1. (2 points) Evaluate $\int \frac{\cos x + \sin x}{\sin 2x} \, dx$. Hint: Multiplying by $\frac{\csc x - \cot x}{\csc x - \cot x}$ may be helpful.

**Solution:** Using the half-angle formula,

$$\int \frac{\cos x + \sin x}{\sin 2x} \, dx = \frac{1}{2} \int \frac{\cos x + \sin x}{\cos x \sin x} = \frac{1}{2} \int (\csc x + \sec x) \, dx.$$

In class we saw

$$\int \sec x \, dx = \ln |\sec x + \tan x|,$$

so we need only concern ourselves with the integral of $\csc x$. Indeed, using the hint,

$$\int \csc x \, dx = \int \frac{\csc x - \cot x}{\csc x - \cot x} \, dx = \int \frac{-\csc x \cot x + \csc^2 x}{\csc x - \cot x} \, dx.$$

Using the substitution $u = \csc x - \cot x$, we have $du = (-\csc x \cot x + \csc^2 x) \, dx$, so

$$\int \csc x \, dx = \int \frac{du}{u} = \ln |u| = \ln |\csc x - \cot x| + C,$$

so

$$\int \frac{\cos x + \sin x}{\sin 2x} = \frac{1}{2} (\ln |\csc x - \cot x| + \ln |\sec x + \tan x|) + C.$$
2. (2 points) Evaluate $\int_0^{\pi/2} \frac{\cos x}{\sqrt{1 + \sin^2 x}} \, dx$ using trigonometric substitution.

\textbf{Solution:} Let $u = \sin x$, so $du = \cos x \, dx$. Then

\[
\int_0^{\pi/2} \frac{\cos x}{\sqrt{1 + \sin^2 x}} \, dx = \int_0^1 \frac{1}{\sqrt{1 + u^2}} \, du.
\]

Now let $u = \tan \theta$. So $du = \sec^2 \theta \, d\theta$ and $\sqrt{1 + u^2} = \sec \theta$. Hence

\[
\int_0^1 \frac{1}{\sqrt{1 + u^2}} \, du = \int_0^{\pi/4} \frac{1}{\sec \theta} \sec^2 \theta \, d\theta
\]

\[
= \int_0^{\pi/4} \sec \theta \, d\theta
\]

\[
= \ln |\sec \theta + \tan \theta| \bigg|_0^{\pi/4}
\]

\[
= \ln(\sqrt{2} + 1) - \ln(1 + 0)
\]

\[
= \ln(\sqrt{2} + 1).
\]
3. (2 points) Evaluate \( \int_0^1 \frac{\ln x}{\sqrt{x}} \, dx \).

**Solution:** We use integration by parts with \( u = \ln x \) and \( dv = \frac{dx}{\sqrt{x}} \). So \( du = \frac{1}{x} \, dx \) and \( v = 2\sqrt{x} \).

We have

\[
\int_0^1 \frac{\ln x}{\sqrt{x}} \, dx = \lim_{t \to 0^+} \int_t^1 \frac{\ln x}{\sqrt{x}} \, dx = \lim_{t \to 0^+} \left( 2\sqrt{x} \ln x \bigg|_t^1 - 2 \int_t^1 \frac{dx}{\sqrt{x}} \right).
\]

Evaluating the integral on the right we have

\[
= \lim_{t \to 0^+} \left( -2\sqrt{t} \ln t - 4\sqrt{t} \right) = \lim_{t \to 0^+} (-2\sqrt{t} \ln t - 4 + 4\sqrt{t}).
\]

Then, by L’hôpital’s rule, we have

\[
\lim_{t \to 0^+} \sqrt{t} \ln t = \lim_{t \to 0^+} \frac{\ln t}{t^{-1/2}} = \lim_{t \to 0^+} \frac{1/t}{-t^{-3/2}/2} = \lim_{t \to 0^+} (-2\sqrt{t}) = 0,
\]

so

\[
\int_0^1 \frac{\ln x}{\sqrt{x}} \, dx = \lim_{t \to 0^+} (-2\sqrt{t} \ln t - 4 + 4\sqrt{t}) = 0 - 4 + 0 = -4.
\]
4. (2 points) Evaluate $\int \frac{x - 1}{x^2 + 2x} \, dx$ using partial fractions.

**Solution:** Begin by factoring the denominator: $x^2 + 2x = x(x + 2)$. So using partial fractions, we have

$$\frac{x - 1}{x^2 + 2x} = \frac{x - 1}{x(x + 2)} = \frac{A}{x} + \frac{B}{x + 2}.$$ 

So

$$x - 1 = A(x + 2) + Bx = Ax + 2A + Bx = (A + B)x + 2A$$

allows us to solve

$$A + B = 1$$
$$2A = -1.$$ 

So $A = -1/2$ and $B = 3/2$, and

$$\int \frac{x - 1}{x^2 + 2x} \, dx = -\frac{1}{2} \int \frac{dx}{x} + \frac{3}{2} \int \frac{dx}{x + 2} = -\frac{1}{2} \ln |x| + \frac{3}{2} \ln |x + 2| + C.$$
5. (2 points) If $f$ is a quadratic function such that $f(0) = 1$ and

$$\int \frac{f(x)}{x^2(x+1)^3} \, dx$$

is a rational function (!), find the value of $f'(0)$.

**Solution:** Since $f$ is quadratic, we can write $f(x) = ax^2 + bx + c$ for some $a, b, c \in \mathbb{R}$. From $f(0) = 1$, we have $1 = a(0) + b(0) + c$, so $c = 1$ and $f(x) = ax^2 + bx + 1$. By partial fractions we have

$$\frac{f(x)}{x^2(x+1)^3} = \frac{ax^2 + bx + 1}{x^2(x+1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2} + \frac{E}{(x+1)^3}.$$  

Since the question says that the integral of this function is a rational function, it means that there can be no “funny” functions showing up in the evaluation. In particular, there can be no logarithms in the answer, which implies $A = C = 0$ (just look at the integrals of the corresponding fractions above!). Then, multiplying through by the denominator of the fraction on the left, we have

$$ax^2 + bx + 1 = B(x+1)^3 + Dx^2(x+1) + Ex^2$$

$$= B(x^3 + 3x^2 + 3x + 1) + D(x^3 + x^2) + Ex^2$$

$$= (B + D)x^3 + (3B + D + E)x^2 + 3Bx + B.$$ 

Equating coefficients gives

$$B + D = 0$$

$$3B + D + E = a$$

$$3B = b$$

$$B = 1.$$ 

So $B = 1$ and $b = 3B = 3$. Hence $f'(0) = 2a(0) + 3 = 3.$