1. (3 points) A hole of radius $r$ is bored through the center of a sphere of radius $R > r$. Find the volume of the remaining portion of the sphere.

**Solution:** The line $y = r$ intersects the semicircle $y = \sqrt{R^2 - x^2}$ when $r = \sqrt{R^2 - x^2}$ implies that $r^2 = R^2 - x^2$, so $x^2 = R^2 - r^2$ and $x = \pm \sqrt{R^2 - r^2}$. Rotating the region (bounded by $y = \sqrt{R^2 - x^2}$ and $y = r$) about the $x$-axis gives

$$
V = \int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} \pi \left( (\sqrt{R^2 - x^2})^2 - r^2 \right) \, dx.
$$

Then since the integrand is an even function,

$$
V = 2\pi \int_{0}^{\sqrt{R^2-r^2}} \left( (R^2 - x^2) - r^2 \right) \, dx = 2\pi \int_{0}^{\sqrt{R^2-r^2}} ((R^2 - x^2) - r^2) \, dx.
$$

Then

$$
V = 2\pi \int_{0}^{\sqrt{R^2-r^2}} (R^2 - x^2 - r^2) \, dx = 2\pi \left. \left( (R^2 - r^2)x - \frac{1}{3}x^3 \right) \right|_{0}^{\sqrt{R^2-r^2}}
$$

$$
= 2\pi \left( (R^2 - r^2)\sqrt{R^2-r^2} - r^2 \right) - \frac{2}{3}(R^2 - r^2)\sqrt{R^2-r^2}
$$

$$
= 2\pi \cdot \frac{2}{3}(R^2 - r^2)^{3/2} = \frac{4}{3}(R^2 - r^2)^{3/2}.
$$

This answer should make sense to you, since taking the limit as $r \to 0$ yields $V \to \frac{4}{3}\pi R^3$. 
2. (2 points) Newton’s Law of Gravitation states that two bodies with masses \( m_1 \) and \( m_2 \) attract each other with a force

\[
F = G \frac{m_1 m_2}{r^2},
\]

where \( r \) is the distance between the bodies and \( G \) is the gravitational constant. If one of the bodies is fixed, find the work needed to move the other from \( r = a \) to \( r = b \).

**Solution:** We have

\[
W = \int_a^b F(r) \, dr = \int_a^b G \frac{m_1 m_2}{r^2} \, dr = Gm_1 m_2 \left( \frac{-1}{r} \right) \bigg|_a^b = Gm_1 m_2 \left( \frac{1}{a} - \frac{1}{b} \right).
\]
3. (2 points) If \( f(0) = g(0) = 0 \) and \( f'' \) and \( g'' \) are continuous, show that

\[
\int_0^a f(x)g''(x) \, dx = f(a)g'(a) - f'(a)g(a) + \int_0^a f''(x)g(x) \, dx.
\]

That is, evaluate the integral on the left until you end at the expression on the right.

**Solution:** Suppose \( f(0) = g(0) = 0 \) and let \( u = f(x) \) and \( dv = g''(x) \, dx \). Then \( du = f'(x) \, dx \) and \( v = g'(x) \). So our integral becomes

\[
\int_0^a f(x)g''(x) \, dx = f(x)g'(x) \bigg|_0^a - \int_0^a f'(x)g'(x) \, dx = f(a)g'(a) - \int_0^a f'(x)g'(x) \, dx.
\]

Now we use integration by parts again on the integral on the right. Let \( U = f'(x) \) and \( dV = g'(x) \, dx \). Then \( dU = f''(x) \, dx \) and \( V = g(x) \). Then

\[
\int_0^a f'(x)g'(x) \, dx = f'(x)g(x) \bigg|_0^a - \int_0^a f''(x)g(x) \, dx = f'(a)g(a) - \int_0^a f''(x)g(x) \, dx.
\]

Combining the two lines gives

\[
\int_0^a f(x)g''(x) \, dx = f(a)g'(a) - f'(a)g(a) + \int_0^a f''(x)g(x) \, dx.
\]
4. (3 points) Use the method of cylindrical shells to find the volume generated by rotating the region bounded by \( y = e^x, y = e^{-x}, \) and \( x = 1 \) about the \( y \)-axis.

**Solution:** The volume of the solid is given by

\[
V = \int_0^1 2\pi (e^x - e^{-x}) \, dx = 2\pi \int_0^1 (xe^x - xe^{-x}) \, dx = 2\pi \left( \int_0^1 xe^x \, dx - \int_0^1 xe^{-x} \, dx \right).
\]

We must use integration by parts on both integrals. For the first integral, let \( u = x \) and \( dv = e^x \). Then \( du = dx \) and \( v = e^x \). So

\[
\int xe^x \, dx = xe^x - \int e^x \, dx = xe^x - e^x.
\]

A similar process on the other integral yields

\[
\int xe^{-x} \, dx = -xe^{-x} - e^{-x}.
\]

So

\[
V = 2\pi \left( \int_0^1 xe^x \, dx - \int_0^1 xe^{-x} \, dx \right) = 2\pi \left( \left. \left( xe^x - e^x \right) \right|_0^1 - \left. \left( -xe^{-x} - x \right) \right|_0^1 \right) = 2\pi \left( \frac{2}{e} - 0 \right) = \frac{4\pi}{e}.
\]