You are allowed to work in groups of no more than five (5) people. This quiz is open book and open notes. Each group only has to hand in one (1) quiz, so please make sure it is written in a neat and concise manner.

1. (4 points) An object with weight $W$ is dragged along a horizontal plane by a force acting along a rope attached to the object. If the rope makes an angle $\theta$ with a plane, then the magnitude of the force is

$$ F = \frac{\mu W}{\mu \sin \theta + \cos \theta}, $$

where $\mu$ is a constant called the coefficient of friction. For what value of $\theta$ is $F$ smallest?

**Solution:** The goal is to minimize the function $F(\theta)$ for $\theta \in [0, \pi/2]$. In order to do so, we must take the derivative:

$$ F'(\theta) = \frac{-\mu W (\mu \cos \theta - \sin \theta)}{(\mu \sin \theta + \cos \theta)^2}. $$

Notice the denominator is always positive as the square of a real number, so if $F'(\theta)$ is positive, then it must be the case that $\mu \cos \theta - \sin \theta < 0$. This implies $\mu \cos \theta < \sin \theta$, so dividing both sides by $\cos \theta$ gives $\mu < \tan \theta$, thus $\theta > \arctan \mu$. Using this information, $F$ is decreasing on $(0, \arctan \mu)$ since $F'$ is negative there, and $F$ is increasing on $(\arctan \mu, \pi/2)$. The first derivative test tells us the minimum value of $F$ occurs at $\theta = \arctan \mu$. Thus the minimum value is

$$ F(\arctan \mu) = \frac{\mu W}{\mu \sin(\arctan \mu) + \cos(\arctan \mu)} = \frac{\mu W}{\mu \sqrt{1+\mu^2} + \frac{1}{\sqrt{1+\mu^2}}} = \frac{\mu W}{\sqrt{\mu^2 + 1}}. $$
2. Find $f$ for the following functions.

(a) (2 points) $f'''(x) = x - \sqrt{x}$.

(b) (2 points) $f''(x) = 2e^x + 3\sin x$, $f(0) = 0$, $f(\pi) = 0$.

(c) (2 points) $f'(x) = \frac{2 + x^2}{1 + x^2}$.

**Solution:**

(a) It is beneficial to rewrite $f'''(x) = x - x^{1/2}$. So using our formula for “undoing” the power rule, we have

\[
\begin{align*}
f''(x) &= \frac{x^2}{2} + \frac{2x^{3/2}}{3} + C \\
f'(x) &= \frac{x^3}{6} + \frac{4x^{5/2}}{15} + Cx + D \\
f(x) &= \frac{x^4}{24} + \frac{8x^{7/2}}{105} + \frac{Cx^2}{2} + Dx + E.
\end{align*}
\]

(b) Using antiderivatives we have

\[
\begin{align*}
f'(x) &= 2e^x - 3\cos x + C \\
f(x) &= 2e^x - 3\sin x + Cx + D.
\end{align*}
\]

Now we use the given conditions to solve for $C$. We have $f(0) = 0$, so $0 = 2e^0 - 3(0) + 0 + D$. Thus $2 + D = 0$ and $D = -2$. Then $f(\pi) = 0$ implies

\[
0 = 2e^\pi - 3\sin \pi + C\pi - 2 = 2e^\pi + C\pi - 2 \iff C = \frac{2 - 2e^\pi}{\pi}.
\]

So

\[f(x) = 2e^x - 3\sin x + \frac{2 - 2e^\pi}{\pi}x - 2.\]

(c) There’s a bit of a trick to this one! Note that $2 + x^2 = 1 + (1 + x^2)$, so we can rewrite our derivative as

\[f(x) = 2 + x^2 = 1 + (1 + x^2) = \frac{1}{1 + x^2} + 1.\]

We know that the antiderivative of the first term is $\arctan x$, so

\[f(x) = \arctan x + x + C.\]