1. (3 points) Verify that \( f(x) = \frac{x}{x+2} \) satisfies the hypotheses of the Mean Value Theorem on the interval \([1, 4]\). Then find all numbers \( c \in [1, 4] \) that satisfy the conclusion of the Mean Value Theorem.

**Solution:** \( f \) is continuous on \([1, 4]\) since \( f \) is a rational function defined at all points in \([1, 4]\) and \( f \) is differentiable on \((1, 4)\) by the quotient rule since both \( x \) and \( x+2 \) are differentiable everywhere and \( x+2 \neq 0 \) on \((1, 4)\).

So it remains to find the values of \( c \in [1, 4] \) that satisfy

\[
f'(c) = \frac{f(4) - f(1)}{4 - 1}.
\]

Taking the derivative of \( f \) at \( c \) and evaluating \( f \) at 1 and 4 gives

\[
\frac{2}{(c+2)^2} = \frac{\frac{2}{3} - \frac{1}{3}}{4 - 1} = \frac{1}{9}.
\]

So \((c+2)^2 = 18\). Expanding out the left hand side, subtracting by 18 and using the quadratic formula gives \( c = -2 \pm 3\sqrt{2} \). But only the sum is in \([1, 4]\), so the values of \( c \) that satisfy the Mean Value Theorem are \( c = -2 + 3\sqrt{2} \approx 2.24 \).
2. (3 points) Boyle’s Law states that when a sample of gas is compressed at a constant temperature, the pressure $P$ and volume $V$ satisfy the equation $PV = C$, where $C$ is a constant. Suppose that at a certain instant the volume is 600 cm$^3$, the pressure is 150 kPa, and the pressure is increasing at a rate of 20 kPa/min. At what rate is the volume decreasing at this instant?

**Solution:** Differentiating both sides of $PV = C$ with respect to time $t$ and using the product rule gives

$$P \frac{dP}{dt} + V \frac{dV}{dt} = 0.$$ 

Solving for $\frac{dV}{dt}$ gives

$$\frac{dV}{dt} = -\frac{V}{P} \frac{dP}{dt}.$$ 

When $V = 600$, $P = 150$ and $\frac{dP}{dt} = 20$, we have

$$\frac{dV}{dt} = -\frac{600}{150}(20) = -80.$$ 

So the volume is decreasing at a rate of 80 cm$^3$/min.
3. (2 points) When blood flows along a blood vessel, the flux $F$ (the volume of blood per unit time that flows past a given point) is proportional to the fourth power of the radius $R$ of the blood vessel:

$$F = kR^4.$$ 

(This is known as Poiseuille’s Law.) A partially clogged artery can be expanded by an operation called angioplasty, in which a balloon-tipped catheter is inflated inside the artery in order to widen it and restore the normal blood flow.

Show that the relative change in $F$ is about four times the relative change in $R$. How will a 5% increase in the radius affect the blood flow. (Hint: “Relative change” in this situation is analogous to “relative error.”)

Solution: Since $F = kR^4$, the differential of $F$ is $dF = 4kR^3 \, dR$. So the relative change in $F$ is given by

$$\frac{dF}{F} = \frac{4kR^3 \, dR}{kR^4} = 4 \left( \frac{dR}{R} \right).$$

Thus the relative change in $F$ is about four times the relative change in $R$. So a 5% increase in the radius corresponds to a 20% increase in blood flow.
4. (2 points) Use a linear approximation (or differentials) to estimate \( \tan 44^\circ \).

**Solution:** Let \( y = f(x) = \tan x \). Then \( dy = f'(x) \, dx = \sec^2 x \, dx \) by the definition of the differential. So when \( x = 45^\circ \) and \( dx = -1^\circ \),

\[
dy = \sec^2(45^\circ) \cdot \frac{-\pi}{180} = (\sqrt{2})^2 \frac{-\pi}{180} = -\frac{\pi}{90}.
\]

Thus

\[
\tan(44^\circ) = f(44^\circ) \approx f(45^\circ) + dy = 1 - \frac{\pi}{90} \approx 0.965.
\]

(The real value is 0.9656887748...)

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