Abstract
This study considers the amount of time it takes for frostbite to begin on a finger. It was considered as a transient heat transfer problem, using Euler’s method to solve for a time that it takes to reach the critical temperature, $T_c$. The results show that it will only take minutes for it to happen. It was found that the time decreased as wind speed increased and also as the outside temperature fell, with wind speed having a larger impact on the outcome.

Introduction
Every year hundreds of people suffer from tissue damage due to frostbite. This occurs when bare skin is exposed to cold temperatures for a long enough period to allow the temperature of the tissue to drop past 23 degrees Fahrenheit [2], if this temperature is maintained, damage to the body part will begin. One of the many functions of the blood is to help keep all parts of the body at the correct temperature; this gets increasingly difficult towards the extremities due to the distance that the blood has to travel. It becomes even harder in severe conditions that are experienced during the winter months. The body only produces a certain amount of heat through metabolic processes which is then carried throughout the body by the blood. When temperatures are below 20°F with winds of 20 mph it will take only minutes for frostbite to begin to damage the tissue [3].

As can be seen below in Figure 1 there are two conditions that have a significant effect on the time it takes to cool a finger to the critical frostbite temperature (23 °F), outside temperature and wind speed. One of the initial conditions that were tested was the outside temperature, which showed that as the temperature decreases and/or the wind speed increases the time it takes for the skin temperature to cool decreases.

From the Figure 1, it can be seen that the wind speed is the most important factor in how quickly the finger reaches 23°F. Even at the highest temperature that was considered with a 20 mph wind, the time to reach the frostbite point was reached quicker than the coldest temperature considered with a 5 mph wind. Completely still air was also considered in this study; however, the times that it took to reach the frostbite point were long enough that it was considered unlikely to happen. It also was a fairly unreasonable assumption because even if there was no wind a hand is not likely to be completely motionless when outside.
Figure 1: Summary of results, time to reach frostbite temperatures as a function of wind speed and outside temperature (°F) with a $Q_{gen}$ of 0.317 W

**Problem Description**

The problem that was considered in this study was the time it takes for a finger, outside, in a cold, windy environment to reach a temperature where frostbite would begin to set in. Since the temperature of the finger was dependent on time, the analysis was not steady-state but rather a transient problem. The conditions that were investigated for this analysis were outside temperatures of -5, 0, 5, 10 and 15°F (253, 255, 258, 261 and 263 K) with wind speeds of 5, 10 and 20 mph at each temperature.

Several assumptions were made in order to solve this problem. The most important of these was that a finger was considered to have a constant heat generation regardless of its environment. The heat generation that was solved for was the amount of energy it takes to keep a finger at 33°C (actual measured skin temperature) in a room temperature environment considering convection and radiation. The shape of the finger that was used was a 76.2 mm long cylinder with a diameter of 12.7 mm, which were taken from actual finger measurements. The initial temperature for cooling once the finger moves outside was considered to be 33°C. Radiation heat transfer also had a significant effect and was unable to be ignored in this problem. Another assumption that was made was the makeup of the finger, which was judged to be 90% bone and 10% skin. The properties of these two substances [1, 4] were used to find a density and specific heat capacity of a finger. It was also assumed that $T_{surr}$ was equal to $T_{inf}$ when solving for the heat transfer rates. With the proceeding assumptions the transient heat transfer analysis could be solved.

**Numerical Modeling**

The finger was initially modeled as a thermal resistance network, composed of three resistance values; one for the heat transfer due to convection from the blood to the surrounding tissue, one for conduction between the tissue and surface of the finger, and one for the convection from the
surface to the outside air. The overall heat transfer coefficient could be solved using equation (1):

$$\frac{1}{UA_o} = R_{blood} + R_{cond} + R_{air}$$  \hspace{1cm} (1)$$

In order to apply this method, the finger was simplified as a cylinder with one embedded, parallel tube, or capillary with an averaged diameter offset a specified distance from the center of the tube, similar to the composition of a heat exchanger. A correction for the conduction shape factor for an eccentric isothermal cylinder inside of another cylinder was used for the conduction resistance. A correlation for the internal convection caused by the flow of blood was used to find the heat transfer coefficient for one capillary [5]. This value was then multiplied by an estimated number of capillaries in a typical human finger based on the total length of blood vessels and arteries in a human body and on the length and area of the modeled finger. This method proved to be very difficult and inaccurate assessment. The external forced convection resistance was determined by using the correct correlation for the ambient air flow conditions.

Ultimately this method proved to be unsuccessful mainly due to the number of unknown properties and correlations, which resulted in a number of unrealistic assumptions. Most notable was assuming blood was acting as a Newtonian fluid, which is not the case for capillary flow. The conductive resistance value was most likely not accurate because treating the capillaries in the finger as a series of parallel tubes is not realistic as capillary configuration takes on the form of random branched networks. As a result a different approach was taken

The method actually employed in the numerical modeling was based on modeling the finger as a constant heat generating body. A value for the heat generated by a human finger to keep the surface temperature of the finger was calculated. It was then assumed that this positive energy rate was kept constant by the body during exposure to the ambient conditions as time progressed, while a changing heat rate was being transferred away from the finger by convection and radiation. Using this approach, the surface temperature of the finger with respect to time could be calculated for the different wind chill factors and ambient air temperatures.

The heat generated by the finger was determined using a reverse approach. According to the First Law of Thermodynamics, the net heat transfer for the finger is equal to zero if the temperature of the finger is remaining constant. Therefore it isn’t unrealistic to assume that the heat leaving the finger due to convection and radiation is equal to the heat being generated by the finger. The value for the constant heat generation of the finger was calculated from equation (2) [5]:

$$Q_{gen} = hA_s(T_s - T_{\infty}) - E\sigma A_s(T_{st}^4 - T_{sur}^4)$$  \hspace{1cm} (2)$$

First several fluid and thermal properties were tabulated so that the Reynolds Numbers and Nusselt numbers and thus the convection heat transfer coefficient could be calculated for the varying ambient conditions. The Reynolds number is given by equation (3):

$$Re = \frac{\rho vD}{\mu}$$  \hspace{1cm} (3)$$
For external forced convection across a cylinder, two correlations were used to the then calculate the average Nusselt number, also known as the dimensionless convection heat transfer coefficient. The correlation for the Nusselt number for Reynolds numbers between 40 and 4,000, which was the case for a wind velocity of 2.24 \text{ m/s} is given by equation (4) [5]:

$$Nu_a = 0.683(Re^{446})Pr^{1/3}$$  \hspace{1cm} (4)

For wind velocities of 4.97 and 8.94 \text{ m/s} the Reynolds numbers were between 4,000 and 40,000, so the Nusselt number was calculated from equation (5) [5]:

$$Nu_b = 0.193(Re^{618})Pr^{1/3}$$  \hspace{1cm} (5)

In order to calculate the Nusselt number for the natural convection, or zero air velocity, the kinematic viscosity, volume expansion coefficient and Raleigh numbers must first be calculated using equations (6), (7), and (8). The equation for kinematic viscosity is given by equation (6) [5]:

$$v = \frac{\mu}{\rho}$$  \hspace{1cm} (6)

The volume expansion coefficient relates the dependence of a fluid’s density on temperature difference. For an ideal gas, the volume expansion coefficient is the inverse of the thermodynamics temperature, or in the present case, the average temperature, and is given by equation (7) [5]:

$$\beta = \left(\frac{T_{sl}+T_{∞}}{2}\right)^{-1}$$  \hspace{1cm} (7)

The Raleigh Number relates the buoyancy and viscosity behaviors for a specific fluid. The correlation for the Raleigh number for natural convection over a horizontal cylinder was calculated using equation (8) [5]:

$$Ra = \frac{g\beta(T_{sl}-T_{∞})D^3}{v^2}$$  \hspace{1cm} (8)

The Nusselt number for the natural convection was then calculated using equation (9) [5]:

$$Nu_{nat} = \left[0.6 + \left(\frac{0.387Ra^{16}}{1+\left(\frac{559}{Pr}\right)^{9/16}\left(\frac{Pr}{Pr}^{9/16}\right)^{8/27}}\right)^2\right]$$  \hspace{1cm} (9)

The corresponding heat transfer coefficients for the three Nusselt numbers were then calculated using equation (10) [5]:

$$h = \frac{Nu_k}{D}$$  \hspace{1cm} (10)
The surface area for the finger was determined so that the heat transfer due to convection could be calculated. The finger was modeled as a cylinder and half sphere configuration, and the relation is represented by equation (11):

$$A_s = \pi \frac{D^2}{2} + \pi DL$$

(11)

The volume of the finger modeled was calculated using equation (12):

$$V = \pi \frac{D^3}{12} + \pi \frac{D^2}{4} L$$

(12)

The measured surface temperature for the finger at room conditions was $33^\circ C$. Using equations (6), (7), (8), (9), and (10) the natural convection heat transfer coefficient for $Q_{gen}$ was calculated to be 3.078 W/m$^2$·K. Then by using equation (2) $Q_{gen}$ was calculated to be 0.321 W. This value for $Q$ was then added to the derived differential equation for the time dependent surface temperature represented by equation (13) [5]:

$$\rho V C_p \left( \frac{dT_s}{dt} \right) = Q_{gen} - hA_s(T_s - T_\infty) - ES\tau A_s(T_s^4 - T_{surr}^4)$$

(13)

Before this equation could be solved a specified finger density, volume, and specific heat for a finger were needed. Approximations were made for density and specific heat using values found in Table 1:

<table>
<thead>
<tr>
<th>Material</th>
<th>$\rho$</th>
<th>$C_p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bone</td>
<td>1850</td>
<td>440</td>
</tr>
<tr>
<td>Blood</td>
<td>1004</td>
<td>1004</td>
</tr>
<tr>
<td>Skin</td>
<td>-</td>
<td>3600</td>
</tr>
</tbody>
</table>

Table 1: Tissue Properties [1,4]

The necessary values needed to solve equation (13) are listed in Table 2:

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$1700 \text{ kg/m}^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V$</td>
<td>$0.00000858 \text{ m}^3$</td>
</tr>
<tr>
<td>$C_p$</td>
<td>$1200 \text{ J/kg} \cdot \text{K}$</td>
</tr>
<tr>
<td>$E$</td>
<td>$0.95$</td>
</tr>
<tr>
<td>$A_s$</td>
<td>$2.787E-3 \text{ m}^2$</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>$5.67E-8 \text{ W/m}^2 \cdot \text{K}^4$</td>
</tr>
</tbody>
</table>

Table 2: Material properties used when solving the differential equation [1,4]
Equation (13) was solved using a simple numerical method technique known as Euler’s method, which is represented below:

\[
t_i + 1 = t_i + \Delta t \\
T_{si} + 1 = T_{si} + \Delta t \cdot f(t_i, T_{si})
\]

For this given analysis, the numerical technique involves using \( T_{si} \) which was 33 °C, and a specified delta time value, which was varied from 0.01 to 0.1 seconds depending on how quickly convergence was achieved, so that the surface temperature could be solved for the next time step. This surface temperature as then used as the \( T_{si} \) for the next iteration. The process was repeated until \( T_c \) was reached. The resulting first order differential equation was approximated using Euler’s method with forward differencing into the form represented by equation (14):

\[
\rho V C_p \left( \frac{T_{si+1} - T_i}{\Delta t} \right) = Q_{gen} - hA_s(T_s - T_\infty) - E\sigma A_s(T_s^4 - T_{sturr}^4)
\]

An Engineering Equation Solver (EES) program was developed to solve for all of the heat transfer coefficients and then Excel was used to calculate the elapsed time to reach \( T_c \).

**Results**

The time it takes for the finger to cool to a temperature where frostbite will begin to develop was solved for as described in the previous section. The shortest time seen for frostbite to begin damaging tissue was roughly 90 seconds, achieved with a wind speed of 20 mph and an outside temperature of -5°F. The longest time calculated was 477 seconds or 8 minutes, achieved with a wind speed of 5 mph and an outside temperature of 15°F. The results from numerical analysis are summarized in Table 3:

<table>
<thead>
<tr>
<th>Outside Temperature (°F)</th>
<th>Wind Speed (mph)</th>
<th>Wind Speed (mph)</th>
<th>Wind Speed (mph)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>213</td>
<td>130</td>
<td>90.1</td>
</tr>
<tr>
<td>0</td>
<td>252</td>
<td>141</td>
<td>98.3</td>
</tr>
<tr>
<td>10</td>
<td>298</td>
<td>165</td>
<td>114</td>
</tr>
<tr>
<td>15</td>
<td>376</td>
<td>203</td>
<td>138</td>
</tr>
<tr>
<td>15</td>
<td>478</td>
<td>245</td>
<td>165</td>
</tr>
</tbody>
</table>

Table 4 shows the effect wind speed had on the cooling of the finger for each of the outside temperatures. Figures 2 and 3 show that when the outside temperature is mild enough, as the finger approaches the frostbite temperature the rate of temperature change begins to decrease. Figures 4, 5 and 6 the heat transfer from the finger is large enough that this slowing of the temperature change never occurs, and the time to reach the critical temperature is significantly faster.
An additional parameter was applied to the analysis to determine the effect of using a different value for the constant heat generation by the finger had on the time to reach the critical temperature. The calculation of the constant heat generation was initially done using 20 °C (68°F) as the fluid temperature. This resulted in a \( Q_{\text{gen}} \) of 0.321 W. When the fluid temperature was decreased to 7°C (44.6°F) the heat required to keep the finger at a constant surface temperature of 33 °C rose significantly to 0.617 W. The elapsed times to reach \( T_c \) for a heat generation of 0.617 W is shown in Table 4:

Table 4: Time elapsed (sec) to reach critical temperature \( Q_{\text{gen}} = 0.617 \) W

<table>
<thead>
<tr>
<th>Outside Temperature (°F)</th>
<th>Wind Speed (mph)</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>-5</td>
<td>5</td>
<td>10</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>262</td>
<td>139</td>
<td>94.2</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>347</td>
<td>165</td>
<td>109</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>367</td>
<td>183</td>
<td>122</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>576</td>
<td>235</td>
<td>152</td>
<td></td>
</tr>
</tbody>
</table>

*Convergence wasn’t achieved, at 10686 sec \( T_c = 25 \) °F, at 15283 sec, \( T_c = 24 \) °F

Table 5 shows the percent difference between the two \( Q_{\text{gen}} \) values that were used. As can be seen when the wind speed is higher, the convection heat transfer has greater control on the outcome, causing the time to \( T_c \) to change less at the higher wind speed. Therefore time to reach the \( T_c \) is less dependent on the convection heat transfer coefficient for the lower wind speeds.

Table 5: Percent difference of time elapsed to achieve critical temperature

<table>
<thead>
<tr>
<th>Outside Temperature (°F)</th>
<th>Wind Speed (mph)</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>-5</td>
<td>20.6</td>
<td>6.69</td>
<td>4.56</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>27.2</td>
<td>15.7</td>
<td>10.6</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>20.8</td>
<td>10.3</td>
<td>6.78</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>42</td>
<td>14.6</td>
<td>9.66</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>*</td>
<td>68.6</td>
<td>13</td>
<td></td>
</tr>
</tbody>
</table>

*Convergence wasn’t achieved, at 10686 sec \( T_c = 25 \) °F, at 15283 sec, \( T_c = 24 \) °F
The results for the various ambient air temperature cases are summarized below. Figures 2-6 display the elapsed time for convergence to $T_c$:

Figure 2: Time to reach the critical temperature with an ambient temperature of 15 °F

Figure 3: Time to reach the critical temperature with an ambient temperature of 10 °F
Figure 4: Time to reach the critical temperature with an ambient temperature of 5 °F

Figure 5: Time to reach the critical temperature with an ambient temperature of 0 °F
Conclusion
This model was meant to reasonably predict the time it would take for bare skin to reach a temperature that where frostbite would begin to damage to the skin when exposed to cold and windy conditions. The body part that was investigated was the smallest finger on the hand. This was chosen because the extremities are at the most danger for frostbite due to the distance from the core of the body. When the body tries to retain heat blood is more restricted to the core and thus the extremities receive less blood flow and less heat.

The trend in results in this model was not surprising. The temperature difference and wind speed are both directly related to the heat transfer rate. When the ambient air temperature was lowest, the time it took for the finger to reach the frostbite temperature was decreased. When the wind speed was increased, the time to reach the critical temperature also decreased. This can be explained by looking at the heat transfer rate equations. A higher wind speed correlates to a higher convection coefficient, which increases the amount of heat transferred away from the finger. The same can be said for a larger temperature difference between the outside air and the surface of the finger, as it increases the more easily heat is able to be transferred. The values seen in the results section are quite reasonable. As previously mentioned, frostbite can take only minutes to begin when there are ambient air temperatures below 20°F and winds speeds of 20 mph, which is confirmed in the results. However since most people would not allow their hand to be exposed to these temperatures and conditions for a long enough period it is unlikely that these conditions would cause damage unless there were some extenuating circumstances, such as in an emergency. It should be noted that simply rubbing your hands together, shielding them from the wind, or placing them close to your body will significantly slow the heat transfer rate to the surroundings, which would effectively delay frostbite.

The results that have been obtained for the elapsed time for the finger to reach the critical frostbite temperature rely significantly on the calculation of the constant heat generation from the finger. First off, the body does demonstrate the ability through metabolism to adapt or adjust its
rate of heat transfer to different parts of the body either involuntarily by metabolic means or from an increase in heart rate and thus blood flow. Therefore in different environments and surroundings the heat generated by the body fluctuates quite significantly. A more complex analysis could be performed by creating a second order differential equation with a fluctuating heat generation. The properties for the finger itself that were assumed could also have a large percentage of error depending on the different individual properties of the finger being analyzed. Adjusting the density and specific heat of the finger has a relatively significant effect on the calculated elapsed time to reach the critical temperature.

Acknowledgments
We would like to acknowledge the help of Dr. Sawyers in creating a reasonable numerical model for the analysis of the finger.

References