonian circle actions, each with 16 fixed points all lying in the momentum map level set of ± 1.

For $(\widetilde{M}_+, \widetilde{\omega}_+)$, the symplectic quotient at each value of the momentum map less than 1 is a symplectic Kummer surface; for values greater than 1, we obtain a tamed $K3$ surface. (A similar property holds for $(\widetilde{M}_-, \widetilde{\omega}_-)$.)

Gluing

All five pieces glue together via symplectic diffeomorphisms that intertwine the momentum maps.

Moreover, the two “ends” of this resulting manifold can be further glued together, forming a symplectic manifold equipped with a symplectic non-Hamiltonian circle action: the old momentum map becomes a “generalised” momentum map, taking values in the circle.

Gluing

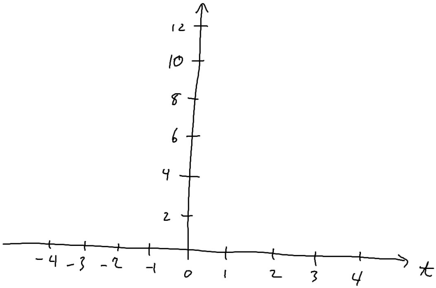

Figure: Duistermaat-Heckman Graph I.

Gluing

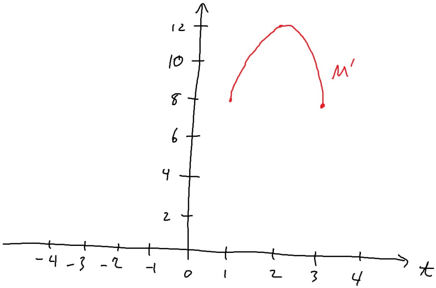

Figure: Duistermaat-Heckman Graph II.

Gluing

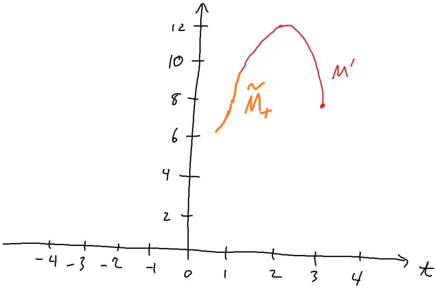

Figure: Duistermaat-Heckman Graph III.

Gluing

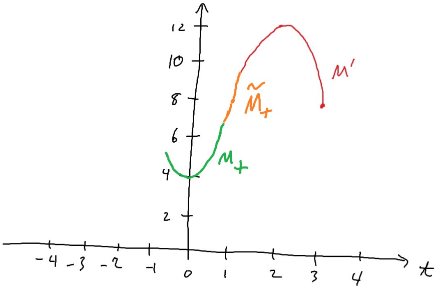

Figure: Duistermaat-Heckman Graph IV.

Gluing

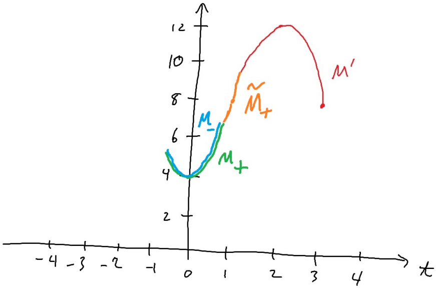

Figure: Duistermaat-Heckman Graph V.

Gluing

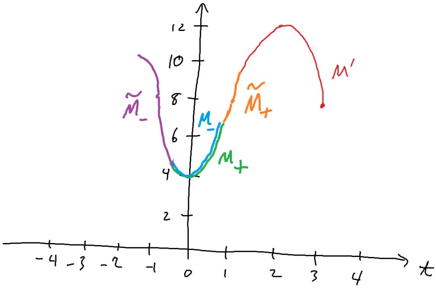

Figure: Duistermaat-Heckman Graph VI.

Gluing

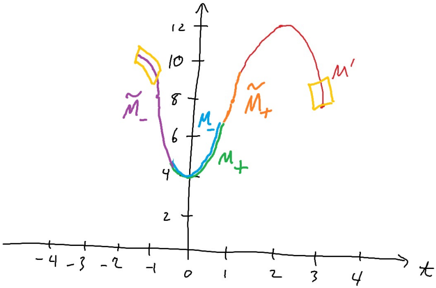

Figure: Duistermaat-Heckman Graph VII.

Back to the Construction!

We now will focus on the construction of $(\widetilde{M}_+, \widetilde{\omega}_+)$ out of (M_+, ω_+). (The construction of $(\widetilde{M}_-, \widetilde{\omega}_-)$ is similar.)

The Hamiltonian circle action on M_+ is **tame**.

(For the purposes of this talk) we say that a circle action on a symplectic complex manifold (M, ω, J) is **tame** if it is the restriction of a holomorphic $\mathbb{C}^*$-action that respects the symplectic structure, and $\omega(\xi_M, J\xi_M) > 0$ on the complement of the fixed-point set.

Here, ξ_M refers to the vector field on M induced by the circle action.

Back to the Construction!

To obtain $(\widetilde{M}_+, \widetilde{\omega}_+)$, first do a symplectic cut at the momentum map value of 1. The result is a symplectic orbifold with isolated $\mathbb{Z}_2$-singularities.

Then, blow up each of these singularities to obtain a new symplectic manifold.

Wait: we need to do these operations in both the symplectic and complex categories. This would be fine in the Kähler category, but *we are not in the Kähler category*. We need to do these constructions in the tame setting.

Wait Again: we also need to show that if for $t < 1$ the symplectic quotient $\widetilde{M}_+ //_t \mathbb{S}^1$ is a symplectic Kummer surface, then passing through the critical value $t = 1$ to positive t yields the $K3$ surface; *i.e.* the blow-up of the Kummer surface at its singularities. This requires a tame version of the birational equivalence theorem of Guillemin and Sternberg.

Main Results

Theorem: (Tolman-W., 2015)

1. Symplectic cutting works in the tame setting.
2. Starting with a symplectic complex manifold/orbifold (M, ω, J) in the tame setting, if J tames ω at a fixed point x, then the blow-up of M at x is obtained in the tame setting.
3. Let (M, ω, J) be a symplectic complex manifold in the tame setting. Assume that $\dim_{\mathbb{C}}(M) > 1$, and that the action has exactly k fixed points, all in the 0-level set, and all with weights $\{-1, 1, \ldots, 1\}$ (or all $\{-2, 1, \ldots, 1\}$). Then $M/\!/_{t>0}\,\mathbb{S}^1$ is the blow-up of $M/\!/_{t<0}\,\mathbb{S}^1$ at the k singularities in the sense of Guillemin and Sternberg's birational equivalence theorem.

Local Normal Forms and Reduction

Focusing on birational equivalence, the proof requires two main ingredients: reduction in the tame setting, and a local normal form theorem around fixed points in the tame setting.

Reduction follows from ingredients used to get the local normal form theorem.

The Local Normal Form Theorem: Let (M, ω, J) be a symplectic complex manifold with a holomorphic $\mathbb{C}^*$-action in which the restricted circle action is tame and Hamiltonian with momentum map Ψ. Assume $\{p_1, \ldots, p_k\}$ is a subset of the fixed points contained in $\Psi^{-1}(0)$. Then there exists a $\mathbb{C}^*$-invariant open neighbourhood of $\{p_1, \ldots, p_k\}$ which is $\mathbb{C}^*$-equivariantly biholomorphic to an open neighbourhood of $\bigsqcup_{j=1}^{k} \{0\}$ in $\bigsqcup_{j=1}^{k} \mathbb{C}^n$, on which $\mathbb{C}^*$ acts linearly.

Idea of Proof

The Observation: Given a Hamiltonian circle action on (M, ω, J) with momentum map Ψ that preserves J, the property $\omega(\xi_M, J\xi_M) > 0$ on the complement of the fixed point set is equivalent to the function $t \mapsto \Psi(e^t \cdot x)$ strictly increasing for each non-fixed point $x \in M$.

Bochner Linearisation Theorem: In the setting of the local normal form theorem, we have an $\mathbb{S}^1$-equivariant biholomorphism between $\mathbb{S}^1$-invariant neighbourhoods.

Definition: Let (M, J) be a complex manifold with a holomorphic $\mathbb{C}^*$-action. Then a subset $A \subseteq M$ is **orbitally convex** if it is $\mathbb{S}^1$-invariant and the set

$$\{t \in \mathbb{R} \mid e^t \cdot x \in A\}$$

is connected for all $x \in A$.

Orbital Convexity

If the $\mathbb{S}^1$-invariant neighbourhoods in the Bochner linearisation theorem can be chosen to be orbitally convex, then the $\mathbb{S}^1$-equivariant biholomorphism can be extended to a $\mathbb{C}^*$-invariant biholomorphism between $\mathbb{C}^*$-invariant neighbourhoods.

Let $\mathbb{C}^*$ act on $(\mathbb{C}^n, \omega)$ (where ω is *not necessarily standard*), in which the restricted $\mathbb{S}^1$-action is tame. Let U be an $\mathbb{S}^1$-invariant open neighbourhood of 0. Then there is an orbitally convex open neighbourhood $V \subseteq U$ of 0.

Moreover, The Observation allows us to ensure that we can transport this orbitally convex neighbourhood via (the inverse of) the biholomorphism from Bochner's linearisation theorem to our manifold.

Orbital Convexity

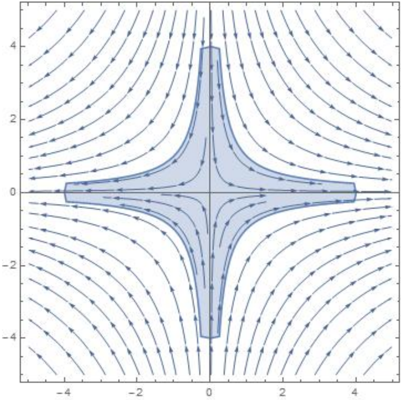

Figure: Orbitally Convex Neighbourhood - Weights 1 and -1.

Orbital Convexity

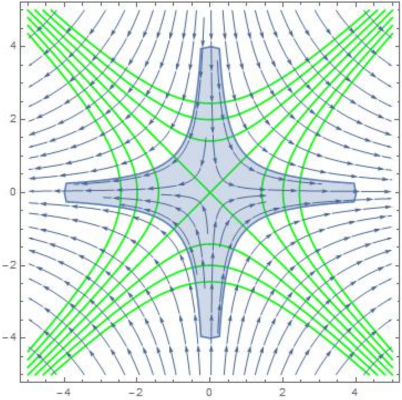

Figure: Orbitally Convex Neighbourhood - Weights 1 and -1 with Level Sets.

Orbital Convexity

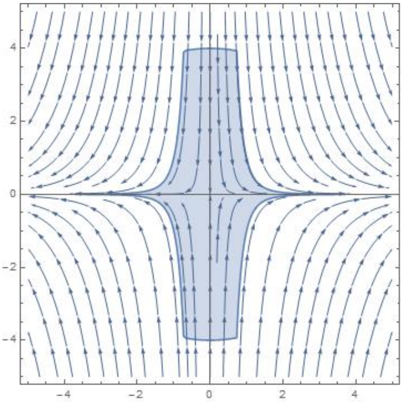

Figure: Orbitally Convex Neighbourhood - Weights 3 and -13.

Orbital Convexity

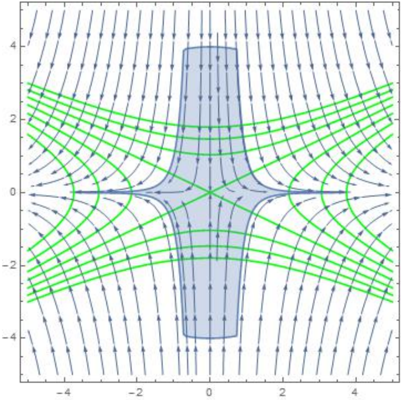

Figure: Orbitally Convex Neighbourhood - Weights 3 and -13 with Levels Sets.

Thank you!

References

- Susan Tolman, “Non-Hamiltonian Actions with Isolated Fixed Points”
  `http://arxiv.org/pdf/1510.02829.pdf`
- Susan Tolman and Jordan Watts, “Tame Circle Actions”
  `http://arxiv.org/pdf/1510.01721.pdf`