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The Calculus on Subcartesian Spaces

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Abstract

Subcartesian spaces are mathematical entities that arise in studies such as Mechanics, and are much more general than manifolds. The fact that dimension can change on these spaces leads to many questions about how one might calculate various quantities on them. In particular, this paper is concerned with properties that a subcartesian space possesses, especially at points where dimension changes; as well as examining the behaviour of various definitions of differential forms on such a space.

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Chapter 1

Introduction

In 1967, Roman Sikorski began to generalise differential geometry, where instead of using the typical ring of smooth functions, he used rings of functions that satisfied more general properties. This eventually lead to his definition of the category of differential spaces ([11] and [12]) which is the category in which we shall work. Chapter 2 will deal with this category in detail.

Also in 1967, Aronszajn introduced the notion of a subcartesian space [1], which is a Hausdorff space locally diffeomorphic to a differential subspace of Euclidean space. In particular, he gave an example of a square (boundary). The four corners must be diffeomorphic to a differential subspace of $\mathbb{R}^2$, whereas the sides away from the corners are diffeomorphic to a differential subspace of $\mathbb{R}$. Together with Szeptycki, Aronszajn also generalised differential geometry in order to study these spaces. Chapter 3 will define these spaces and introduce concepts such as structural dimension, structurally regular points and structurally singular points. Much of this was introduced by Aronszajn and Szeptycki in [2] and [3], although they used different terminology and utilised charts and atlases (whereas we shall use Sikorski's approach). The major result is the fact that the structurally regular part of a subcartesian space is open and dense in the space. This was stated in the two references above, however it is proven independently in this paper in detail.

Chapter 4 introduces the tangent space at a point, and the corresponding tangent (pseudo-)bundle. This was studied by Marshall in [8] and Aronszajn and Szeptycki in [3], but we shall use a more convenient definition of the space: the vector space of derivations at a point. The bundle itself is also a subcartesian space (as shown by Walczak [17]). The main results of this chapter are the fact that any derivation at a point x of $\mathbb{R}^n$ defines a derivation at x on a differential subspace of $\mathbb{R}^n$ containing x if and only if it annihilates the ideal of functions that go to zero on the differential subspace, and that the structural dimension of a point is equal to the dimension of the Zariski tangent space at that point (this was established by Aronszajn and Szeptycki in [3]). The latter result will be proved independently from the work of Aronszajn and Szeptycki as we shall work in a different category with different definitions. Finally, we explore differential structures on the fibred product, tensor product and wedge product of tangent bundles.

Next, we discuss global derivations (or derivations on the ring of smooth functions) in Chapter 5. As proven by Sikorski in [12], we see that the module of global derivations is exactly the module of smooth sections from S to TS. Analogous to the result in the previous chapter, we show that a global derivation of $\mathbb{R}^n$ defines (locally) a global derivation on a differential subspace of $\mathbb{R}^n$ if and only if it annihilates the ideal of functions that go to zero on the differential subspace. As well, expanding on the work of Sikorski and Walczak in [12] and [17], we show that the restriction of the tangent bundle projection to the structurally regular part of a subcartesian space is a locally trivial fibration. We then describe the behaviour of global derivations at structurally singular points. We finish the chapter with some discussion on Lie brackets and vector fields.

In chapter 6 we introduce a definition of a Riemannian metric on a subcartesian space that, although more restrictive than the Riemannian metric defined by

Kowalczyk [6], allows us to equip the cotangent bundle with a differential structure isomorphic to the differential structure on the tangent bundle. Also, we introduce differential structures on tensor and wedge products of the cotangent bundle.

Chapter 7 introduces differential forms on a subcartesian space; in particular, Zariski differential forms. These were studied in detail by Marshall [8] and Sasin [10] (the former does not use differential spaces, and the latter does use differential spaces along with sheaves), where they describe how the canonical definition of the exterior derivative on Zariski forms does not commute with pullback maps on Zariski forms. This problem was solved by use of equivalence classes of "differentials of zero", in which the resulting quotient forms a new module of Marshall differential forms on which we define the Marshall exterior derivative, which does commute with pullbacks. We take the time to prove many of the statements made by Marshall in [8], and make notes on the behaviour of these forms as smooth sections. We end this chapter with a note on Stokes' theorem, both for Zariski and Marshall forms (inspired by the closing notes in [8]).

Finally, chapter 8 introduces another notion of differential forms, which we dub Koszul differential forms. This is a purely algebraic definition, and many of the properties were generalisations of results on manifolds, taken from [5]. These forms have a well-defined exterior derivative, but pullbacks are not well-defined. We end with a comparison of Koszul forms with Zariski and Marshall forms similar to Sasin's work in [10], and a note on smooth sections from a subcartesian space to wedge products of its cotangent bundle.

Chapter 2

Differential Spaces

2.1 The Category of Differential Spaces

We begin by defining the category of differential spaces; in particular, its objects.

Definition 2.1.1. A *differential space* (sometimes called a *Sikorski space*) X is a topological space equipped with a differential structure. A *differential structure* is a family of functions, denoted $C^\infty(X)$, that satisfies the following:

1. The set $\{f^{-1}(a,b) \subseteq X | f \in C^\infty(X), (a,b) \subseteq \mathbb{R}\}$ is a sub-basis for the topology of X.
2. If $f_1, f_2, ..., f_n \in C^\infty(X)$ and $F \in C^\infty(\mathbb{R}^n)$ then $F(f_1, f_2, ..., f_n) \in C^\infty(X)$.
3. If $f : X \to \mathbb{R}$ is such that for any $x \in X$ there exists a neighbourhood $U \subseteq X$ of x and a function $f_x \in C^\infty(X)$ so that $f|_U = f_x|_U$, then $f \in C^\infty(X)$.

Hence we have a generalised notion of smooth (or infinitely differentiable) functions closed under localisation for arbitrary topological spaces. The topology may be conceived as the weakest topology on X such that the functions in $C^\infty(X)$ are continuous.

Next, we must define the morphisms of our category.

Definition 2.1.2. Let X and Y be differential spaces.

1. A map $\varphi : X \to Y$ is *smooth* if for every $f \in C^\infty(Y)$,

$$\varphi^* f := f \circ \varphi \in C^\infty(X).$$

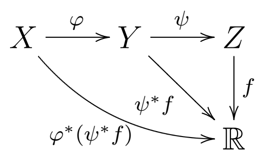
2. A map $\varphi : X \to Y$ is a *diffeomorphism* if it is a smooth homeomorphism with smooth inverse.

It is not difficult to see that smooth maps are continuous from the first condition of a differential space. Let $\varphi : X \to Y$ be a smooth map. For any $(a, b) \subseteq \mathbb{R}$, by definition of a sub-basis for a topology, we have that $f^{-1}((a, b))$ and $(\varphi^* f)^{-1}((a, b))$ are open sets in Y and X, respectively. But, $(\varphi^* f)^{-1} = \varphi^{-1} \circ f^{-1}$, so $\varphi^{-1}(f^{-1}((a, b)))$ is open for any sub-basis element $f^{-1}((a, b))$ of Y, and hence for any open set of Y.

In order to show that differential spaces form a category, we must show that the composition of smooth maps is smooth, that composition is associative, and that there exists a smooth identity map.

The composition of two smooth maps is smooth. Let X, Y and Z be differential spaces and let $\varphi : X \to Y$ and $\psi : Y \to Z$ be smooth maps. Then for any $f \in C^\infty(Z)$,

$$(\psi \circ \varphi)^* f = \varphi^*(\psi^* f) \in C^\infty(X).$$



From the fact that the composition of continuous maps is associative, we know that the composition of smooth maps are associative. Finally, the identity map is clearly a smooth map, and hence we have our category. Next, it is natural to define the sub-objects of the category.

Definition 2.1.3. Let X be a differential space and let $Y \subseteq X$ be a topological subspace of X. Then we can define a family of functions $C^\infty(Y)$ on Y as follows: $f \in C^\infty(Y)$ if and only if for any $x \in Y$ there exists a neighbourhood $U \subseteq X$ of x and a function $f_x \in C^\infty(X)$ such that $f|_{U\cap Y} = f_x|_{U\cap Y}$. With this induced structure, we call Y a *differential subspace* of X. We say that $C^\infty(Y)$ is *generated* by the restrictions of functions in $C^\infty(X)$ to Y.

We must still prove that a differential subspace is indeed a differential space.

Proposition 2.1.4. *Given a differential space X and a differential subspace $Y \subseteq X$, Y is a differential space.*

Proof. All we need to show is that $C^\infty(Y)$ is a differential structure on Y. This comprises of proving the three conditions outlined previously.

Condition 1: Let $f \in C^\infty(Y)$ and $\{x_i\}_{i\in I}$ be a net in Y such that $x_i \to x \in Y$. (If no (non-constant) nets converge in Y, then Y is a set of isolated points, and the condition is clear.) Let $U \subseteq X$ be a neighbourhood of x and $F \in C^\infty(X)$ such that $f|_{U\cap Y} = F|_{U\cap Y}$ (which exists by definition of a differential subspace). Now, we know that there exists $\alpha \in I$ such that for every $\beta \geq \alpha$, we have $x_\beta \in U$. Thus for every $\beta \geq \alpha$,

$$f(x_\beta) = F(x_\beta) \to F(x) = f(x)$$

since F is continuous. Thus, f is continuous.

We have yet to show that $C^\infty(Y)$ is rich enough to give a sub-basis for the subspace topology on Y. Let $y \in Y$ and $V \subseteq Y$ a neighbourhood of y. Then, there exists some open $U \subseteq X$ such that $V = U \cap Y$. But since $C^\infty(X)$ induces a sub-basis for the topology on X, we have that for some $(a,b) \subset \mathbb{R}$ and $f \in C^\infty(X)$, $y \in$

$f^{-1}((a,b)) \subseteq U$. But then $y \in (f|_Y)^{-1}((a,b)) \subseteq V$, and noting that $f|_Y \in C^\infty(Y)$, we are done of this condition.

Condition 2: Let $f_1, f_2, ..., f_m \in C^\infty(Y)$ and $G \in C^\infty(\mathbb{R}^m)$. Letting $x \in Y$, define $U_1, U_2, ..., U_m \subseteq X$ as neighbourhoods of x and $F_1, F_2, ..., F_m \in C^\infty(X)$ such that for every $i = 1, 2, ...m$, we have

$$f_i|_{U_i \cap Y} = F_i|_{U_i \cap Y}.$$

Let $U := \bigcap_{i=1}^m U_i$, which is an open neighbourhood of x. Then, for every $i = 1, 2, ..., m$,

$$f_i|_{U \cap Y} = F_i|_{U \cap Y}$$

But then,

$$G(f_1, f_2, ..., f_m)|_{U \cap Y} = G(F_1, F_2, ..., F_m)|_{U \cap Y} \in C^\infty(\mathbb{R}^m).$$

Condition 3: Let $f : Y \to \mathbb{R}$ be a function where for every $x \in Y$ there is an open neighbourhood $V \subseteq Y$ of x and $f_x \in C^\infty(Y)$ such that $f|_V = f_x|_V$. We wish to show that $f \in C^\infty(Y)$. Let $V = W \cap Y$ for some open set $W \subseteq X$. Since $f_x \in C^\infty(Y)$ we have for some open neighbourhood $U \subseteq X$ of x and $F_x \in C^\infty(X)$,

$$f_x|_{U \cap Y} = F_x|_{U \cap Y}.$$

Since $x \in W \cap U \subseteq U$, we have

$$f_x|_{W \cap U \cap Y} = F_x|_{W \cap U \cap Y}.$$

Finally, since $x \in W \cap U \cap Y \subseteq V$, we conclude

$$f|_{W \cap U \cap Y} = F_x|_{W \cap U \cap Y},$$

and so all three conditions of a differential structure are met. $\square$

So a differential subspace $Y \subseteq X$ is a topological subspace with a differential structure whose elements are locally restrictions of elements of $C^\infty(X)$. If a differential subspace Y of a differential space X is closed in X, then $C^\infty(Y)$ will simply contain restrictions of functions from $C^\infty(X)$. However, since we may not have a closed subset, we need a local definition. The next example justifies this.

Example 2.1.5. Consider the open interval $(0,1) \subset \mathbb{R}$ with the usual Euclidean topology. The function $f : x \mapsto 1/x$ is not in $C^\infty(\mathbb{R})$, since it is not bounded as it approaches $x = 0$. However, $f|_{(0,1)} \in C^\infty((0,1))$, since for every point in $(0,1)$, there exists a neighbourhood of that point contained in $(0,1)$ upon which f is a restriction of some smooth function in $C^\infty(\mathbb{R})$.

2.2 The Ring $C^\infty(X)$

With the obvious point-wise operations, $C^\infty(X)$ for a differential space X is a ring: the sum and product of functions in $C^\infty(X)$ are in $C^\infty(X)$ by condition 2 of a differential space, and these point-wise operations are, of course, local. In fact, these are commutative rings with unity, where the unity 1_X in this case is the function that maps all points to $1 \in \mathbb{R}$.

Lemma 2.2.1. *Let X, Y be differential spaces, and $\varphi : X \to Y$ a smooth map. Then, $\varphi^* : C^\infty(Y) \to C^\infty(X)$ is a ring homomorphism and maps 1_Y to 1_X.*

Proof. Let $f, g \in C^\infty(Y)$ and $x \in X$. Then,

$$\begin{aligned} \varphi^*(f+g)(x) =& (f+g) \circ \varphi(x) \\ =& f \circ \varphi(x) + g \circ \varphi(x) \\ =& (\varphi^* f + \varphi^* g)(x). \end{aligned}$$

A similar argument holds for point-wise multiplication. Finally, since $\varphi^* 1_Y = 1_Y \circ \varphi$, we see that $\varphi^* 1_Y = 1_X$. □

Definition 2.2.2. Let X be a differential space with a differential subspace $Y \subseteq X$. Then let

$$R(Y) := \{f \in C^\infty(Y) |\ f = g|_Y, g \in C^\infty(X)\}, \tag{2.2.1}$$

i.e. restrictions of functions in $C^\infty(X)$ to Y, and let

$$N(Y) := \{g \in C^\infty(X) |\ g|_Y = 0\}. \tag{2.2.2}$$

Remark 2.2.3. It is clear that $R(Y)$ is a subring of $C^\infty(X)$. Since the restriction to Y is a ring homomorphism with the kernel being $N(Y)$, by the isomorphism theorem of rings, we have

$$R(Y) \cong C^\infty(X)/N(Y) \tag{2.2.3}$$

Chapter 3

Subcartesian Spaces

3.1 The Subcategory of Subcartesian Spaces

Definition 3.1.1. A *subcartesian space* is a Hausdorff differential space S such that for any $x \in S$ there exists a neighbourhood $U \subseteq S$ of x that is diffeomorphic to some arbitrary differential subspace of $\mathbb{R}^n$ (i.e. there is a diffeomorphism $\varphi : U \to V$ where $V \subseteq \mathbb{R}^n$ is arbitrary). We shall assume all subcartesian spaces are paracompact and second-countable.

Remark 3.1.2. Note that in the definition of a subcartesian space, n is not assumed to be fixed over all points of S. A *smooth n-manifold* is a subcartesian space M that is locally diffeomorphic to $\mathbb{R}^n$. That is, for any $x \in M$ there exists a neighbourhood $U \subseteq M$ of x such that U is diffeomorphic to an open subset of $\mathbb{R}^n$. This fact will fix the dimension n for any point in the manifold.

Example 3.1.3. Consider the set $S := \{(x, y, z) \in \mathbb{R}^3 |\ (x^2 + y^2)z = 0,\ z \geq 0\}$. This is the xy-plane with the positive z-axis attached. It is clearly a subcartesian space due to the fact that it is a subset of $\mathbb{R}^3$. However, we would like to make some observations which will become important later on. Anywhere on the plane away from the half-line, there exist neighbourhoods diffeomorphic to $\mathbb{R}^2$. On the half-line away from the plane, there exist neighbourhoods diffeomorphic to $\mathbb{R}$. Where the plane and half-line intersect, any neighbourhood is diffeomorphic to a differential subspace of $\mathbb{R}^3$.

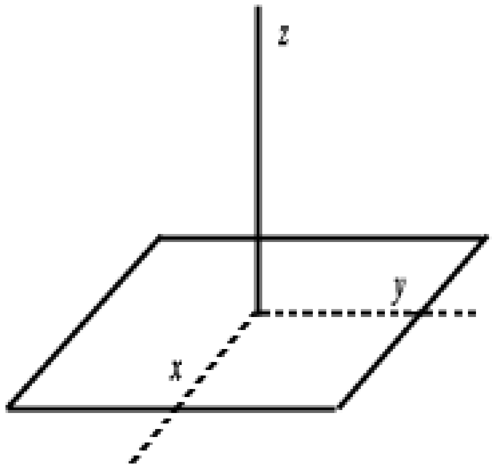


Figure 3.1: Plane and Half-Line

Example 3.1.4. The 2-sphere, the Klein bottle and the torus are examples of 2-manifolds. Any neighbourhood of a point in these spaces is diffeomorphic to the plane.

3.2 Results from the Topology of a Subcartesian Space

We shall begin by constructing some necessary tools that we can use later on: bump functions and partitions of unity.

Lemma 3.2.1. *Let S be a subcartesian space with $x \in S$ and $U \subseteq S$ a neighbourhood of x diffeomorphic to some differential subspace of $\mathbb{R}^n$. Then for any open subset $U' \subseteq U$ containing x there exists a function $f \in R(U)$ (where we are identifying U with the diffeomorphic subspace in $\mathbb{R}^n$) such that $f|_V = 1$ for some neighbourhood $V \subset U'$ of x and $f|_W = 0$ for some open subset $W \subset U$ where $U' \cup W = U$.*

Proof. Assume the hypothesis above. Then, there exist functions $f_1, ..., f_n \in R(U)$ and the smooth map $\Phi : U \to \mathbb{R}^n : x \mapsto (f_1(x), ..., f_n(x))$ such that there is an open set $\tilde{U} \subseteq \mathbb{R}^n$ with $x \in U' := \Phi^{-1}(\tilde{U}) \subseteq U$. Now, there does exist a function $F \in C^\infty(\mathbb{R}^n)$ such that $F|_{\tilde{V}} = 1$ for a neighbourhood $\tilde{V} \subset \tilde{U}$ of $\Phi(x)$ and $F|_{\tilde{W}} = 0$ for an open set $\tilde{W} \subset \mathbb{R}^n$ where $\tilde{U} \cup \tilde{W} = \mathbb{R}^n$. Now let $V := \Phi^{-1}(\tilde{V}) \subset U'$ and $W := \Phi^{-1}(\tilde{W})$. Since $\tilde{U} \cup \tilde{W} = \mathbb{R}^n$ we have $U' \cup W = U$.

Now define $f = F(f_1, f_2, ..., f_n) = F \circ \Phi$, which is in $R(U)$ since it is the composition of $F \in C^\infty(\mathbb{R}^n)$ with smooth functions on $\mathbb{R}^n$ restricted to U. Then,

$$\begin{aligned} f|_V &= F \circ \Phi|_V \\ &= F|_{\Phi(V)} \\ &= F|_{\tilde{V}} \\ &= 1. \end{aligned}$$

As well,

$$\begin{aligned} f|_W &= F \circ \Phi|_W \\ &= F|_{\Phi(W)} \\ &= F|_{\tilde{W}} \\ &= 0. \end{aligned}$$

Hence we have constructed what we need. □

Remark 3.2.2. In the above lemma, it is clear that if we extend f over all of S, letting $f(x) = 0$ for any $x \notin U$, we have a smooth bump function on S.

Proposition 3.2.3. *Let S be a subcartesian space, and let $\mathcal{U}$ be a locally finite open cover of S. Then there is a partition of unity (with functions from $C^\infty(S)$) subordinate to $\mathcal{U}$.*

Proof. This proof is an adaptation of the proof found in [8]; the terminology is modified to satisfy our setting of differential spaces. Assume $\mathcal{U}$ is a locally finite family of neighbourhoods in S that are diffeomorphic to arbitrary subsets of $\mathbb{R}^{n_U}$ (where n_U depends on $U \in \mathcal{U}$). Since S is paracompact and Hausdorff, by the shrinking lemma there exists an open cover $\mathcal{V}$ such that for every $V_U \in \mathcal{V}$, $\bar{V}_U \subset U$ for some $U \in \mathcal{U}$. Let φ_U be the diffeomorphism embedding U into $\mathbb{R}^{n_U}$. Define $f_U \in C^\infty(\mathbb{R}^{n_U})$ such that $f_U(x) > 0$ for all $x \in \varphi_U(V_U)$ and $f_U(x) = 0$ for all $x \notin \varphi_U(V_U)$. Next, define $g_U \in C^\infty(S)$ by $g_U(y) := f_U(\varphi_U(y))$ when $y \in U$, and $g_U(y) := 0$ when $y \notin U$. Now, define $h := \sum_{U \in \mathcal{U}} g_U$. Since $\mathcal{U}$ is locally finite at each x, there are only a finite number of non-zero terms in the sum defining h, and so $h \in C^\infty(S)$. Also, $h(y) > 0$ for all $y \in S$. Thus, the set $P := \{g_U/h |\ U \in \mathcal{U}\} \subset C^\infty(S)$, which gives us a partition of unity subordinate to $\mathcal{U}$.

To generalise $\mathcal{U}$, we take unions of the open sets described above and add their corresponding functions together. □

Given the definition of a subcartesian space with neighbourhoods locally diffeomorphic to arbitrary subsets of $\mathbb{R}^n$, we wish to have a notion of the smallest such n that we can set a neighbourhood to be diffeomorphic to.

Definition 3.2.4. Let S be a subcartesian space, and let $x \in S$. Then the *structural dimension* of S at x, denoted by n_x, is defined as the minimum $n \in \mathbb{N}$ such that there exists a neighbourhood $U \subseteq S$ of x diffeomorphic to a subset of $\mathbb{R}^n$.

We may refer to a subcartesian space S with maximum structural dimension n (over all points in S) to be an *n-dimensional* subcartesian space in a more global sense. For instance, Example 3.1.3 would be a 3-dimensional subcartesian space.

Note, however, that we can have "infinite-dimensional" subcartesian spaces: let $B_m := [m, m+1]^{m+1} \subset \mathbb{R}^{m+1}$, then $B := \bigcup_{m\in\mathbb{N}} B_m$ is such an "infinite" space, but of course all points have finite structural dimension. Hence the structural dimensions of points in a subcartesian space need not be bounded.

Definition 3.2.5. A real-valued function $f : D \to \mathbb{R}$ on a set D is said to be *upper semi-continuous* if the set $\{x \in D|\ f(x) < a\}$ is open for any $a \in \mathbb{R}$.

Proposition 3.2.6. *The function $N : S \to \mathbb{N}$ defined as $N(x) = n_x$ is upper semi-continuous.*

Proof. Let $S_i := \{x \in S|\ n_x \leq i\}$ $(= \{x \in S|\ n_x < a\}$ where $\lceil a \rceil = i + 1)$. Assume that S_i is not open. Then there exists a point $y \in S_i$ such that there is no open neighbourhood $U \subseteq S_i$ of y. But then, there is no open neighbourhood $V \subseteq S$ of y diffeomorphic to an arbitrary subset of $\mathbb{R}^j$ for any $j \leq i$. Hence, $n_y > i$, and so y is not in S_i (a contradiction). Thus, S_i is open, and so the structural dimension serves as an upper semi-continuous function on S. □

Since a subcartesian space in general does not have a fixed structural dimension at every point, we are interested in points where the structural dimension changes from the structural dimension of points about it.

Definition 3.2.7. A point $x \in S$ is called a *structurally regular point* (or just regular if the context is clear) if there is a neighbourhood $U \subseteq S$ of x such that $n_y = n_x$ for all $y \in U$. A point $x \in S$ is called a *structurally singular point* (or just singular) if it is not a structurally regular point. Denote the set of all structurally regular points in S as S_{reg}.

Remark 3.2.8. If R and S are subcartesian spaces with $x \in S$ and $\varphi : S \to R$ is a diffeomorphism, then the structural dimension at x and $\varphi(x)$ are equal. This is because the structural dimension is defined to be the smallest n in which some neighbourhood of x can be mapped to some differential subspace of $\mathbb{R}^n$, and any neighbourhood of $\varphi(x)$ is diffeomorphic to a neighbourhood of x. In particular, this implies that structurally singular points are invariant under diffeomorphism. We can see that this is not true for merely a smooth homeomorphism: consider $S := \{(x, y) \in \mathbb{R}^2 | \; xy = 0, \; x \geq 0, \; y \geq 0\}$ and $R = \mathbb{R}$, and φ the map straightening out S and placing the singular point at $0 \in R$.

Theorem 3.2.9. *S_{reg} is open and dense in S.*

Proof. Let $x \in S_{reg}$. Since x is a regular point, there exists an open neighbourhood $U \subseteq S$ of x such that for every $y \in U$, we have $n_y = n_x$. So, every point of U is a regular point. Hence, $U \subseteq S_{reg}$, and so S_{reg} is open, since every point of S_{reg} is an interior point.

Now, assume that S_{reg} is not dense in S. Thus, there exists a non-empty open subset $U \subseteq S$ such that U contains no regular points (and so every point in U is a singular point). Without loss of generality, assume U is diffeomorphic to a differential subspace of $\mathbb{R}^n$ for some $n > 0$ (if $n = 0$, then U would be a set of isolated points, which are regular by the subspace topology). Define $S_i := \{x \in S | \; n_x \leq i\}$. Now, assume $U \subseteq S_k$ $(k > 0)$. We have that for any open subset $V_1 \subseteq U$, V_1 must contain infinitely many points at which the structural dimension is at least two numbers from 0 to k. Let n_1 be the maximum of these for points in V_1. Now, if every open subset contained in V_1 contained a point at which the structural dimension was n_1, then V_1 would be a set of regular points. Hence, there must exist an open subset $V_2 \subset V_1$ such that for any point in V_2, the maximum structural dimension at the point is $n_2 < n_1$. Similarly, there must be an open subset, $V_3 \subset V_2$ such that for any point in V_3, the maximum structural dimension at the point is $n_3 < n_2$. Thus, continuing this process, we have a resulting chain

$$n_1 > n_2 > ... > n_i.$$

Since we are dealing with finite dimensions, we arrive at some open subset $V_i \subset U$ such that the structural dimension at all points of V_i is $n_i \geq 0$. Hence, these are all regular points.

So U is not a subset of S_k for any $k \geq 0$. Since we are only dealing with finite structural dimensions and U was defined as being diffeomorphic to a differential subspace of $\mathbb{R}^n$ for some n, we have $U \subseteq S_n$, which is a contradiction. Therefore, U does not exist, and S_{reg} is dense in S. □

Proposition 3.2.10. *S_{reg} is a countable union of disjoint open subsets, each of which has a constant structural dimension. Call connected components of these disjoint open subsets* structurally regular components *of S.*

Proof. Let $\mathcal{U} = \{U_\alpha |\ \alpha \in A\}$ (A some index set) be a family of maximal connected sets of S_{reg} covering S_{reg} such that for any $U_\alpha \in \mathcal{U}$, $n_x = n_y$ for all $x, y \in U_\alpha$. But for any $x \in U_\alpha$, since x is regular, there exists a neighbourhood about x with the property that $n_x = n_y$ for all y in that neighbourhood. Hence x is an interior point and U_α is open. It is clear that for any U_α and U_β such that for any $x \in U_\alpha$ and $y \in U_\beta$, if $n_x \neq n_y$ then $U_\alpha \cap U_\beta = \emptyset$. Now define

$$V_i := \bigcup \{U_\alpha \in \mathcal{U} |\ n_x = i \text{ for each } x \in U_\alpha\}.$$

Then we have S_{reg} partitioned into disjoint open subsets. As we have discussed at the beginning of this section, we know that the structural dimension is unbounded in $\mathbb{N}$ generally, but we clearly have at most countably many such open subsets V_i covering S_{reg}. □

Proposition 3.2.11. *Let S be a subcartesian space and let $x \in S$ be singular. Then the structural dimension of S at x is greater than or equal to the structural dimension at each point in each structurally regular component with x in its closure.*

Proof. Let U be a structurally regular component of S with x in its closure. Take any open neighbourhood $V \subseteq S$ of x that is diffeomorphic to a subset of $\mathbb{R}^n$ where $n = n_x$. Then there exists $y \in U$ such that $y \in V$. But then $n_y \leq n_x$. Otherwise, if for any neighbourhood V there exists a $y \in U \cap V$ such that $n_y > n_x$, then n_x would not be the structural dimension at x (contradiction). □

3.3 Examples

Here we shall list some examples to illustrate the concepts defined in the previous section.

Example 3.3.1. Referring to the space in Example 3.1.3, points in the half-line away from the plane give a structural dimension of 1, and points on the plane away from the half-line give a structural dimension of 2. But, the singular point where the half-line and plane intersect gives a structural dimension of 3. One can consider the whole punctured plane as one regular subcomponent, and the (open) half-line as a second regular subcomponent.

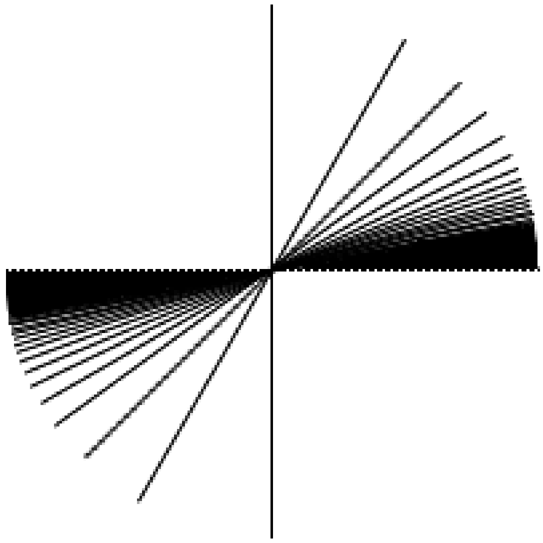


Figure 3.2: Subcartesian space that is not locally compact.

Example 3.3.2. Consider the set

$$S := \{(x, y) \in \mathbb{R}^2 |\ y - \tan(\frac{\pi}{n})x = 0,\ n = 3, 4, 5, ...\} \cup \{(0, y) \in \mathbb{R}^2\},$$

illustrated in Figure 3.2. The structural dimension at the origin is 2, and all points on each of the branches away from the origin give a structural dimension of 1. Note that the origin has an infinite number of structurally regular components with it in their closures. Note also that this space is not locally compact: take any neighbourhood about the origin, and cover this neighbourhood as follows. Put a small open set around the origin, and for each branch have exactly one open set cover the remaining part of the branch not covered by the first open set in the centre. This is an infinite cover that cannot be reduced to a finite subcover.

Example 3.3.3. Consider the following set:

$$A := \{(a, b) \in \mathbb{R}^2 |\ \exists x \in \mathbb{R}(ax = b)\}.$$

Algebraically this is the set of all compatible linear systems in one variable, whereas geometrically it can be viewed as the plane minus the y-axis, but retaining the origin. The structural dimension at the origin is 2, and it is not locally compact there. All other points give a structural dimension of 2, and so all points of the space are regular.

Note that if we twisted one of the half-planes so that we had half of the xy-plane and half of the yz-plane (and the origin), this new space is homeomorphic to A. It is not diffeomorphic, however. Note that in the new space that the origin is a singular point.

Example 3.3.4. The circular cone, $C := \{(x, y, z) \in \mathbb{R}^3 | \ x^2 + y^2 - z^2 = 0, \ z \geq 0\}$, has points at which the structural dimension is 2 everywhere except at the origin, where this singular point gives a structural dimension of 3. Note it has only one regular subcomponent, which can be considered to be the entire cone minus the origin.

Example 3.3.5. Consider the plane $\mathbb{R}^2$ and the Lie group action $\mathrm{SO}(2, \mathbb{R}) \times \mathbb{R}^2 \to \mathbb{R}^2$, which rotates a point in the plane about the origin. The orbits are exactly all concentric circles centred at the origin, and note that $r^2 = x^2+y^2$ is a smooth function that is invariant along any orbit. The corresponding orbit space is $[0, \infty)$, the set of all non-negative real numbers, with the subspace topology from $\mathbb{R}$. Now take as the differential structure on $[0, \infty)$ the ring of functions induced by the quotient map composed with the invariant function:

$$C := \{f : x \mapsto f(x^2) | \ f \in C^\infty(\mathbb{R})|_{[0,\infty)}\}.$$

Although this differential space has the same topology as $[0, \infty)$ as a differential subspace of $\mathbb{R}$, the ring of functions is slightly different (note that the identity map is not in C). However, these are diffeomorphic differential structures: simply send x^2 to x. Since x^2 is a bijection on the non-negative real numbers, this map is a diffeomorphism, and we have that $[0, \infty)$ equipped with differential structure C is diffeomorphic to a subcartesian space, and so it is one itself.

3.4 Maps between Subcartesian Spaces

Before we move on to the next chapter, we should make a note about smooth maps between subcartesian spaces. In particular, we give an extension lemma for smooth maps between subcartesian spaces. Note, however, that this extension is local in general. That is, for any point x of a subcartesian space S and any smooth map $\varphi : S \to R$, there exists a neighbourhood $U \subseteq S$ of x diffeomorphic to a differential subspace of $\mathbb{R}^n$ ($n = n_x$), which we identify with U, and a map $F : \mathbb{R}^n \to \mathbb{R}^m$ ($m = n_{\varphi(x)}$) such that $f|_U = F|_U$.

Lemma 3.4.1. *Let $A \subseteq \mathbb{R}^n$ and let $\varphi : A \to \mathbb{R}^m$ be a smooth map. Then, φ can be locally extended to a smooth map from $\mathbb{R}^n$ to $\mathbb{R}^m$.*

Proof. For any $x \in A$, $\varphi(x) = (p^1(\varphi(x)), ..., p^m(\varphi(x))) \in \mathbb{R}^m$. Thus,

$$\varphi = (p^1 \circ \varphi, ..., p^n \circ \varphi).$$

Note, however, that $p^i \circ \varphi \in C^\infty(A)$ for each $i = 1, ..., n$. Thus, for every $x \in A$, there exists a neighbourhood $U \subset \mathbb{R}^n$ of x and a function $\Phi^i \in C^\infty(\mathbb{R}^n)$ such that

$$(p^i \circ \varphi)|_{U \cap A} = \Phi^i|_{U \cap A}.$$

Define $\Phi := (\Phi^1, ..., \Phi^m)$, which is a smooth map from $\mathbb{R}^m$ to itself. Then,

$$\varphi|_{U\cap A} = \Phi|_{U\cap A},$$

and so φ can be locally extended to a smooth map from $\mathbb{R}^n$ to $\mathbb{R}^m$. □

Proposition 3.4.2. *Let S be a subcartesian space, and let R be another subcartesian space such that $f : S \to R$ is smooth. Then, f can be locally extended to a smooth map between Euclidean spaces.*

Proof. Let $x \in S$ and $y = f(x) \in R$. Let $V \subseteq R$ be a neighbourhood of y and $\psi : V \to \tilde{V} \subseteq \mathbb{R}^m$ a diffeomorphism with $m = n_y$. Let $U \subseteq S$ be a neighbourhood of x with a diffeomorphism $\varphi : U \to \tilde{U} \subseteq \mathbb{R}^n$ where $n = n_x$ and such that $f(U) \subseteq V$.

Define $g_x := \psi \circ f|_U \circ \varphi^{-1}$. This is a smooth map from U to $\mathbb{R}^m$. By Lemma 3.4.1, g_x can be locally extended to a map between $\mathbb{R}^n$ to $\mathbb{R}^m$. In particular, for any $z \in U$, there is a neighbourhood $W \subseteq U$ of z and a smooth map $G_z : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$g_x|_W = \varphi^*(G_z)|_W$$

Hence we are done. □

Chapter 4

The Tangent Bundle

4.1 The Functor T

There are different ways to define a tangent vector on a subcartesian space, and unlike in the case of smooth manifolds, these different definitions result in different tangent spaces and bundles in the general case. We shall use derivations to define the tangent space.

Definition 4.1.1. Let S be a subcartesian space and let $x \in S$. A *derivation* at x is a linear map $u : C^\infty(S) \to \mathbb{R}$ that satisfies Leibniz' Rule:

$$\forall f, g \in C^\infty(S),\ u(fg) = g(x)u(f) + f(x)u(g).$$

The set of derivations at a point x, called the (*Zariski*) *tangent space at* x, is denoted T_xS, or $\mathrm{Der}_x C^\infty(S)$ when the ring of functions over which the derivations are taken is to be emphasised (for example, we shall examine $\mathrm{Der}_x R(S)$ later in this chapter). The set of all tangent spaces over all $x \in S$, called the *tangent bundle*, is the union denoted as:

$$TS := \bigcup_{x \in S} T_xS.$$

(We could use a disjoint union above instead of a regular union, but since each tangent space is disjoint by definition, it is unnecessary.) Finally, let $\tau_S : TS \to S$ be defined as $\tau_S : v \mapsto x$ for any $x \in S$ and $v \in T_xS$. That is, $\tau_S^{-1}(x) = T_xS$.

Remark 4.1.2. The Zariski tangent space at a point x is usually defined in algebraic geometry differently as the dual vector space of $\mathfrak{m}_x/\mathfrak{m}_x^2$ where $\mathfrak{m}_x$ is the maximal ideal of $C^\infty(S)_x$ (the ring of germs at x) of all germs that are equal to 0 at x. We shall state without proof that these are equivalent definitions, but simply note that for any two functions f and g in the same germ at x, we have that if $u \in T_xS$, then $u(f) = u(g)$. This is not important for our purposes beyond our justification for the use of the name "Zariski".

We will later see that the dimension of a tangent space is not necessarily invariant over the whole subcartesian space, and hence some authors refer to such a bundle as a *pseudo-bundle*. However, we shall stick with "bundle" for the purposes of this paper.

It is clear that T_xS is a real vector space, but we would like to examine its structure. In order to do so we shall introduce some basic tools and notations. Let S be a subcartesian space and let $x \in S$ with $n = n_x$, and let $u \in T_xS$. Given a neighbourhood $U \subseteq S$ of x and diffeomorphism $\varphi : U \to \tilde{U} \subseteq \mathbb{R}^n$, there exist canonical coordinate functions $\bar{q}^i := \varphi^*(p^i|_{\tilde{U}})$ where p^i are the canonical coordinate functions on $\mathbb{R}^n$. We can see by their definition that each $\bar{q}^i$ is a smooth map on U. Now let $V \subset U$ be a neighbourhood of x such that $\bar{V} \subset U$, and let $b \in C^\infty(S)$ be a bump function that is equal to 1 for all points in V and 0 for all points on some open set $W \subset S$ so that $U \cup W = S$. Then, the function $q^i := b\bar{q}^i \in C^\infty(S)$ is such that $q^i(x) = 0$ for all $x \in W$ (which is open), and everywhere else q^i is the multiplication of two functions which are locally identical to local restrictions of smooth functions from $\mathbb{R}^n$. (Although abusing notation, we shall often make the identification $q^i = p^i|_U$, assuming appropriate restrictions and diffeomorphisms.) Another useful notation is the following: given $x \in S$, $\partial_i|_x := \frac{\partial}{\partial q^i}|_x$. Finally, define $dg(u) := u(g)$ for any $u \in T_xS$ and $x \in S$. dg is called the *differential* of the function g. It is clearly a linear map from TS to $\mathbb{R}$.

Now, we want be able to relate these local derivations on a subcartesian space S to derivations on Euclidean spaces. Let $x \in S$ with $n_x = n$, then there exists a neighbourhood $U \subseteq S$ of x and a diffeomorphism $\varphi : U \to V \subseteq \mathbb{R}^n$. If $w \in \mathrm{Der}_x(\varphi^* R(V))$, we wish to show that there exists $v \in T_{\varphi(x)}\mathbb{R}^n$ such that for every $f \in C^\infty(\mathbb{R}^n)$

$$v(f) = w(\varphi^*(f|_V)).$$

Define $v : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ such that $v(f) = w(\varphi^*(f|_V))$. Then v is clearly linear due to the linearity of w, and for any $f, g \in C^\infty(\mathbb{R}^n)$

$$\begin{aligned} v(fg) &= w(\varphi^*(f|_V g|_V)) \\ &= \varphi^* g(x) w(\varphi^* f|_V) + \varphi^* f(x) w(g|_V) \\ &= g(x)v(f) + f(x)v(g), \end{aligned}$$

hence satisfying Leibniz' Rule. Thus, $v \in T_{\varphi(x)}\mathbb{R}^n$. Now, any function $h \in C^\infty(S)$ restricts to some neighbourhood $W \subseteq S$ of x so that the restriction is identical to a restriction of some $H \in C^\infty(\mathbb{R}^n)$ to $\varphi(W) \subseteq V$. Thus it is clear that any $u \in T_xS = \mathrm{Der}_x C^\infty(S)$ restricts uniquely to some $w \in \mathrm{Der}_x(\varphi^* R(V))$. So, for any $u \in T_xS$ there exists $v \in T_{\varphi(x)}\mathbb{R}^n$ such that for any $f \in C^\infty(\mathbb{R}^n)$ and $g \in C^\infty(S)$ so that the pullback of $f|_V$ is equal to $g|_U$,

$$u(g) = v(f). \tag{4.1.1}$$

This leads us to an important result: the chain rule for local derivations.

Lemma 4.1.3. *Let S be a subcartesian space with $x \in S$, $n_x = n$ and $u \in T_xS$. Then for every $f_1, f_2, ..., f_n \in C^\infty(S)$ and $F \in C^\infty(\mathbb{R}^n)$,*

$$u(F(f_1, f_2, ..., f_n)) = \sum_{i=1}^{n} (\partial_i|_x F(f_1, f_2, ..., f_n))(u(f_i)).$$

Proof. Let $f_1, ..., f_n \in C^\infty(S)$ and $x \in S$ with $n_x = n$, then there exists a neighbourhood $U \subseteq S$ of x diffeomorphic to some differential subspace of $\mathbb{R}^n$. Let $q_1, ..., q_n$ be the canonical coordinate functions of $\mathbb{R}^n$ restricted to U. Then there exist neighbourhoods $V_1, ..., V_n \subseteq \mathbb{R}^n$ and functions $F_1, ..., F_n \in C^\infty(\mathbb{R}^n)$ such that for every $i = 1, ..., n$,

$$f_i|_{\varphi^{-1}(V_i)\cap U} = F_i(q_1, ..., q_n)|_{V_i \cap \varphi(U)}.$$

Let $V := \bigcap_{i=1}^n V_i$. Then, for any $F \in C^\infty(\mathbb{R}^n)$ we have

$$F(f_1, ..., f_n)|_{\varphi^{-1}(V)\cap U} = F(F_1(q_1, ..., q_n), ..., F_n(q_1, ..., q_n))|_{V\cap\varphi(U)}.$$

So, for any $u \in \mathrm{Der}_x C^\infty(S)$ and letting $v \in \mathrm{Der}_x C^\infty(\mathbb{R}^n)$ satisfy Equation 4.1.1, we get:

$$\begin{aligned} u(F(f_1, ..., f_n)) &= v(F(F_1(q_1, ..., q_n), ..., F_n(q_1, ..., q_n))) \\ &= \sum_{i=1}^{n} (\partial_i|_x F(F_1(q_1, ..., q_n), ..., F_n(q_1, ..., q_n)))(v(F_i(q_1, ..., q_n))) \\ &= \sum_{i=1}^{n} (\partial_i|_x F(f_1, ..., f_n))(u(f_i)). \end{aligned}$$

□

Next, let $f \in C^\infty(S)$. Then without loss of generality, let $F \in C^\infty(\mathbb{R}^n)$ so that $f|_U = \varphi^* F|_U = \varphi^* F(q^1, ..., q^n)$, where U is some neighbourhood diffeomorphic to a differential subspace of $\mathbb{R}^n$ via φ. We have for $u \in T_xS$:

$$\begin{aligned} u(f) =& u(\varphi^* F(q^1, ..., q^n)) \\ =& (\partial_i|_x \varphi^* F)(u(q^i)) \\ =& u^i \partial_i|_x \varphi^* F, \end{aligned}$$

where we have assumed Einstein summation (and shall often do so henceforth unless it is stated otherwise). Hence, any derivation can be represented as $u = u^i \partial_i|_x \in T_xS$. Thus, $\{\partial_i|_x \mid i = 1, ..., n\}$ spans T_xS, and so we can conclude that for any $x \in S$,

$$\dim(T_xS) \leq n_x. \tag{4.1.2}$$

We can say more about the dimension, but first let us examine T as a functor; i.e. given a smooth function between two subcartesian spaces, we would like an induced map between their respective tangent bundles.

Definition 4.1.4. Let R, S be subcartesian spaces and let $f : S \to R$ be a smooth map. Then, define $Tf : TS \to TR$ as follows: for any $u \in T_xS$, $Tf(u) \in T_{f(x)}R$ such that $Tf(u)(g) := u(f^*g) = u(g \circ f)$.

It should be noted that some authors refer to Tf as the pushforward of f, denoted by f_*.

Theorem 4.1.5. *Let Q, R, S be subcartesian spaces, and let $f : S \to R$ and $g : R \to Q$ be smooth maps. Then,*

1. *Tf is fibre-wise linear.*
2. *$Tg \circ Tf = T(g \circ f)$.*

3. $T(\mathrm{id}_S) = \mathrm{id}_{TS}$.

4. If f is a diffeomorphism, then f induces an isomorphism between T_xS and $T_{f(x)}R$ for every $x \in S$.

Proof. 1. Let $u, v \in TS$ such that $\tau_S(u) = \tau_S(v)$, and let $a, b \in \mathbb{R}$ and $h \in C^\infty(R)$. Then,

$$\begin{aligned} Tf(au + bv)(h) =& (au + bv)(h \circ f) \\ =& au(h \circ f) + bv(h \circ f) \\ =& aTf(u)(h) + bTf(v)(h). \end{aligned}$$

Thus, Tf is linear.

2. Let $u \in TS$ and $h \in C^\infty(Q)$. Then,

$$\begin{aligned} (Tg \circ Tf)(u)(h) =& Tf(u)(h \circ g) \\ =& u(h \circ g \circ f) \\ =& T(g \circ f)(u)(h). \end{aligned}$$

3. Let $u \in TS$ and $h \in C^\infty(S)$. Then,

$$\begin{aligned} T(\mathrm{id}_S)(u)(h) =& u(h \circ \mathrm{id}_S) \\ =& u(h) \\ =& \mathrm{id}_{TS}(u)(h). \end{aligned}$$

4. Next, assume $f : S \to R$ is a diffeomorphism. Then both f and f^{-1} induce

linear maps Tf and $T(f^{-1})$. But,

$$\begin{aligned} Tf \circ T(f^{-1}) =& T(f \circ f^{-1}) \\ =& T(\mathrm{id}_R) \\ =& \mathrm{id}_{TR} \\ T(f^{-1}) \circ Tf =& T(f^{-1} \circ f) \\ =& T(\mathrm{id}_S) \\ =& \mathrm{id}_{TS} \end{aligned}$$

Thus, Tf is a linear isomorphism.

□

So, we can treat T as a functor from the category of subcartesian spaces to fibre bundles of linear spaces, taking a subcartesian space to its tangent bundle, and taking maps between subcartesian spaces to the induced fibre-wise linear maps between their tangent bundles. In particular, this is a covariant functor, as it maintains the direction of the arrows in commutative diagrammes. However, we can do more.

Let S be a subcartesian space, and let $x \in S$. Then there exists a neighbourhood $U \subseteq S$ and a diffeomorphism $\varphi : U \to V \subseteq \mathbb{R}^n$ where $n = n_x$. Considering TU and the inequality 4.1.2, we have

$$TU \hookrightarrow V \times \mathbb{R}^n \subseteq \mathbb{R}^{2n}.$$

Thus, for any $u \in TS$, we can find a neighbourhood about u diffeomorphic to some differential subspace of Euclidean space. Hence, TS is a subcartesian space. In particular, we have the following for any $f \in C^\infty(TS)$. For any $u \in TU$ (using the same notation as above) there exist a neighbourhood $W \subseteq \mathbb{R}^{2n}$ and a function $F \in C^\infty(\mathbb{R}^{2n})$ such that

$$\begin{aligned} f|_{W\cap TU} =&(\varphi^* F)|_{W\cap TU} \\ =&\varphi^* F(p^1, ..., p^{2n})|_{W\cap TU} \\ =&\varphi^* F(q^1 \circ \tau_S, ..., q^n \circ \tau_S, dq^1, ..., dq^m)|_{W\cap TU}. \end{aligned}$$

Proposition 4.1.6. *Let R, S be subcartesian spaces and let $f : S \to R$ be a smooth map. Then, $Tf : TS \to TR$ is a smooth map.*

Proof. Let $x \in S$ and $y = f(x) \in R$. Let $U \subseteq S$ be a neighbourhood of x diffeomorphic to a differential subspace of $\mathbb{R}^m$ where $m = n_x$, and let $\bar{q}^i$ be the restrictions of the canonical coordinates of $\mathbb{R}^m$ to U. Similarly, let $V \subseteq R$ be a neighbourhood of y diffeomorphic to a differential subspace of $\mathbb{R}^n$ where $n = n_y$ and let q^j be the restrictions of the canonical coordinates of $\mathbb{R}^n$ to V. Since $q^j \circ f \in C^\infty(S)$, we know that there exists a neighbourhood $W \subseteq \mathbb{R}^m$ of x and a function $F \in C^\infty(\mathbb{R}^m)$ such that

$$(q^i \circ f)|_A = \varphi^* F(\bar{q}^1, ..., \bar{q}^m)|_A \tag{4.1.3}$$

where $A := f^{-1}(V) \cap U \cap W$ and φ is an appropriate diffeomorphism (which we shall drop for the remainder of this proof). Thus, for any $v \in T_x S$

$$\begin{aligned} d(q^i \circ f)(v) =&v(q^i \circ f) \\ =&v(F(\bar{q}^1, ..., \bar{q}^n)) \\ =&\sum_{j=1}^{m} (\bar{\partial}_j F(\bar{q}^1, ..., \bar{q}^n)(x))(v(\bar{q}^j)) \\ =&\sum_{j=1}^{m} (\bar{\partial}_j F(\bar{q}^1, ..., \bar{q}^n)(\tau_S(v)))(d\bar{q}^j(v)), \end{aligned}$$

where $\bar{\partial}_j := \frac{\partial}{\partial \bar{q}^j}$. We conclude that

$$d(q^i \circ f) \in C^\infty(TS). \tag{4.1.4}$$

Now, let $G \in C^\infty(TR)$. Then there exist a neighbourhood $B \subseteq \mathbb{R}^{2n}$ of y and a function $\tilde{G} \in C^\infty(\mathbb{R}^{2n})$ such that

$$G|_{B \cap TS} = \tilde{G}(q^1 \circ \tau_R, ..., q^n \circ \tau_R, dq^1, ..., dq^n)|_{B \cap TS}.$$

Thus,

$$\begin{aligned}
&(G \circ Tf)(v) \\
=&(\tilde{G}(q^1 \circ \tau_R, ..., q^n \circ \tau_R, dq^1, ..., dq^n) \circ Tf)(v) \\
=&\tilde{G}(q^1(y), ..., q^n(y), dq^1(Tf(v)), ..., dq^n(Tf(v))) \\
=&\tilde{G}(q^1 \circ f \circ \tau_S(v), ..., q^n \circ f \circ \tau_S(v), Tf(v)(q^1), ..., Tf(v)(q^n)) \\
=&\tilde{G}(q^1 \circ f \circ \tau_S(v), ..., q^n \circ f \circ \tau_S(v), v(q^1 \circ f), ..., v(q^n \circ f)) \\
=&\tilde{G}(q^1 \circ f \circ \tau_S(v), ..., q^n \circ f \circ \tau_S(v), d(q^1 \circ f)(v), ..., d(q^n \circ f)(v))
\end{aligned}$$

which is a function from $C^\infty(\mathbb{R}^{2m})$ composed with functions from $C^\infty(TS)$ (which we know from Equations 4.1.3 and 4.1.4), resulting in a function from $C^\infty(TS)$ by definition of a differential structure. We have just shown that Tf is a smooth map. □

Thus, T is a functor from subcartesian spaces to subcartesian spaces, taking a subcartesian space to its tangent bundle. If f is a diffeomorphism between two subcartesian spaces, then Tf is a diffeomorphism between the respective tangent bundles, as well as a linear isomorphism on each fibre.

Before we continue, a note on notation. Let R, S be subcartesian spaces with $\varphi : S \to R$ and $F : TR \to K$ smooth, for some subcartesian space K. Then by φ^*F we mean $F \circ T\varphi$.

4.2 Dimension at a Point

So far, we know that for any $x \in S$, $\dim(T_xS) \leq n_x$ (Equation 4.1.2). The goal of this section is to prove that they are in fact equal. To do so, we will need a few lemmas and a proposition. Note that we may drop diffeomorphisms periodically when their presence should be clear, for the sake of avoiding tedious notation.

Lemma 4.2.1. *Let S be a subcartesian space with $x \in S$. If $f \in C^\infty(S)$ is a constant function, then for any $u \in T_xS$, $u(f) = 0$.*

Proof. Assume $f \in C^\infty(S)$ is such that for every $x \in S$, $f(x) = c \in \mathbb{R}$. Now, $f^2 = cf$, and so

$$u(f^2) = 2f(x)u(f) = 2cu(f).$$

But, u is a linear function, and so

$$u(f^2) = u(cf) = cu(f).$$

So, $cu(f) = 0$, and hence either $c = 0$ or $u(f) = 0$. Assume $c = 0$. Then,

$$\begin{aligned} c = 0 &\Rightarrow f = 0 \\ &\Rightarrow cf = f = 0 \\ &\Rightarrow u(f) = u(cf) = cu(f) = 0 \end{aligned}$$

Thus, in either case, $u(f) = 0$ for any $u \in T_xS$, for any $x \in S$. $\square$

Lemma 4.2.2. *Let S be a subcartesian space and let $U \subseteq S$ be open. If $f \in R(U)$ vanishes identically on U, then for any $x \in U$ and any $u \in T_xS$,*

$$u(f) = 0$$

Proof. Let $x \in U$. By Lemma 3.2.1 there exists an open neighbourhood $V \subset U$ of x, an open set $W \subset S$ such that $U \cup W = S$ and $g \in R(U)$ such that $g|_V = 1$ and $g|_W = 0$. Since $f|_U = 0$, we have $fg = 0$. By Lemma 4.2.1 we have the following for all $u \in T_xS$:

$$\begin{aligned} u(fg) &= 0 \\ &= g(x)u(f) + f(x)u(g) \\ &= g(x)u(f) \\ &= u(f) \end{aligned}$$

Thus $u(f) = 0$. □

Lemma 4.2.3. *Let S be a subcartesian space, $x \in S$ with $n_x = n$, $U \subseteq S$ a neighbourhood of x diffeomorphic to a subset of $\mathbb{R}^n$, and let $u \in \mathrm{Der}_x R(U)$. Then u extends uniquely to a derivation $\tilde{u} \in \mathrm{Der}_x C^\infty(S) = T_xS$.*

Proof. For any function $f \in C^\infty(S)$ there exists an open neighbourhood $V \subseteq \mathbb{R}^n$ of x (again, identifying U with its diffeomorphic counterpart in $\mathbb{R}^n$) and a function $F \in C^\infty(\mathbb{R}^n)$ such that

$$f|_{U\cap V} = F|_{U\cap V}.$$

Define $\tilde{u}(f) := u(F|_U)$. We need to show that $\tilde{u}$ is well-defined. Let $W \subseteq \mathbb{R}^n$ be another open neighbourhood of x and let $G \in C^\infty(\mathbb{R}^n)$ be such that

$$f|_{U\cap W} = G|_{U\cap W}.$$

Thus, since $U \cap V \cap W$ is an open subset of S,

$$f|_{U\cap V\cap W} = F|_{U\cap V\cap W} = G|_{U\cap V\cap W}.$$

Consider the function $(F - G)|_U \in R(U)$. This vanishes identically on the open subset $U \cap V \cap W$, and so by Lemma 4.2.2 we have that $u((F - G)|_U) = 0$. Hence,

$$u(F|_U) = u(G|_U)$$

This gives us a well-defined, unique derivation $\tilde{u}$ on $C^\infty(S)$ extending u at x. □

The following proposition is not only used in the main theorem of this section, but can also be used to quickly calculate the tangent space at a point of a subcartesian space.

Proposition 4.2.4. *Let S be a subcartesian space, let $x \in S$ with $n_x = n$, and let $U \subseteq S$ be a neighbourhood of x diffeomorphic to some differential subspace of $\mathbb{R}^n$. Let $u \in \mathrm{Der}_x C^\infty(\mathbb{R}^n)$. Then u defines a derivation of $C^\infty(S)$ if and only if u annihilates $N(U)$.*

Proof. We claim that if for every $f \in N(U)$ we have $u(f) = 0$, then there exists a unique $v \in \mathrm{Der}_x R(U)$ such that

$$v(f + N(U)) := u(f)$$

where by 2.2.3 $f + N(U)$ is identified with $f|_U$. v is well-defined: let $f, g \in C^\infty(\mathbb{R}^n)$;

$$\begin{aligned} f + N(U) = g + N(U) &\Rightarrow f - g \in N(U) \\ &\Rightarrow u(f - g) = 0 \qquad \text{(by the hypothesis)} \\ &\Rightarrow u(f) = u(g) \\ &\Rightarrow v(f + N(U)) = v(g + N(U)). \end{aligned}$$

v is unique: let $v, w \in \mathrm{Der}_x R(U)$;

$$\begin{aligned}
&v(f+N(U)) = w(f+N(U)) = u(f)\ \forall f \in C^\infty(\mathbb{R}^n)\\
&\Rightarrow (v-w)(f+N(U)) := v(f+N(U)) - w(f+N(U)) = 0\ \forall f \in C^\infty(\mathbb{R}^n)\\
&\Rightarrow v - w = 0\\
&\Rightarrow v = w.
\end{aligned}$$

Finally, by Lemma 4.2.3, v extends to a unique derivation of $C^\infty(S)$.

Conversely, let $v \in \mathrm{Der}_x C^\infty(S)$, $f \in C^\infty(S)$ and $q_1, ..., q_n$ be the canonical coordinate functions on $\mathbb{R}^n$ restricted to U. Then there is a neighbourhood $V \subset \mathbb{R}^n$ of x and a function $F \in C^\infty(\mathbb{R}^n)$ such that

$$f|_{U\cap V} = F(q^1, ..., q^n)|_{U\cap V}.$$

Let $g \in N(U)$, and define $h := (F+g)(q^1, ..., q^n) \in C^\infty(\mathbb{R}^n)$. So, by Lemma 4.1.3 we have

$$\begin{aligned}
v(h) =& v((F+g)(q^1, ..., q^n))\\
=& (\partial_i|_x((F+g)(q^1, ..., q^n)))(v(q^i))\\
=& (\partial_i|_x F(q^1, ..., q^n))(v(q^i)) + (\partial_i|_x g(q^1, ..., q^n))(v(q^i))\\
=& v(f) + v(g)\\
=& v(h) + v(g),
\end{aligned}$$

where the last equality comes from the fact that $g(q^1, ..., q^n) = 0$. Hence, $v(h) + v(g) = v(h)$, and so $v(g) = 0$. Since $g \in N(U)$ was arbitrary, we are done. $\square$

Theorem 4.2.5. *Let S be a subcartesian space, and let $x \in S$. Then $n_x = \dim T_x S$.*

Proof. Let $n = n_x$. So, there is a neighbourhood $U \subseteq S$ of x that is diffeomorphic to a differential subspace of $\mathbb{R}^n$. By 4.1.1 and the fact that every derivation of $C^\infty(S)$ extends to a derivation of $C^\infty(\mathbb{R}^n)$, we have $\dim T_xS \leq \dim T_x\mathbb{R}^n = n$.

Now, assume $\dim T_xS < n$. Then, there exists a derivation $u \in T_x\mathbb{R}^n$ such that there is no derivation of $C^\infty(S)$ that extends to u. But by Proposition 4.2.4, this implies that there is a function $f \in N(U)$ such that $u(f) \neq 0$. Hence, letting p^i be the canonical coordinate functions on $\mathbb{R}^n$, we have for some $j \in \{1, ..., n\}$, $\partial_j|_x f \neq 0$ ($\partial_j|_x := \frac{\partial}{\partial p^j}$). Thus, there is a neighbourhood $V \subseteq f^{-1}(0)$ of x that is a *submanifold* (a differential subspace of a manifold that is also a manifold) of $\mathbb{R}^n$, where the structural dimension of points in V are $m < n$. By Remark 3.1.2, there exists an open neighbourhood $\tilde{V} \subseteq V$ of x diffeomorphic to an open set of $\mathbb{R}^m$. As well, since $f \in N(U)$, there exists a neighbourhood $W \subseteq U \subseteq f^{-1}(0)$ of x. Hence, $x \in \tilde{V} \cap W \subseteq \mathbb{R}^m$, where $\tilde{V} \cap W$ is open. But this contradicts the structural dimension $n_x = n > m$. Hence, we must conclude that $\dim T_xS = n_x$. □

Example 4.2.6. Consider the following differential subspace of $\mathbb{R}^2$: $S = \{(x, y) \in \mathbb{R}^2 |\ xy = 0\}$ (the union of the lines $x = 0$ and $y = 0$). Since $\frac{\partial}{\partial t}(xy) = \dot{x}y + x\dot{y}$ (which is the general form for a derivation on this space with $\dot{x}$ and $\dot{y}$ being the values of the derivation when acted on q^x and q^y, respectively), we have

$$TS = \{(x, y, \dot{x}, \dot{y}) \in \mathbb{R}^4 |\ xy = \dot{x}y + x\dot{y} = 0\}.$$

Note that if $x \neq 0$, then $y = 0$, and so one of $\dot{x}y$ or $x\dot{y}$ must be zero given any point in S. Hence, we can rewrite the tangent bundle as

$$TS = \{(x, y, \dot{x}, \dot{y}) \in \mathbb{R}^4 |\ xy = \dot{x}y = x\dot{y} = 0\}.$$

Given a point $(x, 0)$ where $x \neq 0$, we must have $\dot{y} = 0$, leaving $\dot{x}$ free. Hence we have a one-dimensional tangent space at any such point. Similarly, we have a one-dimensional tangent space for any $(0, y)$ where $y \neq 0$. If $x = y = 0$, however, then both $\dot{x}$ and $\dot{y}$ are left free, and we have a two-dimensional tangent space at $(0, 0)$. Note how the structural dimension at any point and the dimension of the corresponding tangent space are equal.

4.3 Differential Structures Built from TS

It will become useful later to have knowledge of the differential structure and the behaviour of $TS \times_S TS$ for a subcartesian space S.

Let $x \in S$ with $n_x = n$, and let $U \subseteq S$ be a neighbourhood of x diffeomorphic to some differential subspace of $\mathbb{R}^n$. We have already shown that TU is diffeomorphic (and linearly isomorphic on each fibre) to a differential subspace of $\mathbb{R}^{2n}$. Hence, we know that $TU \times_U TU \subseteq \mathbb{R}^{4n}$, and in fact since any pair $(u, v) \in TU \times_U TU$ must have the property that $\tau_S(u) = \tau_S(v)$, we can reduce the dimension of the ambient space so that $TU \times_U TU \subseteq \mathbb{R}^{3n}$. So we can treat $TU \times_U TU$ as a differential subspace of $\mathbb{R}^{3n}$, and hence we have constructed $TS \times_S TS$ as a subcartesian space. For any function $f \in C^\infty(TS \times_S TS)$, there exists a function $F \in C^\infty(\mathbb{R}^{3n})$ such that we have the following:

$$\begin{aligned} f|_{TU \times_U TU} =& F|_{TU \times_U TU} \\ =& F(q^1 \circ \tau_S, ..., q^n \circ \tau_S, dq^1, ..., dq^n, dq^1, ..., dq^n) \end{aligned}$$

where the first set $dq^1, ..., dq^n$ act on the first argument u for any $(u, v) \in TU \times_U TU$, and the second set act on v, and $\tau_S : TS \times_S TS \to S : (u, v) \mapsto \tau_S(u) = \tau_S(v)$.

Let S and R be subcartesian spaces and $f : S \to R$. We already know that there is an induced map $Tf : TS \to TR$. Define the map

$$T^{\times} f : TS \times_S TS \to TR \times_R TR : (u, v) \mapsto (Tf(u), Tf(v)).$$

Since Tf is smooth, then $T^{\times} f$ is clearly smooth, using the usual arguments.

The above can be easily generalised to

$$\prod_S^k TS = \underbrace{TS \times_S TS \times_S \ldots \times_S TS}_{k \text{ times}},$$

along with appropriate induced maps $T^{\times} f$.

Now, let S be a subcartesian space with $x \in S$. Then, define

$$TS \otimes_{\mathbb{R}} TS := \bigcup_{x \in S} T_x S \otimes_{\mathbb{R}} T_x S.$$

In the future we shall drop the subscript $\mathbb{R}$. We can generalise the projection τ_S as follows: for any $r_i u^i \otimes v^i \in T_x S \otimes T_x S$ ($r_i \in \mathbb{R}$ and $u^i, v^i \in T_x S$), define $\tau_S(r_i u^i \otimes v^i) = x$.

Let $U \subseteq S$ be a neighbourhood of x diffeomorphic to some differential subspace of $\mathbb{R}^n$ where $n = n_x$. Then since $TU \subseteq \mathbb{R}^{2n}$, then $TU \otimes TU \subseteq \mathbb{R}^{2n} \otimes \mathbb{R}^{2n} \cong \mathbb{R}^{4n^2}$. Thus, $TS \otimes TS$ is a subcartesian space, and for any $f \in C^\infty(TS \otimes TS)$, there exists a function $F \in C^\infty(\mathbb{R}^{4n^2})$ such that

$$f|_{TU \otimes TU} = F(q^1 \circ \tau_S, \ldots, q^n \circ \tau_S, dq^1 \otimes dq^1, dq^1 \otimes dq^2, \ldots, dq^n \otimes dq^n)|_{TU \otimes TU},$$

where the arguments $dq^i \otimes dq^j$ go through all possible pairs of $i, j = 1, \ldots, n$.

Next, note that for any vector space V, $V \wedge V$ is a linear subspace of $V \otimes V$, and so making the following definition:

$$TS \wedge TS := \bigcup_{x \in S} T_x S \wedge T_x S,$$

we can treat this as a differential subspace of $TS \otimes TS$. For any $f \in C^\infty(TS \wedge TS)$, there exists a function $F \in C^\infty(\mathbb{R}^{4n^2})$ such that

$$f|_{TU\wedge TU} = F(q^1 \circ \tau_S, ..., q^n \circ \tau_S, dq^1 \wedge dq^2, dq^1 \wedge dq^3, ..., dq^n \wedge dq^{n-1})|_{TU\wedge TU}.$$

The differential structures on tensor and wedge products of tangent bundles are easy to generalise to $\bigotimes^k TS$ and $\bigwedge^k TS$.

Also, given $f : S \to R$ for subcartesian spaces S and R, we have the following functions defined linearly over couples $a \otimes b$ and $a \wedge b$:

$$T^\otimes f : TS \otimes TS \to TR \otimes TR : a \otimes b \mapsto Tf(a) \otimes Tf(b)$$
$$T^\wedge f : TS \wedge TS \to TR \wedge TR : a \wedge b \mapsto Tf(a) \wedge Tf(b).$$

Pullbacks are defined in the obvious way. If $F : TR \times_R TR \to K$, $G : TR \otimes TR \to K$ and $H : TR \wedge TR \to K$ are smooth, then we have:

$$\varphi^* F = F \circ T^\times \varphi$$
$$\varphi^* G = G \circ T^\otimes \varphi$$
$$\varphi^* H = H \circ T^\wedge \varphi.$$

Chapter 5

Global Derivations

5.1 Derivations on $C^\infty(S)$

Now that we have discussed the tangent bundle on a subcartesian space, it is natural to ask about sections of this bundle, and its relationship to derivations of the ring of smooth functions on the subcartesian space. We will find that there is more than one definition of a tangent space at a point, and more than one type of section, or "vector field", of the tangent bundle. Note that we may often drop diffeomorphisms between differential subspaces in this chapter when they are clearly present.

Definition 5.1.1. A *global derivation* is a linear map $X : C^\infty(S) \to C^\infty(S)$ satisfying Leibniz' Rule,

$$\forall f, g \in C^\infty(S),\ X(fg) = gX(f) + fX(g).$$

The set of all global derivations is denoted $\mathrm{Der}C^\infty(S)$. It is no surprise that this is a $C^\infty(S)$-module. Define addition and scalar multiplication in the usual sense: let $X, Y \in \mathrm{Der}C^\infty(S)$ and let $f, g \in C^\infty(S)$. Then,

$$(X+Y)(f) := X(f) + Y(f)$$
$$gX(f) := g(X(f)).$$

Now, let $x \in S$. Define $X(x) \in T_xS$ as

$$\forall f \in C^\infty(S),\ X(x)(f) := (X(f))(x).$$

We denote all derivations at x defined in such a way as $\check{T}_x S$; i.e.

$$\check{T}_x S := \{u \in T_x S |\ u = X(x) \text{ for some } X \in \mathrm{Der} C^\infty(S)\}.$$

Again, it is no surprise that $\check{T}_x S$ is a real vector space: let $X, Y \in \mathrm{Der}C^\infty(S)$ and let $x \in S$. Then,

$$\begin{aligned}(X+Y)(x) :=& X(x) + Y(x)\\ gX(x) :=& g(x)X(x).\end{aligned}$$

The bundle of all such fibres is denoted as $\check{T}S := \bigcup_{x \in S} \check{T}_x S$. This is clearly a differential subspace of TS (and each fibre in $\check{T}S$ is a linear subspace of a fibre in TS).

Global derivations can be interpreted as vector fields when acting on $C^\infty(S)$. We shall, however, reserve the term "vector field" for something more specific, which we shall define later. Also, a global derivation is a *section* when acting on S, which is a function $F : S \to TS$ such that

$$\tau_S \circ F = \mathrm{id}_S. \qquad (5.1.1)$$

Hence, for any $x \in S$ and $X \in \mathrm{Der}C^\infty(S)$, we have that $\tau_S(X(x)) = x$.

Proposition 5.1.2. *Let S be a subcartesian space, $x \in S$ with $n_x = n$ and let $X \in \mathrm{Der}C^\infty(S)$. Then*

$$X : S \to TS : x \mapsto X(x)$$

is a smooth map between differential spaces S and TS; i.e. X is a smooth section on S.

Proof. We need to show that for any $F \in C^\infty(TS)$, $F \circ X \in C^\infty(S)$. Let $F \in C^\infty(TS)$. Then we know that there exists $G \in \mathbb{R}^{2n}$ such that

$$F = G(q^1 \circ \tau_S, ..., q^n \circ \tau_S, dq^1, ..., dq^n).$$

So, recalling that for any $x \in S$ and $f \in C^\infty(S)$, we have $df(X(x)) = X(x)(f) = X(f)(x)$, then

$$G(q^1 \circ \tau_S, ..., q^n \circ \tau_S, dq^1, ..., dq^n) \circ X = G(q^1, ..., q^n, X(q^1), ..., X(q^n))$$

But since $q^i, X(q^i) \in C^\infty(S)$ for all $i = 1, ...n$, then by condition 2 of a differential structure, we have $G(q^1, ..., q^n, X(q^1), ..., X(q^n)) \in C^\infty(S)$. Thus, $F \circ X \in C^\infty(S)$. $\square$

It is also clear that any smooth section $S \to TS$ that acts linearly on functions and satisfies Leibniz' Rule is a global derivation.

Remark 5.1.3. One of the most important global derivations is $\tilde{\partial}_i = \frac{\partial}{\partial p^i}$ on $\mathbb{R}^n$, which is clearly a linear map from $C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ that obeys Leibniz' Rule. Let S be a subcartesian space with a neighbourhood $U \subseteq S$ of some x diffeomorphic to a differential subspace of $\mathbb{R}^n$. Then for any $f \in C^\infty(S)$ let $\partial_i(f|_U) := (\tilde{\partial}_i F)|_U$ where $F \in C^\infty(\mathbb{R}^n)$ and $f|_U = F|_U$. This gives us partial derivatives locally on a subcartesian space.

Now, for any function $f \in C^\infty(S)$ and point $x \in S$ with $n_x = n$, there exists a neighbourhood $U \subseteq S$ of x diffeomorphic to a differential subspace of $\mathbb{R}^n$ and a function $F \in C^\infty(\mathbb{R}^n)$ such that

$$f|_U = F|_U.$$

So, by Taylor's theorem, we know for any $y \in U$,

$$f(y) = F(y) = F(x) + \sum_{i=1}^{n} \partial_i F(x)(p^i(y) - x^i) + \sum_{i=1}^{n} g^i(y)(p^i(y) - x^i),$$

where $p^1, ..., p^n$ are the canonical coordinates of $\mathbb{R}^n$, $x = (x^1, ..., x^n)$ and $g \in C^\infty(S)$ such that $g(x) = 0$. So, for any global derivation $X \in \mathrm{Der}C^\infty(S)$,

$$\begin{aligned} X(f)(x) =& X(x)(f) \\ =& \sum_{i=1}^{n} \partial_i f(y) X(q^i)(x), \end{aligned}$$

where $q^1, ..., q^n$ are equal to $p^1, ..., p^n$ on U. Thus, we conclude that as a section, $X|_U = X^i \partial_i$ for some $X^i \in C^\infty(S)$. We have just proved the following proposition:

Proposition 5.1.4. *If X is a global derivation, then its component functions X^i when restricted to a neighbourhood diffeomorphic to a differential subspace of Euclidean space are smooth.*

If we think of Der as an operator taking a ring of functions on a subcartesian space to modules, then the next proposition shows that this operator commutes with restriction.

Proposition 5.1.5. *Let $R \subseteq S$. Then,* $\mathrm{Der}(C^\infty(S)|_R) = (\mathrm{Der}C^\infty(S))|_R$.

Proof. Let $X \in \mathrm{Der}C^\infty(S)$, $f \in C^\infty(S)$ and $x \in R$. Then, X restricts uniquely to a global derivation $\tilde{X} \in \mathrm{Der}(C^\infty(S)|_R)$ as follows: let $\tilde{f} = f|_R$. Then $\tilde{X}\tilde{f} = (Xf)|_R$. But then, $(\tilde{X}(x))(\tilde{f}) = (\tilde{X}\tilde{f})(x) = (Xf)(x)$. □

Hence, given a global derivation X, we understand $X|_R$ to be both the restriction of the section to R and the derivation over the restricted ring.

We should make a note about induced maps on global derivations. For a smooth map $f : S \to R$, where S and R are subcartesian spaces, $Tf : \mathrm{Der}C^\infty(S) \to \mathrm{Der}C^\infty(R)$ defined such that $Tf(X)(g) = X(g \circ f)$ for $g \in C^\infty(R)$ is not well-defined (even for manifolds). Consider $S = \mathbb{R}$ and $R = \mathbb{R}^2$, and let f be the embedding into the x-axis. Since these are sections, if $X \in \mathrm{Der}C^\infty(\mathbb{R})$ it is clear what $Tf(X)$ will be on the x-axis. But off the x-axis, it is unknown. Thus we do not have a unique global derivation that X is mapped to. However, if f is a diffeomorphism, then $Tf(X)(g) := X(g \circ f)$ does make sense, and we shall use this fact when needed.

In the previous chapter, we showed that any derivation at a point could be represented by a derivation in $\mathbb{R}^n$ for some n. We will now prove a similar result for global derivations. We then will prove a theorem similar to Proposition 4.2.4, except this theorem will be for global derivations.

Lemma 5.1.6. *Let S be a subcartesian space with $x \in S$ and $n_x = n$. Then any global derivation of S can be locally extended about x to a global derivation of $\mathbb{R}^n$.*

Proof. Let $X \in \mathrm{Der}C^\infty(S)$ and $f \in C^\infty(S)$. Then there exists a neighbourhood $U \subseteq S$ of x diffeomorphic to a differential subspace of $\mathbb{R}^n$, and without loss of generality there is a function $F \in C^\infty(\mathbb{R}^n)$ such that $f|_U = F|_U$. Now, for any global derivation $\tilde{X} \in \mathrm{Der}C^\infty(S)$, we have

$$\tilde{X}(F) = \tilde{X}(F(p^1, ..., p^n)) = (\tilde{\partial}_i F(p^1, ..., p^n))(\tilde{X}(p^i))$$

where $\tilde{\partial}_i = \frac{\partial}{\partial p^i}$. Let $q^i \in C^\infty(S)$ be equal to p^i on U, and define $X^i := X(q^i) \in$

$C^\infty(S)$. Without loss of generality, let $\tilde{X}^i \in C^\infty(\mathbb{R}^n)$ such that $X^i|_U = \tilde{X}^i|_U$. Then,

$$\begin{aligned}\tilde{X}(F)|_U =&(\tilde{\partial}_i F(p^1, ..., p^n))|_U \tilde{X}^i|_U \\ =&(\tilde{\partial}_i F)|_U (X^i)|_U \\ =&\partial_i (f|_U)(X^i)|_U.\end{aligned}$$

Now, for any $x \in U$,

$$\begin{aligned}X(f)(x) =&X(x)(f) \\ =&(\partial_i|_x f)(X(x)(q^i)) \\ =&(\partial_i|_x f)(X^i(x)) \\ =&(\partial_i(f|_U)(x))(X^i(x)) \\ =&\tilde{X}(F)(x).\end{aligned}$$

Hence, $X(f)|_U = \tilde{X}(F)|_U$. □

Theorem 5.1.7. *Let S be a differential subspace of $\mathbb{R}^n$. $X \in \mathrm{Der}C^\infty(\mathbb{R}^n)$ restricts to some $Y \in \mathrm{Der}C^\infty(S)$ if and only if $Xf = 0$ for all $f \in N(S)$.*

Proof. Assume $X \in \mathrm{Der}C^\infty(\mathbb{R}^n)$ restricts to some $Y \in \mathrm{Der}C^\infty(S)$. Then for any $f \in C^\infty(\mathbb{R}^n)$, $X|_S(f|_S) = Y(f|_S)$. But if $f \in N(S)$, then for any $x \in S$,

$$\begin{aligned}Y(f|_S)(x) =&Y(x)(f|_S) \\ =&0 \qquad\qquad \text{by Proposition 4.2.4.}\end{aligned}$$

Thus $Xf(x) = 0$ for any $f \in N(S)$ and $x \in S$.

Conversely, assume $Xf = 0$ for all $f \in N(S)$. Then we claim that there exists a unique $Y \in \mathrm{Der}C^\infty(S)$ such that for any $f \in C^\infty(\mathbb{R}^n)$,

$$Y(f|_S) = X|_S(f|_S). \tag{5.1.2}$$

Define Y so that it satisfies Equation 5.1.2. It is well-defined: let $f, g \in C^\infty(\mathbb{R}^n)$; then

$$\begin{aligned} f|_S = g|_S \Rightarrow\ & (f-g)|_S = 0 \\ \Rightarrow\ & f - g \in N(S) \\ \Rightarrow\ & X|_S((f-g)|_S) = 0 && \text{by hypothesis} \\ \Rightarrow\ & X|_S(f|_S) = X|_S(g|_S) \\ \Rightarrow\ & Y(f|_S) = Y(g|_S). \end{aligned}$$

It is unique: say Z also satisfied Equation 5.1.2 and $f \in C^\infty(\mathbb{R}^n)$; then

$$\begin{aligned} & (Y-Z)(f|_S) = Y(f|_S) - Z(f|_S)\ \forall f \in C^\infty(\mathbb{R}^n) \\ \Rightarrow\ & Y - Z = 0 \\ \Rightarrow\ & Y = Z. \end{aligned}$$

And so the theorem is proved. □

If S was diffeomorphic to a differential subspace of a subcartesian space R, then using bump functions, we could extend the pushforward of Y locally over all of R.

It is natural to ask whether or not $\tau : TS \to S$ is a locally trivial fibration; that is, if the tangent space at a point x is spanned by vectors in $\check{T}_x S$. It turns out this is true on structurally regular points, but false on singular points.

Theorem 5.1.8. *Let S be a subcartesian space. Then the restriction of the tangent bundle projection $\tau : TS \to S$ to $T(S_{reg})$ is a locally trivial fibration over S_{reg}.*

Proof. Let $x \in S_{reg}$ and let $n_x = n$. Then there exists a neighbourhood $U \subseteq S_{reg}$ of

x (since S_{reg} is open) such that $n_y = n$ for all $y \in U$. Now, let $V \subset U$ be open such that $\bar{V} \subset U$ and $x \in V$ and V is diffeomorphic to some differential subspace of $\mathbb{R}^n$.

From the proof of Proposition 4.2.4, we know that for every $f \in N(V)$, $\partial_i f|_V = 0$ for every $i = 1, ..., n$. So, by Theorem 4.2.5, we know that for any $y \in V$, $\partial_1|_y, ..., \partial_n|_y$ span T_yS. The remaining question is whether ∂_i $(i = 1, ..., n)$ are global derivations. Let $b \in C^\infty(S)$ be a bump function such that for some neighbourhood $W \subset V$ of x with $\bar{W} \subset V$, $b(W) = 1$ and $\text{supp}(b) \subset V$. Then $b\partial_i$ is a global derivation over all of S, and the collection of them for $i = 1, ..., n$ span the tangent space of all points in W. □

The last theorem of this section (and its corollary) will tie together structurally regular points and $\check{T}S$.

Theorem 5.1.9. *Let $x \in S$. Then, $\check{T}_xS = T_xS$ if and only if $x \in S_{reg}$.*

Proof. If $x \in S_{reg}$, then by Theorem 5.1.8 we are done. Conversely, let $U \subseteq S$ be a neighbourhood of x diffeomorphic to some differential subspace of $\mathbb{R}^n$ $(n = n_x)$. Let $\{X_1(x), ..., X_n(x)\}$ be a basis of $\check{T}_xS = T_xS$. Then, for some function $f \in C^\infty(S)$, $X_i(x)(f) \neq 0$ for all $i = 1, ..., n$, and so since $X_i(x)(f) = X_i(f)(x)$, there exists a neighbourhood $V \subseteq U$ about x such that $X_i(f)(y) \neq 0$ for all $y \in V$. That is, $X_i(y) \neq 0$ for all $y \in V$. And so, $\{X_1(y), ..., X_n(y)\}$ is a linearly independent subset of $\check{T}_yS$, and in fact is a basis by Theorem 5.1.8. Now, for any $y \in V$, if $n_y > n$, then this contradicts n_x being the structural dimension at x. If $n > n_y$, then $n = \dim \check{T}_yS > \dim T_yS$, which is absurd, since for any $y \in S$, $\check{T}_yS \leq T_yS$. Hence, $n = n_y$, and so $x \in S_{reg}$. □

	Plane Away From Origin	Half-Line Away From Origin	Origin
Structural Dimension	2	1	3
Tangent Space	$\cong \mathbb{R}^2$	$\cong \mathbb{R}$	$\cong \mathbb{R}^3$
Global Derivations	$\text{span}_{C^\infty(S)}\{\partial_x, \partial_y\}$	$\text{span}_{C^\infty(S)}\{\partial_z\}$	vanish

Table 5.1: Data for Plane and Half-Line I

Corollary 5.1.10. *Let $x \in S$. Then x is a structurally singular point if and only if $\dim \check{T}_x S < \dim T_x S$.*

Proof. Take the contrapositive of the above theorem and note that $\check{T}_x S \leq T_x S$ for all $x \in S$. □

Example 5.1.11. Consider again the plane and half-line $S := \{(x, y, z) \in \mathbb{R}^3 | \ (x^2 + y^2)z = 0, \ z \geq 0\}$. Table 5.1 collects some of the data relating to the concepts we have covered so far. Since all derivations at points along the half-line are spanned by ∂_z and in the plane by ∂_x and ∂_y, there is no smooth way for the global derivations as sections to make this change at the origin, and so we may conclude that all global derivations must vanish at the origin. In fact, this can all be seen using Theorem 5.1.7. Given an arbitrary global derivation of $\mathbb{R}^3$, $a\partial_x + b\partial_y + c\partial_z$ where $a, b, c \in C^\infty(\mathbb{R}^3)$, and applying this to xz and yz in $N(S)$, we have:

$$(a\partial_x + b\partial_y + c\partial_z)(xz) = az + cx = 0$$

and

$$(a\partial_x + b\partial_y + c\partial_z)(yz) = bz + cy = 0.$$

Note that when $z = 0$ and $(x, y) \neq (0, 0)$, we must have $c|_S(x, y, 0) = 0$, and continuity demands $c(0, 0, 0) = 0$ as well. Similarly, for $(x, y) = (0, 0)$ and $z \neq 0$, we have $a(0,0,z) = 0 = b(0,0,z)$, and again by continuity these must vanish at the origin. Hence, all global derivations vanish at the origin on S.

5.2 The Lie Bracket

We will need to have the Lie bracket defined for use in chapter 8, and so we shall define it here. It is essentially the same as it is defined on smooth manifolds, as it shall be defined here completely algebraically, and hence independent of the peculiarities of the space.

Definition 5.2.1. Let S be a subcartesian space and let $X, Y \in \mathrm{Der}C^\infty(S)$. Define the *Lie bracket* of X and Y as

$$[X,Y]f := X(Yf) - Y(Xf)$$

where $f \in C^\infty(S)$ and $Xf = X(f)$ (for the sake of brevity).

Remark 5.2.2. Note that the Lie bracket is not well-defined on derivations at a point.

Lemma 5.2.3. *Let S be a subcartesian space and let $X, Y \in \mathrm{Der}C^\infty(S)$. Then, $[X,Y] \in \mathrm{Der}C^\infty(S)$.*

Proof. Let $f, g \in C^\infty(S)$ and let $a, b \in \mathbb{R}$. Then,

$$\begin{aligned}
[X,Y](af+bg) =& X(Y(af+bg)) - Y(X(af+bg)) \\
=& X(aYf + bYg) - Y(aXf + bXg) \\
=& aX(Yf) + bX(Yg) - aY(Xf) - bY(Xg) \\
=& a[X,Y]f + b[X,Y]g.
\end{aligned}$$

Thus, $[X, Y]$ is linear on smooth functions. Now,

$$\begin{aligned}[X,Y](fg) =& X(Y(fg)) - Y(X(fg)) \\ =& X(gYf + fYg) - Y(gXf + fXg) \\ =& (Xg)(Yf) + gX(Yf) + (Xf)(Yg) + fX(Yg) \\ & - (Yg)(Xf) - gY(Xf) - (Yf)(Xg) - fY(Xg) \\ =& gX(Yf) - gY(Xf) + fX(Yg) - fY(Xg) \\ =& g[X,Y]f + f[X,Y]g.\end{aligned}$$

Hence, $[X, Y]$ satisfies Leibniz' Rule. Thus, $[X, Y]$ is a global derivation. □

Lemma 5.2.4. *The Lie bracket is bilinear, antisymmetric, satisfies the Jacobi identity and for $g, h \in C^\infty(S)$ and $X, Y \in \mathrm{Der}C^\infty(S)$,*

$$[gX, hY] = gh[X, Y] + (gXh)Y - (hYg)X.$$

Proof. Let S be a subcartesian space, $X, Y, Z \in \mathrm{Der}C^\infty(S)$, $a, b \in \mathbb{R}$ and $f, g, h \in C^\infty(S)$. Then,

$$\begin{aligned}[aX + bY, Z]f =& (aX + bY)(Zf) - Z((aX + bY)(f)) \\ =& aX(Zf) + bY(Zf) - aZ(Xf) - bZ(Yf) \\ =& a[X, Z]f + b[Y, Z]f.\end{aligned}$$

A similar proof can be done for the second argument. Hence, the Lie bracket is bilinear.

$$
\begin{aligned}
[X, Y]f =& X(Yf) - Y(Xf) \\
=& -(Y(Xf) - X(Yf)) \\
=& -[Y, X]f.
\end{aligned}
$$

Hence the Lie bracket is antisymmetric.

$$
\begin{aligned}
& [X, [Y, Z]]f + [Y, [Z, X]]f + [Z, [X, Y]]f \\
=& X([Y, Z]f) - [Y, Z](Xf) + Y([Z, X]f) - [Z, X](Yf) \\
& + Z([X, Y]f) - [X, Y](Zf) \\
=& X(Y(Zf)) - X(Z(Yf)) - Y(Z(Xf)) + Z(Y(Xf)) \\
& + Y(Z(Xf)) - Y(X(Zf)) - Z(X(Yf)) + X(Z(Yf)) \\
& + Z(X(Yf)) - Z(Y(Xf)) - X(Y(Zf)) + Y(X(Zf)) \\
=& 0.
\end{aligned}
$$

Hence the Jacobi identity is satisfied.

$$
\begin{aligned}
[gX, hY]f =& gX(hYf) - hY(gXf) \\
=& (gXh)Yf + ghX(Yf) - (hYg)Xf - ghY(Xf) \\
=& gh[X, Y]f + (gXh)Yf - (hYg)Xf.
\end{aligned}
$$

This completes the proof. □

5.3 Vector Fields

We mentioned earlier that we are reserving the term "vector field" for a specific kind of global derivation. The purpose of this section is mainly to give readers familiar with the concept an idea of how global derivations and vector fields on subcartesian spaces differ.

One may recall from smooth manifold theory that global derivations admit local one-parameter groups of diffeomorphisms. Visually, this can be imagined as the following: given a non-trivial integral curve of a global derivation containing points x and y, there must be a diffeomorphism φ taking x to y (and hence an open neighbourhood about x to an open neighbourhood about y). This is not true for all global derivations on a subcartesian space, and so we shall define a *vector field* as a global derivation that does.

Example 5.3.1. Consider the half-line $[0, \infty) \subset \mathbb{R}$ (recall this from Example 3.3.5 – it is the orbit space from the Lie group action of rotations in the plane), and the global derivation $-\partial_x$. The tangent space at 0 is another copy of $\mathbb{R}$, and 0 is a regular point. And thus $-\partial_x|_0$ is in the image of $-\partial_x$. However, take any neighbourhood about 0. There is no diffeomorphism that takes 0 to another point on the half-line since any neighbourhood about 0 is of the form $[0, a)$ for some $a \in \mathbb{R}$, and a diffeomorphism would take this to $[b, c)$ for some $0 < b < c$, which is not an open neighbourhood for any point $x > 0$. Thus, $-\partial_x$ is not a vector field. However, $-x\partial_x$ is. The maximal integral curve starting at any non-zero point from $[0, \infty)$ will never quite reach 0. Also, the integral curve containing 0 is the point 0 itself. Since it has a zero velocity, the curve cannot go anywhere.

We refer [14] for more details on vector fields on subcartesian spaces and their applications.

Chapter 6

Riemannian Metrics and the Cotangent Bundle

6.1 The Cotangent Bundle and its Differential Structure

Now that we have studied the tangent bundle and the sections of it, we would like to study the dual notion of these things: the cotangent bundle and differential forms. But in order to do this, we initially shall extend the definition of a Riemannian metric to subcartesian spaces.

Recall that a Riemannian metric on some Euclidean space $\mathbb{R}^n$ is defined to be a map $G : T\mathbb{R}^n \times_{\mathbb{R}^n} T\mathbb{R}^n \to \mathbb{R}$, which is a smooth positive-definite covariant symmetric 2-tensor field on $\mathbb{R}^n$, which we can express as $G = G_{ij} dp^i \otimes dp^j$ where $G_{ij} \in C^\infty(\mathbb{R}^n)$. Then we can make the following definition.

Definition 6.1.1. Let S be a subcartesian space. Define a *Riemannian metric* on S as a map $g : TS \times_S TS \to \mathbb{R}$ which will satisfy the following. Let $x \in S$ ($n_x = n$) with neighbourhood $U \subseteq S$ and diffeomorphism $\varphi : U \to \tilde{U} \subseteq \mathbb{R}^n$. Then there exists $g_{ij} \in C^\infty(S)$ so that $g|_{TU \times_U TU} = g_{ij}|_U dq^i \otimes dq^j$. Also, $g_{ij}|_U = \varphi^* G_{ij}$ for some $G_{ij} \in C^\infty(\mathbb{R}^n)$ where $G = G_{ij} dp^i \otimes dp^j$ is a Riemannian metric on $\mathbb{R}^n$. Thus g is locally symmetric and positive definite.

We do not claim that this is the best way to generalise this notion to subcartesian spaces, as it may be too restrictive. However, it will serve our purpose, which will become clear later in this chapter. First, we need to show that this definition is well-defined. In order to do this, we need a lemma.

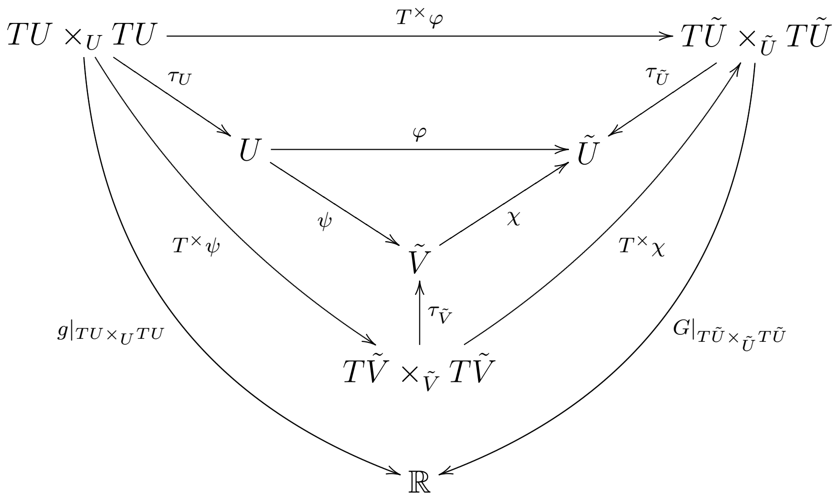
Lemma 6.1.2. *Let S be a subcartesian space and let $x \in S$ ($n = n_x$). Given a neighbourhood $U \subseteq S$ of x with diffeomorphisms $\varphi : U \to U_1 \subseteq \mathbb{R}^n$ and $\psi : U \to U_2 \subseteq \mathbb{R}^n$, the diffeomorphism $\varphi \circ \psi^{-1} : U_2 \to U_1$ can be extended locally to a diffeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$.*

Proof. Define $\chi = \varphi \circ \psi^{-1}$. Then by Lemma 3.4.1 there exist a neighbourhood of $\varphi(x)$ (without loss of generality, let this be U_2) and a smooth function $F : \mathbb{R}^n \to \mathbb{R}^n$ so that $F|_{U_2} = \chi$ (where we identify χ as being composed with the inclusion of U_1 into $\mathbb{R}^n$). Now, $T\chi : T_{\psi(x)}U_2 \to T_{\varphi(x)}U_1$ is a linear isomorphism. However, $\dim(T_{\psi(x)}U_2) = n$ and so we may conclude that $TF : T_{\psi(x)}\mathbb{R}^n \to T_{\varphi(x)}\mathbb{R}^n$ is also a linear isomorphism. But the matrix $TF = F'$. Hence, $F' : \mathbb{R}^n \to \mathbb{R}^n$ is a linear isomorphism. Then, by the inverse function theorem, there exists an open ball $B \subseteq \mathbb{R}^n$ of $\psi(x)$ so that $F|_B$ is diffeomorphic onto its image, and in particular,

$$F|_{B \cap U_2} = \chi|_{B \cap U_2}.$$

Since any open ball is diffeomorphic to the Euclidean space in which it is defined, $B \cong \mathbb{R}^n$, and so we can expand $F|_B$ using this relation. □

Let $x \in S$ with $n = n_x$ and let $U \subseteq S$ be a neighbourhood of x with diffeomorphism $\varphi : U \to \tilde{U} \subseteq \mathbb{R}^n$ such that $g|_{TU \times_U TU} = \varphi^* G|_{T\tilde{U} \times_{\tilde{U}} T\tilde{U}}$ for some Riemannian metric G on $\mathbb{R}^n$. Now, choose another neighbourhood of x diffeomorphic to $\mathbb{R}^n$ (without loss of generality, let this neighbourhood also be U) with diffeomorphism $\psi : U \to \tilde{V} \subseteq \mathbb{R}^n$. Define $\chi := \varphi \circ \psi^{-1}$.



Next, by Lemma 6.1.2, there exist a neighbourhood $W \subseteq \mathbb{R}^n$ of $\psi(x)$ and a diffeomorphism $F : \mathbb{R}^n \to \mathbb{R}^n$ so that

$$F|_{W\cap\tilde{V}} = \chi|_{W\cap\tilde{V}}$$

where we identify χ with the composition of the inclusion of $\tilde{U}$ into $\mathbb{R}^n$ with χ. Define $H := G \circ T^\times F$. Then it is clearly smooth, and we only need to show that it is symmetric and positive definite. Let $u, v \in T\mathbb{R}^n$. Then,

$$\begin{aligned} H(u,v) =& G \circ T^\times F(u,v) \\ =& G(TF(u), TF(v)) \\ =& G(TF(v), TF(u)) \\ =& H(v,u), \end{aligned}$$

so H is symmetric; and

$$\begin{aligned} H(u,u) =& G \circ T^\times F(u,u) \\ =& G(TF(u), TF(u)), \end{aligned}$$

which is positive unless $TF(u) = 0$, in which case $u = 0$ since TF is a diffeomorphism (since F is a diffeomorphism). Thus, H is a Riemannian metric on $\mathbb{R}^n$. Also, let $W' = \psi^{-1}(W \cap \tilde{V})$. Then, $g|_{TW' \times_{W'} TW'} = \psi^* H$. We have just proven that the Riemannian metric g is independent of local diffeomorphisms defining S, and so g is well-defined.

Since for a point x of a subcartesian space S the tangent space T_xS is a finite vector space, we may talk about its dual as follows.

Definition 6.1.3. Let S be a subcartesian space and let $x \in S$. The *cotangent space* at x, denoted T_x^*S, is defined to be the dual vector space to the tangent space T_xS. That is, T_x^*S is the vector space of all linear functionals on the tangent space at x. Similar to the tangent bundle, we define the *cotangent bundle*:

$$T^*S := \bigcup_{x \in S} T_x^*S.$$

As well, define the *cotangent bundle projection*,

$$\pi_S : T^*S \to S : (x, p) \mapsto x,$$

and hence $\pi_S^{-1}(x) = T_x^*S$.

Remark 6.1.4. Define $dq^i|_x := dq^i|_{T_xS}$, where q^i are identical to canonical coordinates of $\mathbb{R}^n$ ($n = n_x$) in some neighbourhood $U \subseteq S$ of x. Then, $dq^i|_x \in T_x^*S$. Now, let $a_1, a_2 \in \mathbb{R}$ and let $u \in T_xS$ be arbitrary. Then for $i \neq j$,

$$\begin{aligned} & (a_1 dq^i|_x + a_2 dq^j|_x)(u) = 0 \\ \Rightarrow\ & a_1 dq^i|_x(u) + a_2 dq^j|_x(u) = 0 \\ \Rightarrow\ & a_1 u^i + a_2 u^j = 0 \\ \Rightarrow\ & a_1 u^i = -a_2 u^j \end{aligned}$$

Then, since u is arbitrary, and hence u^i and u^j are arbitrary, we conclude that $a_1 = a_2 = 0$, and so $dq^i|_x$ and $dq^j|_x$ are linearly independent. Thus, $T_x^*S = \text{span}_{\mathbb{R}}\{dq^i|_x|\ i = 1, ..., n\}$. In particular, for any $\alpha \in T_x^*S$, $\alpha = \alpha_i dq^i|_x$, for $\alpha_i \in \mathbb{R}$. Note also that

$$\begin{aligned}\alpha(\partial_j|_x) =&\alpha_i dq^i|_x(\partial_j|_x)\\ =&\alpha_i \partial_j(q^i)\\ =&\alpha_i \delta_j^i\\ =&\alpha_j\end{aligned}$$

where δ_j^i is the Kronecker delta.

Now we shall use the Riemannian metric to build a differential structure of T^*S, and then show that this structure is independent of the choice of the metric used to construct it.

Let g be a Riemannian metric on S. Hence, for $x \in S$, we have a linear map $g_x^\flat : T_xS \to T_x^*S$ given by

$$(g_x^\flat(u))(v) = \langle g_x^\flat(u)|v\rangle = g(u, v),$$

where $u, v \in T_xS$. Note we shall use the evaluation notation $\langle\cdot|\cdot\rangle$ often. $g_x^\flat$ is an injective (and hence bijective) map:

$$\begin{aligned}\langle g_x^\flat(u_1)|v\rangle = \langle g_x^\flat(u_2)|v\rangle \Rightarrow&\ \langle g_x^\flat(u_1 - u_2)|v\rangle = 0\\ \Rightarrow&\ g(u_1 - u_2, v) = 0\ \forall v \in T_xS\\ \Rightarrow&\ u_1 - u_2 = 0.\end{aligned}$$

But since $v \in T_xS$ is arbitrary, let $v = u$, and thus $g_x^\flat(u)(v) = 0 = g(u, u)$, and so $u = 0$, which implies that $u^i = 0$ for each i. But from above, this implies that $g_x^\flat(\partial_i|_x)$,

$i = 1, ..., n$ are linearly independent. Thus, $T_x^*S = \text{span}_{\mathbb{R}}\{g_x^\flat(\partial_i|_x) |\ i = 1, ..., n\}$, and the image of $g_x^\flat$ is T_x^*S. Thus, we have a linear isomorphism. Denote the inverse of $g_x^\flat$ as $g_x^\sharp$. Thus, one can define the following functions:

$$g^\flat : TS \to T^*S : v \mapsto g_x^\flat(v) \qquad v \in T_xS$$
$$g^\sharp : T^*S \to TS : p \mapsto g_x^\sharp(p) \qquad p \in T_x^*S.$$

We define the differential structure on T^*S so that $g^\sharp$ is a diffeomorphism. Noting that since τ_S is a smooth map, we have that $\pi_S = \tau_S \circ g^\sharp$ is smooth; we can say that the differential structure on T^*S is generated by functions $q^i \circ \pi_S$ for $i = 1, ..., n$ and $dq^i \circ g^\sharp$ for $i = 1, ..., n$. This is to say that for any $f \in C^\infty(T^*S)$ and for any $x \in S$, there exists a neighbourhood $U \subseteq S$ of x and a function $F \in C^\infty(\mathbb{R}^{2n})$ such that,

$$\begin{aligned} f|_{T^*U} =& F(q^1 \circ \pi_S, ..., q^n \circ \pi_S, dq^1 \circ g^\sharp, ..., dq^n \circ g^\sharp)|_{T^*U} \\ =& (F(q^1 \circ \tau_S, ..., q^n \circ \tau_S, dq^1, ..., dq^n) \circ g^\sharp)|_{T^*U} \\ =& (F(q^1 \circ \tau_S, ..., q^n \circ \tau_S, dq^1, ..., dq^n))|_{TU}. \end{aligned}$$

We still must show that $C^\infty(T^*S)$ is independent of the Riemannian metric chosen.

Theorem 6.1.5. *$C^\infty(T^*S)$ is independent of the choice of the metric used to construct it for a subcartesian space S.*

Proof. Let g, h be Riemannian metrics on S. Define the diffeomorphism $L : TS \to TS : u \mapsto h^\sharp \circ g^\flat(u)$. Now let $x \in S$ $(n = n_x)$ with neighbourhood $U \subseteq S$ and diffeomorphism $\varphi : U \to \tilde{U} \subseteq \mathbb{R}^n$ such that there exists $\tilde{L} \in C^\infty(\mathbb{R}^{2n}, \mathbb{R}^{2n})$ satisfying

$$L|_{TU} = \varphi^*\tilde{L},$$

which exists by Proposition 3.4.2. Then for each $i = 1, ..., n$ (and identifying U with $\tilde{U}$),

$$dq^i \circ L|_{TU} = dp^i \circ \tilde{L}|_{TU}$$
$$= (\tilde{c}^i_j dp^j)|_{TU},$$

where $\tilde{c}^i_j \in C^\infty(\mathbb{R}^n)$ (since $\tilde{L}$ is only a change in coordinate system in TU). Define $c^i_j \in C^\infty(S)$ so that for some neighbourhood $V \subseteq U$ of x, we have $c^i_j|_V = (\varphi^* \tilde{c}^i_j)|_V$. Then,

$$dq^i \circ L|_{TV} = (c^i_j dq^j)|_{TV}. \tag{6.1.1}$$

Also, note that for each $i = 1, ..., n$, we have

$$q^i \circ \pi_S = q^i \circ \tau_S \circ h^\sharp$$
$$= q^i \circ \tau_S \circ g^\sharp$$
$$= q^i \circ \pi_S.$$

So $q^i \circ \pi_S$ is independent of the metric for each i.

Now, for any $f \in C^\infty(T^*S)$, there exist $F \in C^\infty(\mathbb{R}^{2n})$ and a neighbourhood of $\varphi(x)$ (which we shall assume is $\tilde{U}$ as defined above without loss of generality, which we will identify as U) such that

$$f|_{T^*U} = F(q^1 \circ \pi_S, ..., q^n \circ \pi_S, dq^1 \circ h^\sharp, ..., dq^n \circ h^\sharp)|_{T^*U}$$
$$= F(q^1 \circ \pi_S, ..., q^n \circ \pi_S, dq^i \circ L \circ g^\sharp, ..., dq^n \circ L \circ g^\sharp)|_{T^*U}$$
$$= F(q^1 \circ \pi_S, ..., q^n \circ \pi_S, c^1_j dq^j \circ g^\sharp, ..., c^n_j dq^j \circ g^\sharp)|_{T^*U}$$
$$= \tilde{F}(q^1 \circ \pi_S, ..., q^n \circ \pi_S, dq^1 \circ g^\sharp, ..., dq^n \circ g^\sharp)|_{T^*U}$$

for some new $\tilde{F} \in C^\infty(\mathbb{R}^{2n})$. □

6.2 Generalised Structures

Following the way we constructed the differential structure on $TS \otimes TS$ and $TS \wedge TS$, we would like to do the same for $T^*S \otimes T^*S$ and $T^*S \wedge T^*S$. To do so, we shall extend the Riemannian metric to suit these new spaces.

Let S be a subcartesian space with $x \in S$, and let $u, v \in T_xS$. Also, let g be a Riemannian metric on S. Then, define the map linearly over $u \otimes v \in TS \otimes TS$ such that

$$g^\flat : TS \otimes TS \to T^*S \otimes T^*S : u \otimes v \mapsto g^\flat(u) \otimes g^\flat(v).$$

We can likewise do the same for the wedge product, and assume a similar generalisation for $g^\sharp$.

To generalise even further, define the following:

$$\bigotimes^k TS := \bigcup_{x \in S} \bigotimes^k T_xS,$$

$$\bigotimes^k T^*S := \bigcup_{x \in S} \bigotimes^k T_x^*S,$$

$$\bigwedge^k TS := \bigcup_{x \in S} \bigwedge^k T_xS,$$

$$\bigwedge^k T^*S := \bigcup_{x \in S} \bigwedge^k T_x^*S.$$

Define the following linear map similarly as above

$$g^\flat : \bigotimes^k TS \to \bigotimes^k T^*S : \bigotimes_{i=1}^k u_i \mapsto \bigotimes_{i=1}^k g^\flat(u_i).$$

Again, we can likewise do the same for the wedge product, and assume a similar generalisation for $g^\sharp$.

Let $\tau_S : \bigotimes^k TS \to S$ and $\pi_S : \bigotimes^k T^*S \to S$ be the respective projections onto S. Now, since for any $(df_1|_x, ..., df_k|_x) \in \bigwedge^k T^*S$ and $v_1, ..., v_k \in T_xS$ we have

$$\begin{aligned}(df_1|_x, ..., df_k|_x)(v_1, ..., v_k) =& v_1(f_1)...v_k(f_k) \\ =& (df_1|_x(v_1))...(df_k|_x(v_k)),\end{aligned}$$

it is clear that elements of $\bigotimes^k T^*S$ are smooth functions from $\prod_S^k TS$ to $\mathbb{R}$; a similar statement can be made about elements of $\bigwedge^k T^*S$.

We can now put differential structures on $\bigotimes^k T^*S$ and $\bigwedge^k T^*S$ using the induced maps $g^\flat$ and $g^\sharp$. For the case where $k = 2$, we have for any $f \in C^\infty(T^*S \otimes T^*S)$, there exists a function $F \in C^\infty(\mathbb{R}^{n^2+n})$ and a neighbourhood $U \subseteq S$ such that

$$\begin{aligned}&f|_{T^*U\otimes T^*U} \\ =& F(q^1 \circ \pi_S, ..., q^n \circ \pi_S, (dq^1 \otimes dq^1) \circ g^\sharp, (dq^1 \otimes dq^2) \circ g^\sharp, ..., (dq^n \otimes dq^n) \circ g^\sharp)|_{T^*U\otimes T^*U}.\end{aligned}$$

Similarly, for $f \in C^\infty(T^*S \wedge T^*S)$ there exists a function $F \in C^\infty(\mathbb{R}^{n^2})$ such that

$$\begin{aligned}&f|_{T^*U\wedge T^*U} \\ =& F(q^1 \circ \pi_S, ..., q^n \circ \pi_S, (dq^1 \wedge dq^2) \circ g^\sharp, (dq^1 \wedge dq^3) \circ g^\sharp, ..., (dq^n \wedge dq^{n-1}) \circ g^\sharp)|_{T^*U\wedge T^*U}.\end{aligned}$$

Chapter 7

Zariski & Marshall Differential Forms

7.1 Zariski Differential Forms

One of the main goals of this paper is to study differential forms on subcartesian spaces. One may recall that on manifolds, there is more than one definition, all of which are equivalent. On subcartesian spaces we will see that they are not all equivalent, and some have well-defined pullback or exterior derivative, while others do not.

It shall be useful to introduce the following notation:

$$\sum\nolimits_I' := \sum_{\{I:1\leq i_1<\ldots<i_k\leq n\}},$$

where $I = i_1 i_2 \ldots i_k$ is a multi-index.

Definition 7.1.1. A *Zariski differential k-form* over a subcartesian space S is a smooth k-linear antisymmetric map from TS to $\mathbb{R}$.

Example 7.1.2. $df : TS \to \mathbb{R} : u \mapsto \langle df|u\rangle = u(f)$ for $f \in C^\infty(S)$ is a one-form; and for $f, g \in C^\infty(S)$,

$$df \wedge dg : (u, v) \mapsto u(f)v(g) - u(g)v(f),$$

this is a two-form.

Remark 7.1.3. Note that we could have defined Zariski differential k-forms as smooth linear maps from $\bigwedge^k TS$ to $\mathbb{R}$. The k-linearity and antisymmetry follow from the domain, making this an equivalent definition. When necessary (in particular, when we wish to use the domain $\bigwedge^k TS$) we will switch between the two definitions.

The set of all Zariski k-forms form a module over $C^\infty(S)$. Let $g \in C^\infty(S)$, $u \in \bigwedge^k TS$, and let ω and μ be k-forms. Then,

$$\begin{aligned}(\omega + \mu)(u) :=& \omega(u) + \mu(u)\\ (g\omega)(u) :=& g(\tau_S(u))\omega(u)\end{aligned}$$

Denote this module as $\Omega^k(S)$. Then it is clear that $\Omega(S) := \bigoplus_{i=1}^{\infty} \Omega^i(S)$ is an exterior algebra. Conventionally, $\Omega^0(S) := C^\infty(S)$.

Since any vector $u \in T_xS$, for some $x \in S$, can be expressed in terms of components, we can do the same for a one-form α. That is, given $x \in S$ with coordinate functions $q^1, ..., q^n$ $(n = n_x)$ in some neighbourhood $U \subseteq S$ of x, we have $u = u^i \partial_i|_x$. So, define $\alpha_i(x) := \alpha(\partial_i|_x) \in \mathbb{R}$. Then,

$$\begin{aligned}\alpha(u) =& \alpha(u^i \partial_i|_x)\\ =& u^i \alpha(\partial_i|_x)\\ =& u^i \alpha_i(x)\\ =& \alpha_i(x) dq^i(u),\end{aligned}$$

and so we have $\alpha = \alpha_i(x)dq^i$ when restricted to T_xS. Extending this over all of TU gives us real-valued functions α_i on U. This can be generalised to any k-form ω, which can be expressed as

$$\omega = \sum\nolimits_I' \omega_I dq^{i_1} \wedge ... \wedge dq^{i_k},$$

in U where ω_I is a real-valued function on U. But, since ω is a smooth map from TS to $\mathbb{R}$, and locally it is a linear combination of $dq^{i_1} \wedge ... \wedge dq^{i_k}$ (for all I), then its coefficient functions must be smooth maps from TU to $\mathbb{R}$. In particular, ω_I is identified as $\omega_I \circ \tau_U$, and so $\omega_I \in C^\infty(U)$ for all I.

Example 7.1.4. For any function $f \in C^\infty(S)$, we have $df \in \Omega^1(S)$. In particular, we have dq^i a Zariski one-form for local coordinate functions q^i, and any wedge product of these is also a smooth multi-linear map from TS to $\mathbb{R}$. Thus, for any function $f \in C^\infty(S)$, $df = (\partial_i f)dq^i$ locally.

Next, we shall discuss pullbacks of Zariski differential forms.

Definition 7.1.5. Let R and S be subcartesian spaces and let $f : S \to R$ be a smooth map. Then define $f^* : \Omega^k(R) \to \Omega^k(S)$ such that for any $\omega \in \Omega^k(R)$ and $u \in T_xS$,

$$f^*(\omega)(u) = \omega(Tf(u)).$$

Theorem 7.1.6. *Let R and S be subcartesian spaces and let $f : S \to R$ be a smooth map, with $g \in C^\infty(R)$, $\omega, \eta \in \Omega^k(R)$ and $\mu \in \Omega^l(R)$. Then,*

1. $f^*(g\omega + \eta) = (f^*g)f^*(\omega) + f^*(\eta)$.
2. $f^*(dg) = d(f^*g)$.
3. $f^*(\omega \wedge \mu) = f^*\omega \wedge f^*\mu$.

Proof. Let $x \in S$ and $u \in \bigwedge^k T_xS$. Then,

$$\begin{aligned} f^*(g\omega + \eta)(u) =& (g\omega + \eta)(Tf(u)) \\ =& g(\tau_R(Tf(u)))\omega(Tf(u)) + \eta(Tf(u)) \\ =& g(f(x))f^*(\omega)(u) + f^*(\eta)(u) \\ =& (f^*g)f^*(\omega)(u) + f^*(\eta)(u). \end{aligned}$$

Let $dg \in \Omega^1(S)$ where $g \in C^\infty(R)$ and let $u \in T_xS$. Then,

$$\begin{aligned} f^*(dg)(u) =& dg(Tf(u)) \\ =& Tf(u)(g) \\ =& u(f^*g) \\ =& d(f^*g)(u). \end{aligned}$$

Let $u_1, ..., u_{k+l} \in T_xS$ for some $x \in S$. Then,

$$\begin{aligned} & f^*(\omega \wedge \mu)(u_1, ..., u_{k+l}) \\ =& (\omega \wedge \mu)(Tf(u_1), ..., Tf(u_{k+l})) \\ =& \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)\omega(Tf(u_{\sigma(1)}), ..., Tf(u_{\sigma(k)}))\mu(Tf(u_{\sigma(k+1)}), ..., Tf(u_{\sigma(k+l)})) \\ =& \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma) f^*(\omega)(u_{\sigma(1)}, ..., u_{\sigma(k)}) f^*(\mu)(u_{\sigma(k+1)}, ..., u_{\sigma(k+l)}) \\ =& (f^*\omega \wedge f^*\mu)(u_1, ..., u_{k+l}). \end{aligned}$$

□

7.2 Local Representatives

It will be to our advantage to relate Zariski forms to differential forms on Euclidean space. We will gain this relation by use of the following definition.

Definition 7.2.1. Consider a linear antisymmetric map $\phi : \bigwedge^k TS \to \mathbb{R}$. If for every $x \in S$ there exists a neighbourhood $U \subseteq S$ and a diffeomorphism $\psi : U \to V \subseteq \mathbb{R}^n$ where $n = n_x$, and a k-form $\upsilon \in \Omega^k(\mathbb{R}^n)$ such that

$$\phi|_{TU} = \psi^* \upsilon$$

then ϕ is *locally representable.* The k-form υ is referred to as the local representative of ϕ relative to U.

Remark 7.2.2. Note that we shall make the identification $\phi|_{TU} = \phi|_{\bigwedge^k TU}$, which is justified by Remark 7.1.3.

We need to guarantee that the locally representable map defined above is independent of the local diffeomorphisms chosen to define S.

Proposition 7.2.3. *Let $x \in S$, and let ϕ be a locally representable linear map from $\bigwedge^k TS \to \mathbb{R}$. Then, for some neighbourhood $U \subseteq S$ of x, there exists a diffeomorphism $\psi_1 : U \to V_1 \subseteq \mathbb{R}^n$ where $n = n_x$, and a form $\upsilon \in \Omega^k(\mathbb{R}^n)$ such that $\phi|_U = \psi_1^* \upsilon$. We may assume without loss of generality that there is a second diffeomorphism $\psi_2 : U \to V_2 \subseteq \mathbb{R}^n$. Then there is a form in $\Omega^k(\mathbb{R}^n)$ such that its pullback by ψ_2 is equal to ϕ on some neighbourhood of x contained in U.*

Proof. The map $\chi := (\psi_1 \circ \psi_2^{-1})|_{\psi_2(U)}$ is a diffeomorphism. By Lemma 3.4.1, for every $y \in \psi_2(U)$ there exists a neighbourhood $W_y \subseteq \mathbb{R}^n$ of y and a smooth map

$F_y : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\chi|_{W_y \cap \psi_2(U)} = F_y|_{W_y \cap \psi_2(U)}.$$

There exists a neighbourhood $W_{\psi_2(x)} \subseteq \mathbb{R}^n$ of $\psi_2(x)$ such that

$$(\chi|_{W_{\psi_2(x)} \cap \psi_2(U)})^* \upsilon = (F_{\psi_2(x)}|_{W_{\psi_2(x)} \cap \psi_2(U)})^* \upsilon \in \Omega^(\mathbb{R}^n).$$

Thus,

$$\begin{aligned}
&(\psi_2)^*(F_{\psi_2(x)}|_{W_{\psi_2(x)} \cap \psi_2(U)})^* \upsilon \\
=&(\psi_2)^*(\chi|_{W_{\psi_2(x)} \cap \psi_2(U)})^* \upsilon \\
=&(\chi|_{W_{\psi_2(x)} \cap \psi_2(U)} \circ \psi_2|_{\psi_2^{-1}(W_{\psi_2(x)} \cap \psi_2(U))})^* \upsilon \\
=&(\psi_1 \circ \psi_2^{-1} \circ \psi_2|_{\psi_2^{-1}(W_{\psi_2(x)} \cap \psi_2(U))})^* \upsilon \\
=&(\psi_1|_{\psi_2^{-1}(W_{\psi_2(x)} \cap \psi_2(U))})^* \upsilon \\
=&\psi_1^* \upsilon|_{\psi_2^{-1}(W_{\psi_2(x)} \cap \psi_2(U))}
\end{aligned}$$

where of course $\psi_2^{-1}(W_{\psi_2(x)} \cap \psi_2(U))$ is an open neighbourhood of x contained in U. □

Any Zariski differential form is locally representable. Recall that for any $\omega \in \Omega^k(S)$ and some neighbourhood $U \subseteq S$ of $x \in S$ (with diffeomorphism $\psi : U \to V \subseteq \mathbb{R}^n$ where $n = n_x$) with coordinate functions $q^1, ..., q^n$,

$$\omega|_{TU} = {\sum_I}' \omega_I dq^{i_1} \wedge ... \wedge dq^{i_k}.$$

But each ω_I is locally a restriction of a function $\bar{\omega}_I \in C^\infty(\mathbb{R}^n)$, and so without loss of generality, let $\omega_I|_{TU} = \psi^*\bar{\omega}_I$. So,

$$\begin{aligned}\omega|_{TU} =& \sum\nolimits_I' \psi^*\bar{\omega}_I d(\psi^* p^{i_1}) \wedge ... \wedge d(\psi^* p^{i_k}) \\ =& \sum\nolimits_I' \psi^*\bar{\omega}_I (\psi^* dp^{i_1}) \wedge ... \wedge (\psi^* dp^{i_k}) \\ =& \psi^*(\sum\nolimits_I' \bar{\omega}_I dp^{i_1} \wedge ... \wedge dp^{i_k}).\end{aligned}$$

We shall prove next that locally representable linear maps from $\bigwedge^k TS$ to $\mathbb{R}$ are smooth (and hence they are Zariski forms).

Theorem 7.2.4. *Let S be a subcartesian space. The module of all locally representable linear maps from $\bigwedge^k TS$ to $\mathbb{R}$ is exactly the module of Zariski differential forms on S.*

Proof. We already know that any Zariski differential form is locally representable. Now, let ϕ be a locally representable linear map from $\bigwedge^k TS \to \mathbb{R}$ for a subcartesian space S. Then, for every $x \in S$ there exists a neighbourhood $U \subseteq S$ of x diffeomorphic by ψ to some $V \subseteq \mathbb{R}^n$, and a k-form $\upsilon \in \Omega^k(\mathbb{R}^n)$ (where $n = n_x$) such that $\phi|_{TU} = \psi^*\upsilon$. But, we know that $\upsilon = \sum_I' \upsilon_I dp^{i_1} \wedge ... \wedge dp^{i_k}$, and so,

$$\begin{aligned}\phi_{|TU} =& \psi^*\upsilon \\ =& \psi^*(\sum\nolimits_I' \upsilon_I dp^{i_1} \wedge ... \wedge dp^{i_k}) \\ =& \sum\nolimits_I' \psi^*\upsilon_I d(\psi^* p^{i_1}) \wedge ... \wedge d(\psi^* p^{i_k}).\end{aligned}$$

Since $\psi^*\upsilon_I, \psi^* p^j \in C^\infty(S)$ for all $j = 1, ..., n$, and for any function $f \in C^\infty(S)$, $df = (\partial_i f) dq^i$ locally, it is clear that $\phi \in C^\infty(\bigwedge^k TS)$. Thus ϕ is a smooth k-linear antisymmetric map from TS to $\mathbb{R}$, and so it is a Zariski differential form. $\square$

Definition 7.2.5. Let S be a subcartesian space and let $x \in S$ with a neighbourhood $U \subseteq S$ diffeomorphic by ψ to some $V \subseteq \mathbb{R}^n$ where $n = n_x$. Then define

$$N^k(U) := \{\upsilon \in \Omega^k(\mathbb{R}^n) | \ \psi^*\upsilon = 0\}.$$

Remark 7.2.6. Note the following:

1. $N^k(U)$ is dependent on the diffeomorphism ψ; however, since the pullback of $N^k(U)$ is 0, it is clear that $\psi^*(N^k(U))$ is independent of the diffeomorphism.
2. $N^0(U) = N(U)$.
3. $N^k(U)$ is the kernel of the map $\psi^* : \Omega^k(\mathbb{R}^n) \to \Omega^k(S)|_{TU}$.

Corollary 7.2.7. *Let S be a subcartesian space and let $x \in S$ with a neighbourhood $U \subseteq S$ diffeomorphic by ψ to some $V \subseteq \mathbb{R}^n$ where $n = n_x$. Then, $\Omega^k(S)|_{TU} \cong \Omega^k(\mathbb{R}^n)/N^k(U)$.*

Proof. This is simply the first isomorphism theorem for modules. □

Remark 7.2.8. Note that Equation 2.2.3 is a special case of this corollary.

7.3 Marshall's Ideal

We wish to define the exterior derivative for Zariski forms. However, it turns out that what one might try to use as a definition does not behave well. Consider the following example.

Example 7.3.1. Looking at the plane and half-line example, $S = \{(x, y, z) \in \mathbb{R}^3 | \ (x^2 + y^2)z = 0, \ z \geq 0\}$, consider the form $xdz \in \Omega^1(S)$, which is equal to 0 everywhere on S. Assuming $S \hookrightarrow \mathbb{R}^3$ is the usual inclusion, this 2-form is the pullback of $xdz \in \Omega^1(\mathbb{R}^3)$. Note, on $\mathbb{R}^3$, $d(xdz) = dx \wedge dz$. We would like the pullback of $dx \wedge dz$ from $\mathbb{R}^3$ to be 0 everywhere on S as well, but as we can see,

$$dx \wedge dz(\partial_x|_{(0,0,0)}, \partial_z|_{(0,0,0)}) = 1.$$

In fact, $dx \wedge dz$ is 0 everywhere on S but the origin.

The goal of this section is to characterise these "problem Zariski forms", and in the next section we shall define a new type of differential form on which the exterior derivative behaves properly.

Definition 7.3.2. Let $x \in S$ with a neighbourhood $U \subseteq S$ diffeomorphic by ψ to some $V \subseteq \mathbb{R}^n$ where $n = n_x$. Then define

$$M^{k+1}(U) := \text{span}_{C^\infty(\mathbb{R}^n)}\{dv \in \Omega^{k+1}(\mathbb{R}^n)|\ v \in N^k(U)\}.$$

Note that in general, $M^{k+1}(U) \nsubseteq N^{k+1}(U)$. An example is the subcartesian space $S = \{(x, y) \in \mathbb{R}^2|\ xy = 0\}$. We can let $U = S$ in this case, and let the diffeomorphism be the inclusion into $\mathbb{R}^2$. Consider the one-form $xdy \in \Omega^1(\mathbb{R}^2)$. This is identically zero when pulled back to S, however, its differential $dx \wedge dy$ is not zero when pulled back to S. Thus, $dx \wedge dy \in M^2(S)$, but $dx \wedge dy \notin N^2(S)$. This fact is the source of our problems in defining the exterior derivative on Zariski differential forms.

We already know that $N^k(U)$ is a submodule of $\Omega^k(\mathbb{R}^n)$, as it is a kernel of the module homomorphism $\psi^* : \Omega^k(\mathbb{R}^n) \to \Omega^k(S)|_{TU}$. Since $M^k(U)$ is defined as the span of particular forms, it also is a submodule of $\Omega^k(\mathbb{R}^n)$.

Definition 7.3.3. Define the set $\mathfrak{m}^{k+1}(S) \subseteq \Omega^{k+1}(S)$ as follows: a Zariski $(k+1)$-form ϕ is in $\mathfrak{m}^{k+1}(S)$ if for every $x \in S$ there exists a neighbourhood $U \subseteq S$ of

x, a diffeomorphism $\psi : U \to V \subseteq \mathbb{R}^n$ ($n = n_x$), $a_i \in C^\infty(\mathbb{R}^n)$ ($i = 1, ..., m$) and $\omega^i \in \Omega^k(\mathbb{R}^n)$ ($i = 1, ..., m$) such that $\psi^*\omega^i = 0$ for each i and $\phi|_{TU} = \psi^* \sum_{i=1}^m a_i d\omega^i$ (we will henceforth drop the summation sign and the use of m). That is, $\phi|_{TU} \in \psi^* M^{k+1}(U)$. We refer to the set $\mathfrak{m}^{k+1}(S)$ as the $(k+1)^{\text{th}}$ *Marshall submodule* of S. We define $\mathfrak{m}^0(S) := \{0\}$ and $\mathfrak{m}(S) := \bigoplus_i^\infty \mathfrak{m}^i(S)$.

Example 7.3.4. Consider the space $S := \{(x, y) \in \mathbb{R}^2 |\ xy = 0\}$ and the form $xdy \in \Omega^1(\mathbb{R}^2)$. Let $i : S \to \mathbb{R}^2$ be the usual inclusion. Then $i^*(xdy) = 0$. However, $d(xdy) = dx \wedge dy$, and $i^*(dx \wedge dy) \neq 0$. In particular, although for any non-zero $u \in \bigwedge^2 T_{(x,y)}S$ where $(x, y) \neq (0, 0)$,

$$i^*(dx \wedge dy)(u) = 0,$$

if $(x, y) = (0, 0)$, then

$$i^*(dx \wedge dy)(u) \neq 0.$$

For example, let $u = \partial_x \wedge \partial_y$. Then

$$i^*(dx \wedge dy)(u) = 1.$$

So, $i^*(dx \wedge dy) \in \mathfrak{m}^2(S)$ is a "differential of 0" on S.

Proposition 7.3.5. $\mathfrak{m}^1(S) = \{0\}$.

Proof. Let $\phi \in \mathfrak{m}^1(S)$. Then for every $x \in S$ there exists a neighbourhood $U \subseteq S$ of x, a diffeomorphism $\psi : U \to V \subseteq \mathbb{R}^n$ ($n = n_x$) and functions $F^i \in C^\infty(\mathbb{R}^n)$ such that $\psi^* F^i = 0$ and $\psi^*(a_i dF^i) = \phi|_{TU}$ where $a_i \in C^\infty(\mathbb{R}^n)$. But, we already know that $\psi^*(a_i dF^i) = \psi^* a_i(\psi^* dF^i) = \psi^* a_i d\psi^* F = 0$. And so $\phi = 0$. □

Lemma 7.3.6. *The Marshall submodule is indeed a submodule of* $\Omega^k(S)$.

Proof. Let $f \in C^\infty(S)$ and $\phi_1, \phi_2 \in \mathfrak{m}^{k+1}(S)$. We wish to show that $f\phi_1 + \phi_2 \in \mathfrak{m}^{k+1}(S)$. Choose any $x \in S$. Since $\phi_1 \in \mathfrak{m}^{k+1}(S)$ there exists a neighbourhood $U_1 \subseteq S$ of x, a diffeomorphism $\psi_1 : U_1 \to V_1 \subseteq \mathbb{R}^n$ where $n = n_x$, $a_i \in C^\infty(\mathbb{R}^n)$ and forms $\omega_1^i \in \Omega^k(\mathbb{R}^n)$ such that $\psi_1^*\omega_1^i = 0$ for each i and $\psi^*(a_i d\omega_1^i) = \phi_1|_{TU_1}$.

Since $f \in C^\infty(S)$, we know that there exists a function $F \in C^\infty(\mathbb{R}^n)$ such that (without loss of generality) $f|_{U_1} = \psi_1^*(F)$. Thus,

$$(f\phi_1)|_{TU_1} = \psi_1^*(F a_i d\omega_1^i).$$

Now, there exists a neighbourhood $U_2 \subseteq S$ of x, a diffeomorphism $\psi_2 : U_2 \to V_2 \subseteq \mathbb{R}^n$, $b_j \in C^\infty(\mathbb{R}^n)$ and forms $\omega_2^j \in \Omega^k(\mathbb{R}^n)$ such that $\psi_2^*\omega_2^j = 0$ and $\psi_2^*(b_j d\omega_2^j) = \phi_2|_{TU_2}$. Letting $U_{12} := U_1 \cap U_2$, we know that there is a diffeomorphism $\chi = (\psi_2 \circ \psi_1^{-1})|_{\psi_1(U_{12})}$. By Lemma 3.4.1 there exists a neighbourhood $W \subseteq \mathbb{R}^n$ of $\psi_1(x)$ and a function $\bar{\chi} \in C^\infty(\mathbb{R}^n)$ such that $\bar{\chi}|_{W \cap \psi_1(U_{12})} = \chi|_{W \cap \psi_1(U_{12})}$. Thus, letting $W' := \psi_1^{-1}(W) \cap U_{12} \subseteq S$,

$$\begin{aligned}\phi_2|_{T(W')} =& (\psi_2|_{W'})^*(b_j d\omega_2^j)\\ =& (\psi_1|_{W'})^*(\chi|_{\psi_1(W')})^*(b_j d\omega_2^j)\\ =& (\psi_1|_{W'})^*(\bar{\chi}|_{\psi_1(W')})^*(b_j d\omega_2^j).\end{aligned}$$

Finally,

$$(f\phi_1 + \phi_2)|_{TW'} = \psi_1^*|_{W'}(F a_i d\omega_1^k + (\bar{\chi}|_{\psi_1(W')})^*(b_j d\omega_2^j))$$

where $F a_i d\omega_1^i + \bar{\chi}^*(b_j d\omega_2^j) \in \Omega^k(\mathbb{R}^n)$. □

Lemma 7.3.7. *Let $x \in S$. If $\phi \in \Omega^k(S)$ $(k \geq 1)$ and $\phi|_{TU} \in \psi^* M^k(U)$ for some neighbourhood $U \subseteq S$ of x diffeomorphic by ψ to some differential subspace of $\mathbb{R}^n$ $(n = n_x)$, then for every point in U, ϕ is locally extendable to a form in $\mathfrak{m}^k(S)$.*

Proof. Without loss of generality, let U be a neighbourhood so that there exist forms $\omega^i \in \Omega^k(\mathbb{R}^n)$ with $\psi^*\omega^i = 0$ and $a_i \in C^\infty(\mathbb{R}^n)$ such that $\psi^*(a_i d\omega^i) = \phi|_{TU}$.

By Lemma 3.2.1, there exist a function $f \in \psi^*(R(U))$, a neighbourhood $W_1 \subset U$ of x so that $f(W_1) = 1$ and an open set $W_2 \subseteq S$ so that $f(W_2) = 0$ and $W_2 \cup U = S$. For every $y \in U$, there is a neighbourhood $W_y \subseteq S$ of y and a function $F_y \in C^\infty(\mathbb{R}^n)$ so that $f|_{W_y} = \psi^* F_y|_{W_y}$.

We have the following:

$$\begin{aligned}\phi|_{T(U\cap W_1\cap W_y)} =& (f\phi)|_{T(U\cap W_1\cap W_y)}\\ =& \psi^*(F_y|_{\psi(U\cap W_1\cap W_y)} a_i d\omega^i).\end{aligned}$$

But $F_y|_{\psi(U\cap W_1\cap W_y)} a_i d\omega^i \in M^k(U)$, and so we are done. □

Lemma 7.3.8. *Let* $\phi_1 \in \mathfrak{m}^k(S)$ *and* $\phi_2 \in \Omega^l(S)$*. Then,*

$$\phi_1 \wedge \phi_2 \in \mathfrak{m}(S).$$

Proof. Since we know that locally representable forms are independent of the diffeomorphisms chosen, we may without loss of generality find a neighbourhood $U \subseteq S$ of x, a diffeomorphism $\psi : U \to V \subseteq \mathbb{R}^n$ ($n = n_x$) and forms $\omega_1^i \in \Omega^{k-1}(\mathbb{R}^n)$ and $\omega_2 \in \Omega^l(\mathbb{R}^n)$ such that $\psi^*\omega_1^i = 0$ for each i, $\psi^*(a_i d\omega_1^i) = \phi_1|_{TU}$ for some $a_i \in C^\infty(\mathbb{R}^n)$ and $\psi^*\omega_2 = \phi_2|_{TU}$. Then,

$$\begin{aligned}(\phi_1 \wedge \phi_2)|_{TU} =& \psi^*(a_i d\omega_1^i) \wedge \psi^*\omega_2\\ =& \psi^* a_i \psi^*(d\omega_1^i \wedge \omega_2)\\ =& \psi^* a_i \psi^*(d(\omega_1^i \wedge \omega_2) - (-1)^{k-1}\omega_1^i \wedge d\omega_2)\\ =& \psi^* a_i(\psi^* d(\omega_1^i \wedge \omega_2) + (-1)^k \psi^*\omega_1^i \wedge \psi^* d\omega_2)\\ =& \psi^* a_i \psi^* d(\omega_1^i \wedge \omega_2).\end{aligned}$$

But $\psi^*(\omega_1^i \wedge \omega_2) = \psi^*\omega_1^i \wedge \psi^*\omega_2$, and hence is zero for each i. Thus, $\psi^* a_i \psi^* d(\omega_1^i \wedge \omega_2) \in \mathfrak{m}^{k+l}(S)$. $\square$

We have just shown that $\mathfrak{m}(S)$ is an ideal of the exterior algebra $\Omega(S)$.

Lemma 7.3.9. *Let R, S be subcartesian spaces and let $\varrho : S \to R$ be a smooth map. Then $\varrho^*\mathfrak{m}^k(R) \subseteq \mathfrak{m}^k(S)$.*

Proof. Let $\phi \in \mathfrak{m}^k(R)$ ($k \geq 1$) and let $x \in S$ with $y = \varrho(x)$. Then there exist a neighbourhood $U_y \subseteq R$ of y, a diffeomorphism $\psi_y : U_y \to V_y \subseteq \mathbb{R}^n$ ($n = n_y$) and $\omega^i \in \Omega^{k-1}(\mathbb{R}^n)$ such that $\psi_y^*\omega^i = 0$ for each i and $\psi_y^*(a_i d\omega^i) = \phi|_{TU_y}$ for some $a_i \in C^\infty(\mathbb{R}^n)$.

Since $\varrho^*\phi \in \Omega^k(S)$, we know there exists a neighbourhood $U_x \subseteq S$ of x, a diffeomorphism $\psi_x : U_x \to V_x \subseteq \mathbb{R}^m$ ($m = n_x$) and $\bar{\omega} \in \Omega^k(\mathbb{R}^m)$ such that $\psi_x^*\bar{\omega} = \varrho^*\phi|_{TU_x}$.

Now, let $W := \varrho^{-1}(U_y) \cap U_x$ and define the smooth map $\chi : \mathbb{R}^m \to \mathbb{R}^n$ as

$$\chi := \psi_y \circ \varrho \circ \psi_x^{-1}|_{\psi_x(W)}.$$

Then,

$$\begin{aligned}\bar{\omega}|_{T(\psi_x(W))} =&((\psi_x^{-1})^*\varrho^*\phi)|_{T(\psi_x(W))}\\ =&(\chi^*(a_i d\omega^i))|_{T(\psi_x(W))}\\ =&(\chi^*(a_i) d\chi^*\omega^i)|_{T(\psi_x(W))}.\end{aligned}$$

Hence,

$$(\psi_x^*\bar{\omega})|_{TW} = (\varrho^*\phi)|_{TW} = (\psi_x^*(\chi^*(a_i)d(\chi^*\omega^i)))|_{TW}.$$

Note that

$$
\begin{aligned}
(\psi_x^*(\chi^*(a_i)\chi^*\omega^i))|_{TW} =&(\psi_x^*(\chi^*(a_i))\psi^*(\psi_x^{-1})^*\varrho^*\psi_y^*\omega^i)|_{TW}\\
=&(\psi_x^*(\chi(a_i))\varrho^*0)|_{TW}\\
=&0.
\end{aligned}
$$

And so, $\varrho^*\phi \in \mathfrak{m}^k(S)$. □

7.4 The Marshall Exterior Derivative

Definition 7.4.1. We shall call $\Omega^k_M(S) := \Omega^k(S)/\mathfrak{m}^k(S)$ the module of *Marshall differential k-forms*. It is a module because $\mathfrak{m}^k(S)$ is a submodule of $\Omega^k(S)$. Similar to Zariski differential forms, we let $\Omega_M(S) := \bigoplus_{i=0}^{\infty} \Omega^i_M(S)$. Note that $\Omega^0_M(S) = C^\infty(S)$ and $\Omega^1_M(S) = \Omega^1(S)$ since $\mathfrak{m}^0(S) = \mathfrak{m}^1(S) = \{0\}$.

Let $\varrho : S \to R$ be a smooth map between subcartesian spaces. Due to Lemma 7.3.9, we can define the induced map $\bar{\varrho}^* : \Omega^k_M(R) \to \Omega^k_M(S) : \phi + \mathfrak{m}^k(R) \mapsto \varrho^*\phi + \mathfrak{m}^k(S)$. This is clearly linear. More precisely, $\bar{\varrho}^*$ is the unique map making the following diagramme commute:

$$
\begin{array}{ccc}
\Omega^k(R) & \xrightarrow{\varrho^*} & \Omega^k(S) \\
\downarrow & & \downarrow \\
\Omega^k_M(R) & \xrightarrow[\bar{\varrho}^*]{} & \Omega^k_M(S)
\end{array}
$$

where the two maps pointing down are quotient maps. For any form $\phi \in \Omega^k(R)$, the pre-image of $\phi + \mathfrak{m}^k(R)$ by the quotient map is $\phi + \mathfrak{m}^k(R) \subseteq \Omega^k(R)$. Then $\varrho^*(\phi + \mathfrak{m}^k(R)) = \varrho^*\phi + \varrho^*\mathfrak{m}^k(R)$, and applying the righthand side quotient map, we get

$$\varrho^*\phi + \varrho^*\mathfrak{m}^k(R) + \mathfrak{m}^k(S) = \varrho^*\phi + \mathfrak{m}^k(S).$$

We shall henceforth denote $\bar{\varrho}^*$ as ϱ^*.

Definition 7.4.2. Let $\phi \in \Omega^k(S)$. Then we know that for every $x \in S$, there exists a neighbourhood $U \subseteq S$ of x, a diffeomorphism $\psi : U \to V \subseteq \mathbb{R}^n$ $(n = n_x)$ and a form $\omega \in \Omega^k(\mathbb{R}^n)$ so that $\phi|_{TU} = \psi^*\omega$. Define the *Marshall exterior derivative* of ϕ, denoted $d_M\phi$, so that:

$$d_M : \Omega(S) \to \Omega_M(S) : (d_M\phi)|_{TU} := \psi^* d\omega + \mathfrak{m}^{k+1}(S).$$

We shall identify $d_M f = df$ for any $f \in C^\infty(S)$ since $\Omega^1_M(S) = \Omega^1(S)$.

Remark 7.4.3. Let $\phi \in \mathfrak{m}^k(S)$. Using the notation utilised above, we know that $\phi|_{TU} = \psi^*(a_i d\omega^i)$. Thus, $(d_M\phi)|_{TU} = \psi^*(da_i \wedge d\omega^i) + \mathfrak{m}^{k+1}(S) = \mathfrak{m}^{k+1}(S)$. Thus, we have $d_M(\mathfrak{m}^k(S)) = \mathfrak{m}^{k+1}$, and so we can define d_M as a map

$$d_M : \Omega^k_M(S) \to \Omega^{k+1}_M(S) : \phi \mapsto d_M\phi.$$

Theorem 7.4.4. *Let $\phi \in \Omega^k(S)$ and $\phi' \in \Omega(S)$. Then:*

1. *d_M is a linear map over $\mathbb{R}$.*
2. *$d_M(\sum'_I \phi_I dq^{i_1} \wedge ... \wedge dq^{i_k}) = \sum'_I d\phi_I \wedge dq^{i_1} \wedge ... \wedge dq^{i_k} + \mathfrak{m}^{k+1}(S)$*
3. *$d_M(\varrho^*\phi) = \varrho^*(d_M\phi) + \mathfrak{m}^{k+1}(S)$ for any smooth map $\varrho : S \to R$.*
4. *$d_M(\phi \wedge \phi') = d_M\phi \wedge \phi' + (-1)^k \phi \wedge d_M\phi'$*
5. *$d^2_M\phi = \mathfrak{m}^{k+2}(S)$.*

Proof. 1. Let $a, b \in \mathbb{R}$ and $\phi_1, \phi_2 \in \Omega^k(S)$. Then for any $x \in S$, there exists a neighbourhood $U \subseteq S$ of x, a diffeomorphism $\psi : U \to V \subseteq \mathbb{R}^n$ ($n = n_x$) and $\tilde{\phi}_1, \tilde{\phi}_2 \in \Omega^k(\mathbb{R}^n)$ such that

$$\begin{aligned}(a\phi_1 + b\phi_2)|_{TU} =& \psi^*(a\tilde{\phi}_1 + b\tilde{\phi}_2)\\ =& a\psi^*\tilde{\phi}_1 + b\psi^*\tilde{\phi}_2.\end{aligned}$$

Thus,

$$\begin{aligned}d_M(a\phi_1 + b\phi_2) =& \psi^* d(a\tilde{\phi}_1 + b\tilde{\phi}_2) + \mathfrak{m}^{k+1}(S)\\ =& a\psi^* d\tilde{\phi}_1 + b\psi^* d\tilde{\phi}_2 + \mathfrak{m}^{k+1}(S)\\ =& a d_M\phi_1 + b d_M\phi_2 + \mathfrak{m}^{k+1}(S).\end{aligned}$$

2. Let $\phi \in \Omega^k(S)$. Then, for any $x \in S$, there exists a neighbourhood $U \subseteq S$ of x, a diffeomorphism $\psi : U \to V \subseteq \mathbb{R}^n$ ($n = n_x$) and $\omega \in \Omega^k(\mathbb{R}^n)$ such that $\phi|_{TU} = \psi^*\omega$. That is, there exist $\omega_I \in C^\infty(\mathbb{R}^n)$ such that

$$\begin{aligned}\phi|_{TU} =& {\sum_I}' \phi_I dq^{i_1} \wedge ... \wedge dq^{i_k}\\ =& \psi^*({\sum_I}' \omega_I dp^{i_1} \wedge ... \wedge dp^{i_k})\\ =& {\sum_I}' \psi^*\omega_I dq^{i_1} \wedge ... \wedge dq^{i_k}.\end{aligned}$$

Hence, $\phi_I = \psi^*\omega_I$. Now,

$$\begin{aligned}&(d_M\phi)|_{TU}\\ =& \psi^* d\omega + \mathfrak{m}^{k+1}(S)\\ =& {\sum_I}' \psi^* d\omega_I \wedge dq^{i_1} \wedge ... \wedge dq^{i_k} + \mathfrak{m}^{k+1}(S)\\ =& {\sum_I}' d\psi^*\omega_I \wedge dq^{i_1} \wedge ... \wedge dq^{i_k} + \mathfrak{m}^{k+1}(S)\\ =& {\sum_I}' d_M\phi_I \wedge dq^{i_1} \wedge ... \wedge dq^{i_k} + \mathfrak{m}^{k+1}(S).\end{aligned}$$

3. Let $\phi \in \Omega^k(R)$ $(k \geq 1)$ and let $x \in S$ with $y = \varrho(x)$. Then there exist a neighbourhood $U_y \subseteq R$ of y, a diffeomorphism $\psi_y : U_y \to V_y \subseteq \mathbb{R}^n$ $(n = n_y)$ and $\omega \in \Omega^k(\mathbb{R}^n)$ such that $\psi_y^* \omega = \phi|_{TU_y}$.

   Since $\varrho^*\phi \in \Omega^k(S)$, we know there exist a neighbourhood $U_x \subseteq S$ of x, a diffeomorphism $\psi_x : U_x \to V_x \subseteq \mathbb{R}^m$ $(m = n_x)$ and $\bar{\omega} \in \Omega^k(\mathbb{R}^m)$ such that $\psi_p^* \bar{\omega} = \varrho^* \phi|_{TU_x}$.

   Now, let $W := \varrho^{-1}(U_y) \cap U_x$ and define the smooth map $\chi : \mathbb{R}^m \to \mathbb{R}^n$ as

$$\chi := \psi_y \circ \varrho \circ \psi_x^{-1}|_{\psi_x(W)}.$$

   Then,

$$\begin{aligned} \varrho^*((d_M\phi)|_{T(\varrho(W))}) =& \varrho^*((\psi_y^* d\omega + \mathfrak{m}^{k+1}(R))|_{T(\varrho(W))}) \\ =& \varrho^*((\psi_y^* d\omega)|_{T(\varrho(W))}) + \varrho^* \mathfrak{m}^{k+1}(R) \end{aligned}$$

   Hence, by Remark 7.4.3 and Lemma 7.3.9 we have:

$$\begin{aligned} \varrho^*(d_M(\phi)|_{T(\varrho(W))}) + \mathfrak{m}^{k+1}(S) =& \varrho^*((\psi_y^* d\omega)|_{T(\varrho(W))}) + \mathfrak{m}^{k+1}(S) \\ =& (\psi_p^* \chi^* d\omega)|_{TW} + \mathfrak{m}^{k+1}(S) \\ =& d_M(\varrho^*\phi)|_{TW}. \end{aligned}$$

4. Let $\phi \in \Omega^k(S)$ and $\phi' \in \Omega^l(S)$. Then, for any $x \in S$, there exists a neighbourhood $U \subseteq S$ of x, a diffeomorphism $\psi : U \to V \subseteq \mathbb{R}^n$ $(n = n_x)$, $\tilde{\phi} \in \Omega^k(\mathbb{R}^n)$

and $\tilde{\phi}' \in \Omega^l(\mathbb{R}^n)$ such that $\phi|_{TU} = \psi^*\tilde{\phi}$ and $\phi'|_{TU} = \psi^*\tilde{\phi}'$. Then,

$$\begin{aligned}
d_M(\phi \wedge \phi')|_{TU} =& \psi^* d(\tilde{\phi} \wedge \tilde{\phi}') + \mathfrak{m}^{k+l+1}(S) \\
=& \psi^*(d\tilde{\phi} \wedge \tilde{\phi}' + (-1)^k \tilde{\phi} \wedge d\tilde{\phi}') + \mathfrak{m}^{k+l+1}(S) \\
=& \psi^* d\tilde{\phi} \wedge \psi^*\tilde{\phi}' + (-1)^k \psi^*\tilde{\phi} \wedge \psi^* d\tilde{\phi}' + \mathfrak{m}^{k+l+1}(S) \\
=& (d_M\phi - \mathfrak{m}^{k+1}(S)) \wedge \phi' + (-1)^k \phi \wedge (d_M\phi' - \mathfrak{m}^{l+1}(S)) + \mathfrak{m}^{k+l+1}(S) \\
=& d_M\phi \wedge \phi' + (-1)^k \phi \wedge d_M\phi' + \mathfrak{m}^{k+l+1}(S),
\end{aligned}$$

where the last step follows from Lemma 7.3.8.

5. Let $\phi \in \Omega^k(S)$. Then using the usual notation where ω is a local representative of ϕ relative to $\psi : U \to V$, we have

$$\begin{aligned}
d_M^2(\phi)|_{TU} =& d_M(\psi^* d\omega + \mathfrak{m}^{k+1}(S)) \\
=& \psi^* d^2\omega + \mathfrak{m}^{k+2}(S) \\
=& \mathfrak{m}^{k+2}(S).
\end{aligned}$$

□

Note that if we realise the Marshall exterior derivative as a map $d_M : \Omega_M^k \to \Omega_M^{k+1}$, then the third and fifth parts of the above theorem could be simplified to $\varrho^*(d_M\phi) = d_M(\varrho^*\phi)$ where $\phi \in \Omega_M(S)$, and $d_M^2 = 0$. This is exactly how we would like the exterior derivative to act.

Example 7.4.5. Consider once more the plane and half-line, $S = \{(x, y, z) \in \mathbb{R}^3 | \ (x^2 + y^2)z = 0, \ z \geq 0\}$. We're going to extend Table 5.1 to include data on Zariski and Marshall forms. (See Table 7.1.) For the sake of brevity, we have

	Plane Minus Origin	Half-Line Minus Origin	Origin
Structural Dimension	2	1	3
Tangent Space	$\cong \mathbb{R}^2$	$\cong \mathbb{R}$	$\cong \mathbb{R}^3$
Global Derivations	$\{\partial_x, \partial_y\}$	$\{\partial_z\}$	0
Zariski 1-Forms	$\{dx, dy\}$	$\{dz\}$	$\{dx, dy, dz\}$
Zariski 2-Forms	$\{dx \wedge dy\}$	0	$\{dx \wedge dy, dx \wedge dz, dy \wedge dz\}$
Zariski 3-Forms	0	0	$\{dx \wedge dy \wedge dz\}$
Marshall 2-Submodule	0	0	$\{dx \wedge dz, dy \wedge dz\}$
Marshall 3-Submodule	0	0	$\{dx \wedge dy \wedge dz\}$
Marshall 2-Forms	$\{dx \wedge dy\}$	0	$\{dx \wedge dy\}$
Marshall 3-Forms	0	0	0

Table 7.1: Data for Plane and Half-Line II

listed bases $\{\cdot\}$ in the table and omitted $\text{span}_{C^\infty(S)}\{\cdot\}$. The two Zariski 1-Forms $i^*(xdz)$ and $i^*(ydz)$ (where $i : S \hookrightarrow \mathbb{R}^3$ is the inclusion) are both 0 everywhere on S, but the pullback of their differentials are not 0 at the origin, and in fact together $i^*(dx \wedge dz)$ and $i^*(dy \wedge dz)$ span the Marshall submodule of degree 2. Of course, by Lemma 7.3.8, $i^*(dx \wedge dy \wedge dz) \in \mathfrak{m}^3(S)$.

7.5 Forms as Smooth Sections

Note that on manifolds, global derivations and vector fields coincide. Similarly, all of our definitions of forms coincide, and are in fact smooth sections from S to $\bigwedge^k T^*S$. On more general subcartesian spaces, this is not always true. In this section we shall examine forms that are smooth sections. Note we shall use the notation $\Gamma^k(S)$ to denote the set of smooth sections $S \to \bigwedge^k T^*S$.

$\Gamma^k(S)$ is a $C^\infty(S)$-module: let $\varsigma_1, \varsigma_2 \in \Gamma^k(S)$ and $f, g \in C^\infty(S)$. Then,

$$(f\varsigma_1 + g\varsigma_2)(x) := f(x)\varsigma_1(x) + g\varsigma_2(x),$$
$$((f+g)\varsigma_1)(x) := (f(x) + g(x))\varsigma_1(x).$$

Next we shall relate these smooth sections to smooth sections $S \to \bigwedge^k TS$.

Definition 7.5.1. Let $\varsigma : S \to \bigwedge^k T^*S$ be a section (not necessarily smooth) and let g be a Riemannian metric on S. Define $g^\sharp\varsigma : S \to \bigwedge^k TS$ such that

$$g^\sharp\varsigma(x) := g^\sharp(\varsigma(x)).$$

Hence $g^\sharp\varsigma$ is a section from S to $\bigwedge^k TS$.

Theorem 7.5.2. *A map $\varsigma : S \to \bigwedge^k T^*S$ is smooth if and only if for some Riemannian metric g, $g^\sharp\varsigma$ is a smooth section.*

Proof. Assume that ς is a smooth section. Then since $g^\sharp$ is a smooth map by definition of $C^\infty(\bigwedge^k T^*S)$, their composition is smooth. Conversely, if $g^\sharp\varsigma$ is smooth, the composition with $g^\flat$ is smooth, the result being that ς is smooth since $g^\flat$ and $g^\sharp$ are inverses of each other. □

The above theorem tells us that smooth sections from S to $\bigwedge^k T^*S$ are the dual notion of global derivations (and wedge products thereof). Thus, it is easy to see that the one-forms dq^1 and dq^2 on $S = \{(x, y) \in \mathbb{R}^2 | \ xy = 0\}$ are not smooth sections, as they do not go to 0 at the origin. This is an example of how Zariski and Marshall forms are not generally smooth sections. We can show, however, that $\Gamma^k(S)$ is a subset of $\Omega^k(S)$.

Definition 7.5.3. Define the map $\gamma^k : \Gamma^k(S) \to \Omega^k(S)$ such that for any $u_1, ..., u_k \in T_xS$ and $\varsigma \in \Gamma^k(S)$,

$$\gamma^k(\varsigma)(u_1, ..., u_k) := \varsigma(x)(u_1, ..., u_k).$$

Note that since ς is a smooth section, and $\gamma^k(\varsigma) \in \bigwedge^k T^*S \subset C^\infty(\bigwedge^k TS)$, this map is well-defined. That is, $\gamma^k(\varsigma)$ is indeed a Zariski form, and we thus can identify $\Gamma^k(S)$ as a subset of $\Omega^k(S)$.

7.6 Stokes' Theorem

We can generalise Stokes' Theorem to subcartesian spaces. The singular points do not affect the integration, as we will see, since we shall be mapping simplices in $\mathbb{R}^n$ to the subcartesian space, the image of which are free to pass through singular points. In fact, we need to only check which definition of forms we will need: Zariski or Marshall. Since pullbacks are not well-defined on smooth sections, they will not do. Note that the following presentation is largely based on the proof and presentation of Stokes' Theorem by Spivak in [15].

Let S be a subcartesian space and let $c : A \to S$ be a smooth n-simplex (where $A \subset \mathbb{R}^n$). For any Zariski form $\omega \in \Omega^k(S)$, we have the k-form $c^*\omega \in \Omega^k(\mathbb{R}^n)$. Define:

$$\int_c \omega := \int_A c^*\omega.$$

Then, we can apply Stokes' Theorem in $\mathbb{R}^n$:

$$\int_A dc^*\omega = \int_{\partial A} c^*\omega. \tag{7.6.1}$$

Consider now $\phi \in \mathfrak{m}^k(S)$. For any $x \in \mathrm{im}(c)$, there is a neighbourhood $U \subseteq S$ and a diffeomorphism $\psi : U \to \tilde{U} \subseteq \mathbb{R}^m$ ($m = n_x$) such that $\phi|_{TU} = \psi^*(a_i d\omega^i)$ for some $\omega^i \in \Omega^{k-1}(\mathbb{R}^m)$ where $\psi^*\omega^i = 0$ and $a_i \in C^\infty(\mathbb{R}^m)$. So,

$$\begin{aligned} c^*\phi|_{c^{-1}(U\cap \mathrm{im}(c))} =& c^*\psi^*(a_i d\omega^i)|_{c^{-1}(U\cap \mathrm{im}(c))} \\ =& (c^*\psi^*(a_i)d(c^*\psi^*\omega^i))|_{c^{-1}(U\cap \mathrm{im}(c))} \\ =& (c^*\psi^*(a_i)d(c^*0))|_{c^{-1}(U\cap \mathrm{im}(c))} \\ =& 0, \end{aligned}$$

where one should note that $\psi \circ c$ is a smooth map from A to $\mathbb{R}^m$, and since A is a structurally regular differential subspace of $\mathbb{R}^n$, Marshall and Zariski forms coincide, and so the exterior derivative and pullback can commute.

So we have $c^*(\omega + \mathfrak{m}^k(S)) = c^*\omega$, and so we define

$$\int_c (\omega + \mathfrak{m}^k(S)) := \int_A c^*(\omega + \mathfrak{m}^k(S)) = \int_A c^*\omega.$$

We then have the following:

$$\begin{aligned} \int_c d_M(\omega + \mathfrak{m}^k(S)) &= \int_A c^*(d_M(\omega + \mathfrak{m}^k(S))) \\ &= \int_A dc^*\omega \\ &= \int_{\partial A} c^*\omega && \text{from Equation 7.6.1} \\ &= \int_{\partial A} c^*(\omega + \mathfrak{m}^k(S)) \\ &= \int_{\partial c} (\omega + \mathfrak{m}^k(S)). \end{aligned}$$

Or, we could say $\int_c d_M(\omega) = \int_{\partial c} \omega$, since the Marshall ideal is redundant in this setting. Hence, either definition of form will do.

Chapter 8

Koszul Differential Forms

8.1 Definition of Koszul Forms

Another definition of differential forms shall be discussed in this chapter, this time being pure algebraic.

Definition 8.1.1. A *Koszul differential k-form* is a k-linear (over the ring $C^\infty(S)$) antisymmetric map from $\mathrm{Der}C^\infty(S)$ to $C^\infty(S)$. This forms a $C^\infty(S)$-module, denoted $\Omega^k_K(S)$, by the following operations: let $\omega_1, \omega_2 \in \Omega^k_K(S)$, $f, g \in C^\infty(S)$ and $X_1, ..., X_k \in C^\infty(S)$. Then,

$$(f\omega_1 + g\omega_2)(X_1, ..., X_k) = f\omega_1(X_1, ..., X_k) + g\omega_2(X_1, ..., X_k)$$
$$((f+g)\omega_1)(X_1, ..., X_k) = f\omega_1(X_1, ..., X_k) + g\omega_1(X_1, ..., X_k).$$

Define $\Omega^0_K(S) = C^\infty(S)$, and $\Omega_K(S) := \bigoplus_{i=0}^\infty \Omega^i_K(S)$

It should be noted that although these forms are k-linear over $C^\infty(S)$, one must make sure that the forms are still acting on global derivations, otherwise it is not well-defined. For example, if $f \in C^\infty(S)$, $fX, X_2, ..., X_k \in \mathrm{Der}C^\infty(S)$, but $X \notin \mathrm{Der}C^\infty(S)$, then for $\omega \in \Omega^k_K(S)$, $\omega(fX, X_2, ..., X_k) \neq f\omega(X, X_2, ..., X_k)$.

Now, for a smooth map $f : S \to R$ for subcartesian space R, it is tempting to define $f^* : \Omega^k_K(R) \to \Omega^k_K(S)$ where for $X \in \mathrm{Der}C^\infty(S)$ we have $f^*\omega(X) = \omega(Tf(X))$. However, f^* is not well-defined (since Tf is not well-defined on $\mathrm{Der}C^\infty(S)$). For this reason we do not have Stokes' Theorem for Koszul forms. Note that Tf and f^* are well-defined in these respective cases, however, if f is a diffeomorphism.

In this chapter, we want to define the exterior derivative on Koszul differential forms using a completely algebraic approach. Much of this work was done in [5] for manifolds, but can easily be extended since the algebra does not depend on the underlying space on which the forms are defined. We will reach our goal by first introducing contractions and an algebraic definition of the Lie derivative.

8.2 Contractions

Definition 8.2.1. Let $\omega \in \Omega^k_K(S)$ $(k \geq 1)$ and $X \in \mathrm{Der}C^\infty(S)$. Define the map $i_X : \Omega^k_K(S) \to \Omega^{k-1}_K(S)$ as follows. For any $X_1, ..., X_{k-1} \in \mathrm{Der}C^\infty(S)$,

$$i_X\omega(X_1, ..., X_{k-1}) := \omega(X, X_1, ..., X_{k-1}).$$

By convention, for $f \in C^\infty(S)$, $i_X f := 0$. Call this map *contraction* with X. It is also known as *left interior multiplication* or the *substitution operator*, and sometimes denoted as $X \lrcorner\, \omega$.

Lemma 8.2.2. *Let* $X, Y \in \mathrm{Der}C^\infty(S)$*,* $\alpha \in \Omega^1_K(S)$*,* $\omega, \eta \in \Omega^k_K(S)$*,* $\mu \in \Omega^l_K(S)$ *and* $f, g \in C^\infty(S)$*. Then contraction has the following properties:*

1. $i_X\alpha = \langle\alpha|X\rangle$*.*
2. $i_X df = Xf$*.*
3. $i_X(f\omega + g\eta) = fi_X\omega + gi_X\eta$*.*
4. $i_X(\omega \wedge \mu) = i_X\omega \wedge \mu + (-1)^k\omega \wedge i_X\mu$ *(thus* i_X *is an* antiderivation *in* $\Omega_K(S)$*).*

5. *$i_{fX+gY} = fi_X + gi_Y$.*

6. *$i_X i_Y = -i_Y i_X$.*

7. *If $i_X\omega = 0$ for all $X \in \mathrm{Der}C^\infty(S)$ then $\omega = 0$.*

Proof. 1. $i_X\alpha = \alpha(X) = \langle\alpha|X\rangle$.

2. $i_X df = \langle df|X\rangle = Xf$.

3. Let $X_1, ..., X_{k-1} \in \mathrm{Der}C^\infty(S)$. Then,

$$\begin{aligned} i_X(f\omega + g\eta)(X_1, ..., X_{k-1}) =& (f\omega + g\eta)(X, X_1, ..., X_{k-1}) \\ =& f\omega(X, X_1, ..., X_{k-1}) + g\eta(X, X_1, ..., X_{k-1}) \\ =& fi_X\omega(X_1, ..., X_{k-1}) + gi_X\eta(X_1, ..., X_{k-1}). \end{aligned}$$

4. The following proof of (4) comes from [7]. Let $X_2, ..., X_{k+l} \in \mathrm{Der}C^\infty(S)$ and let $X_1 = X$. Since i_X is linear, we may assume without loss of generality that ω and μ can be decomposed into one-forms: $\omega = \bigwedge_{i=1}^{k} \eta^i$ and $\mu = \bigwedge_{i=k+1}^{k+l} \eta^i$. Thus, we have

$$\begin{aligned} &i_X(\eta^1 \wedge ... \wedge \eta^{k+l})(X_2, ..., X_{k+l}) \\ =&\eta^1 \wedge ... \wedge \eta^{k+l}(X_1, ..., X_{k+l}) \\ =&\det(\mathbb{X}) \end{aligned}$$

where $\mathbb{X}$ is the matrix with $\eta^i(X_j)$ as the $(i, j)^{\text{th}}$ entry. But by definition of the determinant,

$$\det(\mathbb{X}) = \sum_{i=1}^{k+l} (-1)^{i-1}\eta^i(X_1)\det(\mathbb{X}_1^i),$$

where $\mathbb{X}^i_j$ is the minor of $\mathbb{X}$ with the i^{th} row and j^{th} column deleted. So

$$\begin{aligned}
&\det(\mathbb{X})\\
=&\sum_{i+1}^{k+l}(-1)^{i-1}\eta^i(X)\eta^1\wedge...\wedge\hat{\eta}^i\wedge...\wedge\eta^{k+l}(X_2,...,X_{k+l})\\
=&((\sum_{i=1}^{k}(-1)^{i-1}\eta^i(X)\eta^1\wedge...\wedge\hat{\eta}^i\wedge...\wedge\eta^k)\wedge(\bigwedge_{i=k+1}^{k+l}\eta^i)\\
&+(\bigwedge_{i=1}^{k}\eta^i)\wedge(\sum_{i=k+1}^{k+l}(-1)^{i-1}(-1)^k\eta^i(X)\eta^{k+1}\wedge...\wedge\hat{\eta}^i\wedge...\wedge\eta^{k+l}))(X_2,...,X_{k+l})\\
=&(i_X\omega\wedge\mu+(-1)^k\omega\wedge i_X\mu)(X_2,...,X_{k+l}).
\end{aligned}$$

5. Let $X_1, ..., X_{k-1} \in \mathrm{Der}C^\infty(S)$. Then,

$$\begin{aligned}
i_{fX+gY}\omega(X_1,...,X_{k-1}) =&\omega(fX+gY,X_1,...,X_k)\\
=&f\omega(X,X_1,...,X_{k-1})+g\omega(Y,X_1,...,X_{k-1})\\
=&(fi_X\omega+gi_Y\omega)(X_1,...,X_{k-1}).
\end{aligned}$$

6. Let $X_1, ..., X_{k-2} \in \mathrm{Der}C^\infty(S)$. Then,

$$\begin{aligned}
i_X(i_Y\omega)(X_1,...,X_{k-2}) =&\omega(Y,X,X_1,...,X_{k-2})\\
=&-\omega(X,Y,X_1,...,X_{k-2})\\
=&-i_Y(i_X\omega)(X_1,...,X_{k-2}).
\end{aligned}$$

7. Let $X_1, ..., X_{k-1} \in \mathrm{Der}C^\infty(S)$. If for any $X \in \mathrm{Der}C^\infty(S)$, $i_X\omega(X_1, ..., X_{k-1}) = 0$, then $\omega(X, X_1, ..., X_{k-1}) = 0$ for all $X, X_1, ..., X_{k-1} \in \mathrm{Der}C^\infty(S)$. So, $\omega = 0$.

$\square$

8.3 The Lie Derivative

As mentioned before, we shall define the Lie derivative for subcartesian spaces in a completely algebraic context. However, it should be mentioned that a different definition is mentioned by Marshall in [8], where the Lie derivative is calculated in Euclidean space and then a pullback is applied to take this back to the subcartesian space. The relationship between the natural definition of the Lie derivative using one parameter groups of diffeomorphisms from vector fields, the definition that Marshall uses and the definition we are about to present is currently unknown.

Definition 8.3.1. Let $X, X_1, ..., X_k \in \mathrm{Der}C^\infty(S)$ and $\omega \in \Omega^k_K(S)$. Then define a map $\pounds_X : \Omega^k_K(S) \to \Omega^k_K(S)$ called the *Lie derivative* with respect to X as

$$\begin{aligned}\pounds_X\omega(X_1, ..., X_k) :=& X(\omega(X_1, ..., X_k)) \\ &+ \sum_{i=1}^{k}(-1)^i\omega([X, X_i], X_1, ..., \hat{X}_i, ..., X_k).\end{aligned}$$

Remark 8.3.2. It makes sense to define the Lie derivative on global derivations. Let $X, Y \in \mathrm{Der}C^\infty(S)$. Then, define $\pounds_X Y := [X, Y]$. Thus, the above definition of the Lie derivative on forms gives us:

$$\begin{aligned}&\pounds_X(\omega(X_1, ..., X_k)) \\ =&\pounds_X\omega(X_1, ..., X_k) - \sum_{i=1}^{k}(-1)^i\omega(\pounds_X X_i, X_1, ..., \hat{X}_i, ..., X_k).\end{aligned}$$

One justification we must make is that $\pounds_X\omega$ is indeed a form. That is, we must show that $\pounds_X\omega$ is antisymmetric and k-linear over $C^\infty(S)$. Let $X_1, ..., X_k \in$

Der$C^\infty(S)$. Then,

$$
\begin{aligned}
&\pounds_X\omega(X_1, ...X_m, ..., X_n, ..., X_k)\\
=&X(\omega(X_1, ...X_m, ..., X_n, ..., X_k))\\
&+\sum_{i=1}^{k}(-1)^i\omega([X, X_i], X_1, ..., X_m, ..., \hat{X}_i, ..., X_n, ..., X_k)\\
=&-X(\omega(X_1, ..., X_n, ..., X_m, ..., X_k))\\
&-\sum_{i=1}^{k}(-1)^i\omega([X, X_i], X_1, ..., X_n, ..., \hat{X}_i, ..., X_m, ..., X_k)\\
=&-\pounds_X\omega(X_1, ..., X_n, ..., X_m, ..., X_k).
\end{aligned}
$$

And so, $\pounds_X\omega$ is antisymmetric. Next, let $f \in C^\infty(S)$ and $Y \in \text{Der}C^\infty(S)$. Then,

$$
\begin{aligned}
&\pounds_X\omega(fX_1, X_2, ..., X_k)\\
=&X(\omega(fX_1, X_2, ..., X_k)) + \sum_{i=2}^{k}(-1)^i\omega([X, X_i], fX_1, ..., \hat{X}_i, ..., X_k)\\
&-\omega([X, fX_1], X_2, ..., X_k)\\
=&X(f\omega(X_1, ..., X_k)) + f\sum_{i=2}^{k}(-1)^i\omega([X, X_i], X_1, ..., \hat{X}_i, ..., X_k)\\
&-\omega((Xf)X_1 + f[X, X_1], X_2, ..., X_k)\\
=&(Xf)\omega(X_1, ..., X_k) + fX(\omega(X_1, ..., X_k) - (Xf)\omega(X_1, ..., X_k)\\
&+f\sum_{i=1}^{k}(-1)^i\omega([X, X_i], X_1, ..., \hat{X}_i, ..., X_k)\\
=&f\pounds_X\omega(X_1, ..., X_k)).
\end{aligned}
$$

Also,

$$
\begin{aligned}
&\pounds_X\omega(X_1+Y, X_2, ..., X_k)\\
=&X(\omega(X_1+Y, X_2, ..., X_k)) + \sum_{i=2}^{k}(-1)^i\omega([X, X_i], X_1+Y, ..., \hat{X}_i, ..., X_k)\\
&-\omega([X, X_1+Y], X_2, ..., X_k)\\
=&X(\omega(X_1, ..., X_k)) + X(\omega(Y, X_2, ..., X_k))\\
&+\sum_{i=2}^{k}(-1)^i\omega([X, X_i], X_1, ..., \hat{X}_i, ..., X_k) - \omega(X_1, ..., X_k)\\
&+\sum_{i=2}^{k}(-1)^i\omega([X, X_i], Y, X_2, ..., \hat{X}_i, ..., X_k) - \omega(Y, X_2, ..., X_k)\\
=&\pounds_X\omega(X_1, ..., X_k) + \pounds_X\omega(Y, X_2, ..., X_k).
\end{aligned}
$$

Since these proofs are the same for each argument of ω, we have that $\pounds_X\omega$ is k-linear. Thus, $\pounds_X\omega \in \Omega^k(S)$.

Lemma 8.3.3. *Let $X, Y \in \mathrm{Der}C^\infty(S)$, $\alpha \in \Omega^1_K(S)$, $\omega, \eta \in \Omega^k_K(S)$, $\mu \in \Omega^l_K(S)$ and $f \in C^\infty(S)$. Then,*

1. $\langle \pounds_X\alpha | Y\rangle + \langle \alpha |\ [X, Y]\rangle = X(\langle \alpha | Y\rangle)$.
2. $\pounds_X df = d(Xf)$.
3. $\pounds_X(f\omega + \eta) = f\pounds_X\omega + (Xf)\omega + \pounds_X\eta$.
4. $i_{[X,Y]} = \pounds_X i_Y - i_Y\pounds_X$.
5. $\pounds_X(\omega\wedge\mu) = (\pounds_X\omega)\wedge\mu + \omega\wedge(\pounds_X\mu)$.
6. $\pounds_{X+Y} = \pounds_X + \pounds_Y$.
7. $\pounds_{[X,Y]} = \pounds_X\pounds_Y - \pounds_Y\pounds_X$.

8. $\pounds_{fX}\omega = f\pounds_X\omega + df \wedge i_X\omega$.

Proof. 1.

$$\begin{aligned}\langle \pounds_X\omega|Y\rangle =& X(\omega(Y)) - \omega([X,Y]) \\ =& X\langle\omega|Y\rangle - \langle\omega|[X,Y]\rangle\end{aligned}$$

2. From (1),

$$\begin{aligned}\pounds_X(df)(Y) =& X(Yf) - [X,Y](f) \\ =& Y(Xf) \\ =& d(Xf)(Y)\end{aligned}$$

3. Let $X_1, ..., X_k \in \mathrm{Der}C^\infty(S)$. Then,

$$\begin{aligned}&\pounds_X(f\omega+\eta)(X_1,...,X_k) \\ =& X((f\omega+\eta)(X_1,...,X_k)) + \sum_{i=1}^{k}(-1)^i(f\omega+\eta)([X,X_i],X_1,...,\hat{X}_i,...,X_k) \\ =& (Xf)\omega(X_1,...,X_k) \\ &+ fX(\omega(X_1,...,X_k)) + f\sum_{i=1}^{k}(-1)^i\omega([X,X_i],X_1,...,\hat{X}_i,...,X_k) \\ &+ X(\eta(X_1,...,X_k)) + \sum_{i=1}^{k}(-1)^i\eta([X,X_i],X_1,...,\hat{X}_i,...,X_k) \\ =& ((Xf)\omega + f\pounds_X\omega + \pounds_X\eta)(X_1,...,X_k).\end{aligned}$$

4. Let $\omega \in \Omega^k_K(S)$ and $X_1, ..., X_{k-1} \in \text{Der}C^\infty(S)$. Then,

$$\begin{aligned}
&\pounds_X(i_Y\omega)(X_1, ..., X_{k-1}) \\
=&X(i_Y\omega(X_1, ..., X_{k-1})) + \sum_{j=1}^{k-1}(-1)^j i_Y\omega([X, X_j], X_1, ..., \hat{X}_j, ..., X_{k-1}) \\
=&X(\omega(Y, X_1, ..., X_{k-1})) + \sum_{j=1}^{k-1}(-1)^{j+1}\omega([X, X_j], Y, X_1, ..., \hat{X}_j, ..., X_{k-1}),
\end{aligned}$$

and

$$\begin{aligned}
&i_Y(\pounds_X\omega)(X_1, ..., X_{k-1}) \\
=&\pounds_X\omega(Y, X_1, ..., X_{k-1}) \\
=&X(\omega(Y, X_1, ..., X_{k-1})) + \sum_{j=1}^{k-1}(-1)^{j+1}\omega([X, X_j], Y, X_1, ..., \hat{X}_j, ..., X_{k-1}) \\
&- \omega([X, Y], X_1, ..., X_{k-1}).
\end{aligned}$$

Subtracting the second term from the first, we get,

$$\begin{aligned}
\pounds_X(i_Y\omega)(X_1, ..., X_{k-1}) - \pounds_X\omega_(Y, X_1, ..., X_{k-1}) =&\omega([X, Y], X_1, ..., X_{k-1}) \\
=&i_{[X,Y]}\omega(X_1, ..., X_{k-1}).
\end{aligned}$$

5. We shall use mathematical induction. Recalling that $\omega \in \Omega^k(S)$ and $\mu \in \Omega^l(S)$, assume $k + l = 0$. Then $\omega, \mu \in C^\infty(S)$, and

$$\pounds_X(\omega \wedge \mu) = X(\omega\mu) = \mu X(\omega) + \omega X(\mu).$$

Now, assuming the hypothesis for the cases where $k + l < m$, let $k + l = m$.

Then,

$$
\begin{aligned}
& i_Y(\pounds_X(\omega \wedge \mu)) \\
=& \pounds_X(i_Y(\omega \wedge \mu)) - i_{[X,Y]}(\omega \wedge \mu) && \text{(by (4))} \\
=& \pounds_X(i_Y(\omega) \wedge \mu + (-1)^k \omega \wedge (i_Y \mu)) \\
& - i_{[X,Y]}\omega \wedge \mu - (-1)^k \omega \wedge i_{[X,Y]}\mu && \text{(by Lemma 8.2.2)} \\
=& \pounds_X(i_Y\omega) \wedge \mu + i_Y\omega \wedge \pounds_X\mu + (-1)^k \pounds_X\omega \wedge i_Y\mu \\
& + (-1)^k \omega \wedge \pounds_X(i_Y\mu) - i_{[X,Y]}\omega \wedge \mu - (-1)^k \omega \wedge i_{[X,Y]}\mu && \text{(by our assumption)} \\
=& \pounds_X(i_Y\omega) \wedge \mu + i_Y\omega \wedge \pounds_X\mu + (-1)^k \pounds_X\omega \wedge i_Y\mu \\
& + (-1)^k \omega \wedge \pounds_X(i_Y\mu) - \pounds_X(i_Y\omega) \wedge \mu + i_Y(\pounds_X\omega) \wedge \mu \\
& - (-1)^k \omega \wedge \pounds_X(i_Y\mu) + (-1)^k \omega \wedge i_Y(\pounds_X\mu) && \text{(by (4))} \\
=& i_Y\omega \wedge \pounds_X\mu + (-1)^k \pounds_X\omega \wedge i_Y\mu \\
& + i_Y(\pounds_X\omega) \wedge \mu + (-1)^k \omega \wedge i_Y \pounds_X\mu \\
=& i_Y(\omega \wedge \pounds_X\mu) + i_Y(\pounds_X\omega \wedge \mu) && \text{(by Lemma 8.2.2)} \\
=& i_Y(\omega \wedge \pounds_X\mu + \pounds_X\omega \wedge \mu).
\end{aligned}
$$

Thus, we have

$$i_Y(\pounds_X(\omega \wedge \mu) - \omega \wedge \pounds_X\mu - \pounds_X\omega \wedge \mu) = 0,$$

and applying Lemma 8.2.2 (7), we obtain our result.

6. Let $X_1, ..., X_k \in \mathrm{Der}C^\infty(S)$. Then,

$$
\begin{aligned}
&\pounds_{X+Y}\omega(X_1, ..., X_k)\\
=&(X+Y)\omega(X_1, ..., X_k) + \sum_{i=1}^{k}(-1)^i\omega([X+Y, X_i], X_1, ..., \hat{X}_i, ..., X_k)\\
=&X\omega(X_1, ..., X_k) + \sum_{i=1}^{k}(-1)^i\omega([X, X_i], X_1, ..., \hat{X}_i, ..., X_k)\\
&+Y\omega(X_1, ..., X_k) + \sum_{i=1}^{k}(-1)^i\omega([Y, X_i], X_1, ..., \hat{X}_i, ..., X_k)\\
=&\pounds_X\omega(X_1, ..., X_k) + \pounds_Y\omega(X_1, ..., X_k).
\end{aligned}
$$

7. Let $X_1, ..., X_k \in \mathrm{Der}C^\infty(S)$. Then,

$$
\begin{aligned}
&\pounds_X\pounds_Y\omega(X_1, ..., X_k)\\
=&X(\pounds_Y\omega(X_1, ..., X_k)) + \sum_{i=1}^{k}(-1)^i\pounds_Y\omega([X, X_i], X_1, ..., \hat{X}_i, ..., X_k)\\
=&XY(\omega(X_1, ..., X_k)) + \sum_{i=1}^{k}(-1)^iX(\omega([Y, X_i], X_1, ..., \hat{X}_i, ..., X_k))\\
&+\sum_{i=1}^{k}(-1)^iY(\omega([X, X_i], X_1, ..., \hat{X}_i, ..., X_k))\\
&-\sum_{i=1}^{k}(-1)^i\omega([Y, [X, X_i]], X_1, ..., \hat{X}_i, ..., X_k)\\
&+\sum_{1\leq i\neq j\leq k}(-1)^{i+j}\omega([Y, X_i], [X, X_j], X_1, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_k).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\pounds_Y \pounds_X \omega(X_1, ..., X_k) \\
=&Y X(\omega(X_1, ..., X_k)) + \sum_{i=1}^{k} (-1)^i Y(\omega([X, X_i], X_1, ..., \hat{X}_i, ..., X_k)) \\
&+ \sum_{i=1}^{k} (-1)^i X(\omega([Y, X_i], X_1, ..., \hat{X}_i, ..., X_k)) \\
&- \sum_{i=1}^{k} (-1)^i \omega([X, [Y, X_i]], X_1, ..., \hat{X}_i, ..., X_k) \\
&+ \sum_{1 \le i \ne j \le k} (-1)^{i+j} \omega([X, X_i], [Y, X_j], X_1, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_k).
\end{aligned}
$$

Thus, subtracting the second from the first, we get:

$$
\begin{aligned}
&(\pounds_X \pounds_Y - \pounds_Y \pounds_X)(X_1, ..., X_k) \\
=&[X, Y](\omega(X_1, ..., X_k)) \\
&+ \sum_{i=1}^{k} (-1)^i \omega([X, [Y, X_i]], X_1, ..., \hat{X}_i, ..., X_k) \\
&- \sum_{i=1}^{k} (-1)^i \omega([Y, [X, X_i]], X_1, ..., \hat{X}_i, ..., X_k).
\end{aligned}
$$

Now, the Jacobi identity states the following:

$$
\begin{aligned}
&[[X, Y], X_i] + [[X_i, X], Y] + [[Y, X_i], X] \\
=&[[X, Y], X_i] + [Y, [X, X_i]] - [X, [Y, X_i]] \\
=&0.
\end{aligned}
$$

So, considering $\omega([[X,Y],X_i],X_1,...,\hat{X}_i,...,X_k)$,

$$\begin{aligned}
&\omega([[X,Y],X_i],X_1,...,\hat{X}_i,...,X_k)\\
=&\omega([X,[Y,X_i]],X_1,...,\hat{X}_i,...,X_k)\\
&-\omega([Y,[X,X_i]],X_1,...,\hat{X}_i,...,X_k).
\end{aligned}$$

Thus,

$$\begin{aligned}
&(\pounds_X\pounds_Y-\pounds_Y\pounds_X)(X_1,...,X_k)\\
=&[X,Y](X_1,...,X_k)\\
&+\sum_{i=1}^{k}(-1)^i\omega([[X,Y],X_i],X_1,...,\hat{X}_i,...,X_k)\\
=&\pounds_{[X,Y]}\omega(X_1,...,X_k).
\end{aligned}$$

8. Let $X_1,...,X_k\in\mathrm{Der}C^\infty(S)$. Then,

$$\begin{aligned}
&\pounds_{fX}\omega(X_1,...,X_k)\\
=&(fX)\omega(X_1,...,X_k)+\sum_{i=1}^{k}(-1)^i\omega([fX,X_i],X_1,...,\hat{X}_i,...,X_k)\\
=&fX(\omega(X_1,...,X_k))+f\sum_{i=1}^{k}(-1)^i\omega([X,X_i],X_1,...,\hat{X}_i,...,X_k)\\
&-\sum_{i=1}^{k}(-1)^i(X_if)\omega(X,X_1,...,\hat{X}_i,...,X_k).
\end{aligned}$$

As well,

$$
\begin{aligned}
&- (df \wedge i_X\omega)(X_1, ..., X_k) \\
=&i_X(df \wedge \omega)(X_1, ..., X_k) - i_X df \wedge \omega(X_1, ..., X_k) \\
=&(Xf)\omega(X_1, ..., X_k) + \sum_{i=1}^{k}(-1)^i(X_i f)\omega(X, X_1, ..., \hat{X}_i, ..., X_k) \\
&- (Xf)\omega(X_1, ..., X_k) \\
=&\sum_{i=1}^{k}(-1)^i(X_i f)\omega(X, X_1, ..., \hat{X}_i, ..., X_k).
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&\pounds_{fX}\omega(X_1, ..., X_k) - (df \wedge i_X\omega)(X_1, ..., X_k) \\
=&f(X\omega(X_1, ..., X_k) + \sum_{i=1}^{k}(-1)^i\omega([X, X_i], X_1, ..., \hat{X}_i, ..., X_k)) \\
=&f\pounds_X\omega(X_1, ..., X_k).
\end{aligned}
$$

□

8.4 The Exterior Derivative

Definition 8.4.1. Let $\omega \in \Omega_K^{k-1}(S)$ $(k > 0)$ and $X_1, ..., X_k \in \mathrm{Der}C^\infty(S)$. Then define the *exterior derivative* $d\omega \in \Omega_K^k(S)$ as follows:

$$
\begin{aligned}
d\omega(X_1, ..., X_k) := &\sum_{i=1}^{k}(-1)^{i+1}X_i(\omega(X_1, ..., \hat{X}_i, ..., X_k)) \\
&+ \sum_{1\leq i<j\leq k}(-1)^{i+j}\omega([X_i, X_j], X_1, ..., \hat{X}_i, ..., \hat{X}_j, ...X_k),
\end{aligned}
$$

where $\hat{X}_i$ means that X_i was removed from the list *if it appears* in the list (this condition will be important in the proof of the theorem following this definition).

The exterior derivative has the following properties:

Theorem 8.4.2. *Let* $f \in C^\infty(S)$*,* $\omega, \eta \in \Omega^k(S)$ *(*$k > 0$*),* $\mu \in \Omega^l(S)$ *(*$l > 0$*),* $X \in \mathrm{Der}C^\infty(S)$*, and* $a \in \mathbb{R}$*.*

1. $d(f) = df$.
2. $d(a\omega + \eta) = ad\omega + d\eta$.
3. $\pounds_X\omega = i_X(d\omega) + d(i_X\omega)$.
4. $d(\omega \wedge \mu) = d\omega \wedge \mu + (-1)^k\omega \wedge d\mu$.
5. $d\pounds_X\omega = \pounds_X d\omega$.
6. $d^2 = 0$.

Proof. 1. Let $X \in \mathrm{Der}C^\infty(S)$. Then, it is clear using the definition of $d(f)$ that $d(f)(X) = X(f) = df(X)$.

2. Let $X_1, ..., X_k \in \mathrm{Der}C^\infty(S)$. Then,

$$\begin{aligned}
&d(a\omega+\eta)(X_1,...,X_k)\\
=&\sum_{i=1}^{k}(-1)^{i+1}X_i((a\omega+\eta)(X_1,...,\hat{X}_i,...,X_k))\\
&+\sum_{1\le i<j\le k}(-1)^{i+j}(a\omega+\eta)([X_i,X_j],X_1,...,\hat{X}_i,...,\hat{X}_j,...X_k)\\
=&a\sum_{i=1}^{k}(-1)^{i+1}X_i(\omega(X_1,...,\hat{X}_i,...,X_k))\\
&+a\sum_{1\le i<j\le k}(-1)^{i+j}\omega([X_i,X_j],X_1,...,\hat{X}_i,...,\hat{X}_j,...X_k)\\
&+\sum_{i=1}^{k}(-1)^{i+1}X_i(\eta(X_1,...,\hat{X}_i,...,X_k))\\
&+\sum_{1\le i<j\le k}(-1)^{i+j}\eta([X_i,X_j],X_1,...,\hat{X}_i,...,\hat{X}_j,...X_k)\\
=&(ad\omega+d\eta)(X_1,...,X_k)
\end{aligned}$$

3. Let $X_1, ..., X_k \in \mathrm{Der}C^\infty(S)$, and let $X_0 := X$. Then,

$$\begin{aligned}
&d(i_X\omega)(X_1,...,X_k)\\
=&\sum_{i=1}^{k}(-1)^{i+1}X_i(\omega(X_0,...\hat{X}_i,...,X_k))\\
&-\sum_{1\le i<j\le k}(-1)^{i+j}\omega([X_i,X_j],X_0,...,\hat{X}_i,...,\hat{X}_j,...,X_k).
\end{aligned}$$

As well,

$$
\begin{aligned}
i_X(d\omega)(X_1, ..., X_k) =& d\omega(X_0, ..., X_k) \\
=& \sum_{i=0}^{k}(-1)^i X_i(\omega(X_0, ..., \hat{X}_i, ..., X_k)) \\
&+ \sum_{0\leq i<j\leq k}(-1)^{i+j}\omega([X_i, X_j], X_0, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_k).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&(d(i_X\omega) + i_X(d\omega))(X_1, ..., X_k) \\
=& X(\omega(X_1, ..., X_k)) + \sum_{j=1}^{k}(-1)^j\omega([X, X_j], X_1, ..., \hat{X}_j, ..., X_k) \\
=& \pounds_X\omega(X_1, ..., X_k).
\end{aligned}
$$

4. We shall use mathematical induction. First, recalling that $\omega \in \Omega^k(S)$ and $\mu \in \Omega^l(S)$, then assume $k + l = 0$. Then, $d(fg) = gdf + fdg$. Next, assume that the hypothesis is true for all cases $k + l < m$. Now, consider the case

where $k + l = m$.

$$
\begin{aligned}
& i_X(d(\omega \wedge \mu)) \\
=& \pounds_X(\omega \wedge \mu) - d(i_X(\omega \wedge \mu)) && \text{(by (3))} \\
=& \pounds_X\omega \wedge \mu + \omega \wedge \pounds_X\mu - d(i_X(\omega \wedge \mu)) && \text{(by Lemma 8.3.3)} \\
=& \pounds_X\omega \wedge \mu + \omega \wedge \pounds_X\mu \\
& - d(i_X\omega \wedge \mu) - (-1)^k d(\omega \wedge i_X\mu) && \text{(by Lemma 8.2.2)} \\
=& \pounds_X\omega \wedge \mu + \omega \wedge \pounds_X\mu - d(i_X\omega) \wedge \mu \\
& - (-1)^{k-1} i_X\omega \wedge d\mu - (-1)^k d\omega \wedge i_X\mu - \omega \wedge d(i_X\mu) && \text{(induction hypothesis)} \\
=& (\pounds_X\omega - d(i_X\omega)) \wedge \mu + \omega \wedge (\pounds_X\mu - d(i_X\mu)) \\
& + (-1)^k i_X\omega \wedge d\mu - (-1)^k d\omega \wedge i_X\mu \\
=& i_X(d\omega) \wedge \mu + \omega \wedge (i_X d\mu) \\
& + (-1)^k i_X\omega \wedge d\mu + (-1)^{k+1} d\omega \wedge i_X\mu && \text{(by (3))} \\
=& i_X(d\omega \wedge \mu) + (-1)^k i_X(\omega \wedge d\mu) && \text{(by Lemma 8.2.2)} \\
=& i_X(d\omega \wedge \mu + (-1)^k \omega \wedge d\mu).
\end{aligned}
$$

Thus,

$$i_X(d(\omega \wedge \mu) - d\omega \wedge \mu - (-1)^k \omega \wedge d\mu) = 0,$$

and by Lemma 8.2.2 (7), we have our result.

5. Using mathematical induction, consider the base case:

$$\begin{aligned} d\pounds_X f(Y) =& d(Xf)(Y) \\ =& YXf \\ \pounds_X df(Y) =& X(df(Y)) - df([X,Y]) \\ =& XYf - XYf + YXf \\ =& YXf. \end{aligned}$$

Now, assume the hypothesis is true for all $(k-1)$-forms. Then,

$$\begin{aligned} & i_Y(d\pounds_X\omega) \\ =& \pounds_Y\pounds_X\omega - d(i_Y\pounds_X\omega) && \text{(by (3))} \\ =& \pounds_Y\pounds_X\omega - d(-i_{[X,Y]}\omega + \pounds_X i_Y\omega) && \text{(by Lemma 8.3.3)} \\ =& \pounds_Y\pounds_X\omega + d(i_{[X,Y]}\omega) - d(\pounds_X i_Y\omega) \\ =& \pounds_Y\pounds_X\omega + \pounds_{[X,Y]}\omega \\ & - i_{[X,Y]}d\omega - d(\pounds_X i_Y\omega) && \text{(by (3))} \\ =& \pounds_Y\pounds_X\omega + \pounds_{[X,Y]}\omega \\ & - i_{[X,Y]}d\omega - \pounds_X d(i_Y\omega) && \text{(induction hypothesis)} \\ =& \pounds_X\pounds_Y\omega - i_{[X,Y]}d\omega - \pounds_X d(i_Y\omega) && \text{(by Lemma 8.3.3)} \\ =& \pounds_X\pounds_Y\omega - \pounds_X i_Y d\omega \\ & + i_Y\pounds_X d\omega - \pounds_X d(i_Y\omega) && \text{(by Lemma 8.3.3)} \\ =& \pounds_X\pounds_Y\omega - \pounds_X\pounds_Y\omega + i_Y(\pounds_X d\omega) && \text{(by (3))} \\ =& i_Y(\pounds_X d\omega). \end{aligned}$$

Hence, by Lemma 8.2.2, we are done.

6. We again shall use mathematical induction. Consider the base case:

$$\begin{aligned} d^2 f(X,Y) =& X(df(Y)) - Y(df(X)) - df([X,Y]) \\ =& [X,Y]f - [X,Y]f \\ =& 0. \end{aligned}$$

Now, assume the hypothesis is true for all $(k-1)$-forms. Then,

$$\begin{aligned} i_X d^2\omega =& \pounds d\omega - di_X d\omega & \\ =& d\pounds_X\omega - di_X d\omega & \text{(by (5))} \\ =& d^2(i_X\omega) & \text{(by (3))} \\ =& 0 & \text{(induction hypothesis)}. \end{aligned}$$

So, again, by Lemma 8.2.2 (7), we are done.

□

8.5 Comparisons

Given a neighbourhood $U \subseteq S$ with local coordinates $q^1, ..., q^n$; dq^i for $i = 1, ..., n$ are Koszul forms on U. In fact, any wedge product of these are Koszul forms, and so $\sum_I' \omega_I dq^{i_1} \wedge ... \wedge dq^{i_k}$ is a Koszul form. This tells us that any locally representable form is a Koszul form, and so there is a canonical map from Zariski forms to Koszul forms.

In fact, define the map $\kappa^k : \Omega^k(S) \to \Omega^k_K(S) : \omega \mapsto \bar{\omega}$ as follows: for any $X_1, ..., X_k \in \mathrm{Der}C^\infty(S)$,

$$\bar{\omega}(X_1, ..., X_k)(x) := \omega(X_1(x), ..., X_k(x)).$$

We can say something very interesting about this map:

Proposition 8.5.1. *$\mathfrak{m}^k(S) \subseteq \ker(\kappa^k)$.*

Proof. Let $x \in S$ with neighbourhood $U \subseteq S$ and diffeomorphism $\psi : U \to V \subseteq \mathbb{R}^n$ where $n = n_x$, and let $\omega \in \Omega^{k-1}(\mathbb{R}^n)$ such that $\psi^*\omega = 0$ and $\psi^* d\omega = \phi|_{TU}$. Then,

$$\begin{aligned}
&\phi|_{TU}(X_1(x), ..., X_k(x)) \\
=&\psi^* d\omega(X_1, ..., X_k)(x) \\
=&d\omega(T\psi(X_1, ..., X_k))(\psi(x)) \\
=&\sum_{i=1}^{k}(-1)^i T\psi(X_i)(\omega(T\psi(X_1), ..., T\hat{\psi(X_i)}, ..., T\psi(X_k)))(\psi(x)) \\
&+ \sum_{1\le i<j\le k}(-1)^{i+j}\omega([T\psi(X_i), T\psi(X_j)], T\psi(X_1), ..., T\hat{\psi(X_i)}, ..., T\hat{\psi(X_j)}, ..., T\psi(X_k))(\psi(x)),
\end{aligned}$$

recalling that $T\psi$ is well-defined here on global derivations since ψ is a diffeomorphism. Now,

$$\omega(T\psi(Y_1), ..., T\psi(Y_{k-1})) = \psi^*\omega(Y_1, ..., Y_k) = 0,$$

for any $Y_1, ..., Y_k$ and so we can conclude that $\phi|_{TU}(X_1(x), ..., X_k(X)) = 0$. □

Thus κ^k can be extended to a canonical map from Marshall forms to Koszul forms.

Example 8.5.2. Consider $S = \{(x, y) \in \mathbb{R}^2 |\ xy = 0\}$. We will build a table like we did in Example 7.4.5 (see Table 8.1). Again, for the sake of brevity, we have listed bases $\{\cdot\}$ in the table and omitted $\text{span}_{C^\infty(S)}\{\cdot\}$. The structural dimension and tangent spaces are clear. Let's check the global derivations. An arbitrary global derivation in $\mathbb{R}^2$ is $a\partial_x + b\partial_y$ for $a, b \in C^\infty(\mathbb{R}^2)$. $xy \in N(S)$, and so we have

$$(a\partial_x + b\partial_y)(xy) = ay + bx = 0,$$

	x-axis Minus Origin	y-axis Minus Origin	Origin
Structural Dimension	1	1	2
Tangent Space	$\cong \mathbb{R}$	$\cong \mathbb{R}$	$\cong \mathbb{R}^2$
Global Derivations	$\{\partial_x\}$	$\{\partial_y\}$	0
Zariski 1-Forms	$\{dx\}$	$\{dy\}$	$\{dx, dy\}$
Zariski 2-Forms	0	0	$\{dx \wedge dy\}$
Marshall 2-Submodule	0	0	$\{dx \wedge dy\}$
Marshall 2-Forms	0	0	0
Smooth Sections (1-Forms)	$\{dx\}$	$\{dy\}$	0
Smooth Sections (2-Forms)	0	0	0
Koszul 1-Forms	$\{dx\}$	$\{dy\}$	$\{dx, dy\}$
Koszul 2-Forms	0	0	0

Table 8.1: Two Intersecting Lines

and so if $x = 0$ and $y \neq 0$, we get $a(0, y) = 0$, and by continuity, $a(0, 0) = 0$. Likewise, if $y = 0$ and $x \neq 0$, we get $b(x, 0) = 0$, and by continuity, $b(0, 0) = 0$. So, at the origin, global derivations must vanish on S. Next, note that $i^*(xdy)$ and $i^*(ydx)$ are both 0 (where $i : S \hookrightarrow \mathbb{R}^2$ is the inclusion), but $i^*(dx \wedge dy)$ is not 0 at the origin. Hence, $i^*(dx \wedge dy) \in \mathfrak{m}^2(S)$. This results in having no Marshall 2-forms (besides 0). By Theorem 7.5.2, we can use the Euclidean metric ($g = i^*(\delta_{jk} dx^j \otimes dy^k)$ where δ_{ij} is the Kronecker delta) to see that the sections must vanish at the origin, just like the global derivations, and it is clear that there are no 2-forms that are smooth sections. Of course, since all forms on $\mathbb{R}^2$ are smooth sections, it is clear that pullbacks of smooth sections are not necessarily smooth sections. Finally, since we cannot wedge two global derivations on this space together without getting 0 as a result, there are no Koszul 2-forms. Note, however, that there do exist Zariski, Marshall and Koszul 1-forms that are not smooth sections (recalling that $\Omega^1_M(S) = \Omega^1(S)$), since they do not have to vanish at the origin.

To conclude, we see that Zariski forms have well-defined pullbacks, but they fail to have a well-behaved exterior derivative, and they are not smooth sections. Marshall differential forms also have well-defined pullbacks and also a proper exterior derivative which commutes with pullbacks, but again these are not smooth sections. The Koszul forms do not have pullbacks in general nor are they smooth sections, but they do have an exterior derivative. Finally, a pullback of a smooth section is not necessarily a smooth section. Note that if a subcartesian space is structurally regular, then the Marshall ideal is a zero ideal, and so Marshall forms and Zariski forms coincide, and these are smooth sections; also in this case κ^k defined at the beginning of this section is an isomorphism of modules.

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